ON CERTAIN CONFIGURATIONS OF THE CARDINAL POINTS IN PLANE KINEMATICS.

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Abstract. In plane kinematics, the knowledge of the location of the instantaneous centre — deriving from the eighteenth century — affords information which is partial and very incomplete. But it is the case that there always exists an enumerable set of *cardinal points*, as they may be named, having the following characteristic: all the properties of the path of any, and every, element or series of elements, fixed in the moving plane, are completely determined by the configuration of this set of cardinal points. In this manner is given a very simple and complete synthesis of the whole realm of plane kinematics.

Essentially, in plane kinematics, there is a duality; for we deal with the relative coplanar motion of two planes, depending, in the usual mechanism or linkage, upon a single parameter. There is then a dual set of cardinal points, the configuration of each set depending upon the assigned relative motion, and also upon the parameter. And it is the case that, for an assigned relative motion the configuration of either set, for a single value of the parameter, determines the configurations of both sets, for all values of the parameter. These changing configurations, corresponding to varying values of the parameter, possess therefore certain properties which are conserved throughout, and express the underlying unity of the particular relative motion. But, in certain cases, there is a conservation of a more special kind — a conservation of the form of the set itself. In the present paper are considered some cases of this special conservation of form, and we are led to investigate sets of cardinal points which may be named *rectangular*, *linear*, *spiral*, *circular*, and *orthogonal* — together with the kinematical implications of such sets.

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1. Introduction: The Cardinal Points. We consider two planes p and ϖ subject to relative coplanar displacement, or motion, and we assume this displacement, or motion, to have one degree of freedom, as in the usual mechanism, or linkage. As the parameter, or generalized coordinate, of this single degree of freedom, it is convenient to take the angle ϕ between two lines, one fixed in each of the two planes, and the relative displacement is assumed to be a single-valued function of the variable ϕ.

We define (STEWARD 1951) two enumerable sets of *cardinal points*, $\mathcal{A}_n$ and $\mathcal{A}'_n$, $n=1, 2, 3, \ldots$, having the property that the relative path of any, and every, element of either plane is completely determined by the relationship of the element to either set of these cardinal points. It is the case that either set of points is completely determined by the other set; for, clearly we have a duality. Indeed, for an infinitesimal displacement, $\mathcal{A}_1$ and $\mathcal{A}'_1$ coincide with the point commonly known as the instantaneous centre; but, for $n>1$, $\mathcal{A}_n$ and $\mathcal{A}'_n$ do not coincide, in general.

In this way the general relative displacement of the two planes is analysed uniquely into the configuration of either of the two enumerable sets of cardinal points.

Analytically, we take rectangular axes Oxy, fixed in the plane p, and $\Omega\xi\eta$ fixed in the plane ϖ, and for any point P, not in general fixed in either plane, we write $P(z, \zeta)$, $z\equiv x+iy$, $\zeta\equiv\xi+i\eta$. We write also $\bar{z}(\phi)$, $\bar{\zeta}(\phi)$ respectively for the point Ω, referred to the z-axes, and for the point O referred to the ζ-axes, where ϕ is the angle between the lines $\Omega\xi$ and Ox. Then clearly

$$z=\bar{z}+\zeta \exp(i\phi),$$

and

$$\bar{z}(\phi)+\bar{\zeta}(\phi)\exp(i\phi)=0.$$

We introduce the operators ∂ and ∂', defined by

$$\partial\equiv d/d\phi+i \quad \text{and} \quad \partial'\equiv d/d\phi-i,$$

and reducing to $\partial_0\,(\equiv i)$ and $\partial'_0\,(\equiv -i)$ when applied to a quantity independent of ϕ; and we indicate differentiation with respect to the variable ϕ as follows,

$$\bar{z}^{(n)}=d^n\bar{z}/d\phi^n.$$

We may show then that the two sets of cardinal points are given by the dual relations

$$\mathcal{A}_n(z_n,\ \zeta_n):\quad z_n = \bar{z} - \partial_0'^{\,n}\, \bar{z}^{(n)};\quad \zeta_n = \partial_0'^{\,n}\, \partial^n\, \bar{\zeta}^{(n)};$$

$$\mathcal{A}'_n(z'_n,\ \zeta'_n):\quad z'_n = \partial_0^n\, \partial'^{\,n}\, \bar{z};\qquad \zeta'_n = \bar{\zeta} - \partial_0^n\, \bar{\zeta}^{(n)}.$$

The relationship between these two sets of points is given by

$$z'_n = \sum_{r=1}^{n} \sigma_{n,r}\, z_r\,,\qquad \zeta_n = \sum_{r=1}^{n} \sigma_{n,r}\, \zeta'_r\,,$$

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where

$$\sigma_{n,r} = (-)^{r+1}\binom{n}{r},\quad \text{so that}\quad \sum_{r=1}^{n} \sigma_{n,r} = 1.$$

In passing, we notice that $\mathcal{A}'_n$ coincides with the centre of mass of masses proportional to $\sigma_{n,r}$, placed at $\mathcal{A}_r$, $r = 1, 2, \ldots, n$; and dually for the derivation of $\mathcal{A}_n$ from $\mathcal{A}'_1, \mathcal{A}'_2, \ldots, \mathcal{A}'_n$.

Further, we write

$$w_n(\phi) = z_{n+1} - z_n,\quad \text{and}\quad \omega_n(\phi) = \zeta_{n+1} - \zeta_n,$$

so that

$$w_n(\phi) = \omega_n(\phi)\, \exp\,(i\,\phi),$$

and $w_n(\phi)$, $\omega_n(\phi)$ represent the vector $\mathcal{A}_n\mathcal{A}_{n+1}$ referred to the two sets of axes respectively. Dually, for the vector $\mathcal{A}'_n\mathcal{A}'_{n+1}$, we have

$$w'_n(\phi) = z'_{n+1} - z'_n,\quad \omega'_n(\phi) = \zeta'_{n+1} - \zeta'_n$$

and

$$w'_n(\phi) = \omega'_n(\phi)\, \exp\,(i\,\phi).$$

The relations between these several quantities are given by

$$w'_n = \sum_{r=1}^{n} \sigma_{n-1,r-1}\, w_r,\qquad \omega_n = \sum_{r=1}^{n} \sigma_{n-1,r-1}\, \omega'_r,$$

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2. Various Formulae. The cardinal points, in general, are not fixed in either plane; associated with each such point, therefore, are two curves, the corresponding paths in each of the two planes p and ϖ. The simplest case, that of $\mathcal{A}_1\,(\mathcal{A}'_1)$, gives rise to the usual *centrodes*; for the other cardinal points, we have 'centrodes' of higher orders.

If we differentiate μ times with respect to the parameter ϕ, we have

$$z_n^{(\mu)} = \partial_0^\mu (z_{n+\mu} - z_\mu) = \partial_0^\mu \left\{ \sum_{r=1}^{n+\mu} \sigma_{n+\mu,r} z_r' - \sum_{r=1}^{\mu} \sigma_{\mu,r} z_r' \right\},$$

and

$$z_n'^{(\mu)} = \partial_0^\mu \sum_{r=0}^{n} \sigma_{n,r} z_{\mu+r} = -\partial_0^\mu \sum_{r=0}^{\mu} \sigma_{\mu,r} z_{n+r}',$$

together with the dual formulae

$$\zeta_n^{(\mu)} = -\partial_0'^\mu \sum_{r=0}^{\mu} \sigma_{\mu,r} \zeta_{n+r} = \partial_0'^\mu \sum_{r=0}^{n} \sigma_{n,r} \zeta_{\mu+r}'$$

and

$$\zeta_n'^{(\mu)} = \partial_0'^\mu (\zeta_{n+\mu}' - \zeta_\mu') = \partial_0'^\mu \left\{ \sum_{r=1}^{n+\mu} \sigma_{n+\mu,r} \zeta_r - \sum_{r=1}^{\mu} \sigma_{\mu,r} \zeta_r \right\}.$$

We have also

$$z_n^{(\mu)} = \partial_0^\mu \sum_{r=0}^{\mu-1} w_{n+r}, \quad z_n'^{(\mu)} = \partial_0^\mu \sum_{r=0}^{\mu-1} \sigma_{\mu-1,r} w_{n+r}',$$

and

$$\zeta_n^{(\mu)} = \partial_0'^\mu \sum_{r=0}^{\mu-1} \sigma_{\mu-1,r} \omega_{n+r}, \quad \zeta_n'^{(\mu)} = \partial_0'^{(\mu)} \sum_{r=0}^{\mu-1} \omega_{n+r}'.$$

We note the general result

$$\sum_{r=0}^{n} \sigma_{n,r} z_{\mu+r} + \sum_{r=0}^{\mu} \sigma_{\mu,r} z_{n+r}' = 0.$$

We have also

$$w_n^{(\mu)} = \partial_0^\mu w_{n+\mu}, \qquad \omega_n^{(\mu)} = -\partial_0'^\mu \sum_{r=0}^{\mu} \sigma_{\mu,r} \omega_{n+r},$$

$$w_n'^{(\mu)} = -\partial_0^\mu \sum_{r=0}^{\mu} \sigma_{\mu,r} w_{n+r}', \quad \omega_n'^{(\mu)} = \partial_0'^\mu \omega_{n+\mu}'.$$

Latent here are many properties of the 'centrodes' of the several orders. In particular, for $n = 1$, $\mu = 1$, we have

$$z_1^{(1)} = \partial_0 w_1, \quad \zeta_1'^{(1)} = \partial_0' \omega_1', \quad w_1 + w_1' = 0,$$

that is

$$z_1^{(1)} = \zeta_1'^{(1)} \exp(i\phi);$$

or, the familiar rolling property of the usual centrodes, the paths of $A_1 (A_1')$ in the two planes p and ϖ.

The relative path of an arbitray point, fixed in either plane, depends only upon the geometrical relationship between this point and either of the sets of the cardinal points. Indeed, if $P_{\tilde{\omega}}(z)$ be a point fixed in the plane ϖ the variation $\varDelta z$ of z, consequent upon a variation $\varDelta\phi$ of ϕ, is given by

$$\varDelta z = \sum_{n=1}^{\infty} \sigma_n (\varDelta \phi)^n, \quad \text{where} \quad n!\,\sigma_n = \partial_0^n \mathcal{A}_n P_{\tilde{\omega}}.$$

Dually, for a point $P_p(\zeta)$, fixed in the plane p, we have

$$\varDelta \zeta = \sum_{n=1}^{\infty} \sigma'_n (\varDelta \phi)^n, \quad \text{where} \quad n!\,\sigma'_n = \partial_0'^n \mathcal{A}'_n P_p.$$

3. The Rectangular Set. The foot of the perpendicular B, from $\mathcal{A}_1$ upon $\mathcal{A}_1\mathcal{A}_3$, is the *Ball point*, indicating that single element $B_{\tilde{\omega}}$ of the plane ϖ, the path of which, relative to p, has *four* point contact with a straight line of p. And this, in general, for any element of ϖ, is the highest order contact possible with any straight line of p. Further, the Ball point is not, in general, fixed in either plane, so that there are two associated *Ball lines*, the two paths of $B(\phi)$ in the two planes, as ϕ varies.

Dually, there is a second Ball point $B'(\phi)$, derived similarly from $\mathcal{A}'_1$, $\mathcal{A}'_2$ and $\mathcal{A}'_3$, indicating the single element B'_p, of p, the path of which, relative to ϖ, has four point contact with a straight line of ϖ; and there are, in general, two dual Ball lines.

If, as a special case, $\mathcal{A}_4$ lie upon $\mathcal{A}_2 B$, then the Ball point indicates *five* point contact of the relative path of $B_{\tilde{\omega}}$, with a straight line of p; and proceeding step by step it is the case, as may be proved from the formulae of §§ 1, 2, that if $\mathcal{A}_1, \mathcal{A}_3, \ldots, \mathcal{A}_{2n+1}, \ldots$ be collinear, and also $\mathcal{A}_2, \mathcal{A}_4, \ldots, \mathcal{A}_{2n}, \ldots$ be collinear, and if, further, the two lines so arising be perpendicular, meeting in B, then the element $B_{\tilde{\omega}}$ of ϖ, at B, describes accurately a straight line in p. In this case the Ball point is fixed in ϖ, and the associated Ball line in p is the line $\mathcal{A}_2, \mathcal{A}_4, \ldots, \mathcal{A}_{2n}, \ldots$, fixed in p. We regard such a configuration of the points $\mathcal{A}_n$ as constituting a *rectangular* set of cardinal points.

The rectangular set arises here for a single value of the parameter ϕ, but it is kinematically evident, and follows also from the formulae of §§ 1, 2, that the set remains rectangular throughout the motion, or continued displacement, for varying values of ϕ.

For, let $\mathcal{A}_1, \mathcal{A}_3, \ldots, \mathcal{A}_{2n+1}, \ldots$ lie upon a line l_1, and $\mathcal{A}_2, \mathcal{A}_4, \ldots, \mathcal{A}_{2n}, \ldots$ upon a perpendicular line l_2, for a particular value of ϕ; then we may write

$$z_{2n+1} = z_1 + \theta_{2n+1}\,\boldsymbol{s}, \quad \text{and} \quad z_{2n} = z_2 + \theta_{2n}\,\boldsymbol{t},$$

where the θ-coefficients are real, and $\boldsymbol{s}, \boldsymbol{t}$ are unit vectors parallel to l_1, l_2 respectively, so that $\boldsymbol{t} = i\,\boldsymbol{s}$. Differentiating these relations with respect to ϕ, and using the formulae of §§ 1, 2, we have

$$(\theta^{(1)}_{2n+1} + \theta_{2n+2})\,\boldsymbol{s} + \theta_{2n+1}\,\boldsymbol{s}^{(1)} = 0,$$

and

$$(\theta^{(1)}_{2n} - \theta_{2n+1} + \theta_3)\,\boldsymbol{t} + \theta_{2n}\,\boldsymbol{t}^{(1)} = 0.$$

Since $\boldsymbol{s}, \boldsymbol{t}$ are unit vectors, and the θ-coefficients are real, it follows that

$$\boldsymbol{s}^{(1)} = 0 \quad \text{and} \quad \boldsymbol{t}^{(1)} = 0,$$

and so the lines l_1, l_2 are fixed in direction, in the plane p. Incidentally, we notice also that

$$\theta^{(1)}_{2n+1} + \theta_{2n+2} = 0, \quad \text{and} \quad \theta^{(1)}_{2n} - \theta_{2n+1} + \theta_3 = 0.$$

And further, since

$$z^{(1)}_{2n} = \partial_0\,(z_{2n+1} - z_1) = \theta_{2n+1}\,\boldsymbol{t},$$

it follows that the line l_2 is fixed in position, in the z-plane p.

Also, if the lines l_1, l_2 intersect in the point B, we may write

$$B\,\mathcal{A}_{2n+1} = \psi_{2n+1}\,\boldsymbol{s}, \quad \text{and} \quad B\,\mathcal{A}_{2n} = \psi_{2n}\,\boldsymbol{t} = i\,\psi_{2n}\,\boldsymbol{s},$$

where the ψ-coefficients are real functions of ϕ, so that

$$\partial_0^{2n}\,\mathcal{A}_{2n}\,B = (-)^{n+1}\,\psi_{2n}\,\boldsymbol{t},$$

and

$$\partial_0^{2n+1}\,\mathcal{A}_{2n+1}\,B = (-)^{n+2}\,\psi_{2n+1}\,\boldsymbol{t} = -\partial_0^{2n+1}\,\psi_{2n+1}\,\boldsymbol{s}.$$

If, then, we consider a point $B_{\tilde{\omega}}(z)$, fixed in the plane ϖ and coinciding with B for the particular value of ϕ, the variation Δz of z, consequent upon the variation $\Delta\phi$ of ϕ, is given by

$$\Delta z = \sum_{n=1}^{\infty} \sigma_n\,(\Delta\phi)^n = \left\{ \sum_{n=1}^{\infty} \lambda_n\,(\Delta\phi)^n \right\} \boldsymbol{t},$$

from the formulae of §§ 1, 2, the λ-coefficients being real. Thus $B_{\tilde{\omega}}$ moves upon the line l_2, and indeed coincides with B for all values of the parameter ϕ; so we have a point fixed in the plane ϖ describing a line fixed in the plane p.

The converse of the preceding may be derived readily from the formulae of §§ 1, 2; namely, that if a point $B_{\tilde{\omega}}$, fixed in ϖ, describe a line l fixed in p the set $\mathcal{A}_n$ of cardinal points is rectangular. For we may take l to be coincident with the real axis in the z-plane, and write

$$\bar{z}(\phi)=\bar{x}(\phi),$$

a real function of ϕ. Then we have

$$\mathcal{A}_n \mathcal{A}_{n+2}=\partial_0'^n (\bar{x}^{(n)}+\bar{x}^{(n+2)}),$$

so that, for even values of n the points $\mathcal{A}_n$ all lie upon a line parallel to l, and indeed upon the line l itself, and for odd values of n upon a line perpendicular to l; and the set $\mathcal{A}_n$ is rectangular.

The rectangular configuration therefore, once existing, is conserved, and this is an illustration of the general property that the configuration of the cardinal points, for a single value of ϕ, is determined by, and itself determines, the relative motion for all values of ϕ.

It is evident that the dual of a rectangular set cannot itself be rectangular; yet, nevertheless, for each value of ϕ, and throughout its changing configurations for varying values of ϕ, such a dual set indicates the rectangular property of the associated set.

4. The Linear Set. A set of cardinal points, throughout its changing configurations, corresponding to varying values of the parameter ϕ, expresses the underlying unity of the assigned relative motion of the two planes ϖ and p; these changing configurations, therefore, possess properties conserved throughout the relative motion. We desire to consider here certain simple cases of such conservation.

We have seen already that the property of rectangularity is conserved. A special case of this property arises if *two* points, $B_{\tilde{\omega}}$ and $C_{\tilde{\omega}}$, each fixed in ϖ, describe lines l and m respectively, relative to p. We have then *elliptical* displacement of ϖ relative to p, and, dually, *cardioid* displacement of p relative to ϖ. And here, clearly, the cardinal points $\mathcal{A}_2, \mathcal{A}_4, \ldots, \mathcal{A}_{2n}, \ldots$ coincide with the intersection of l and m, while the cardinal points $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{2n+1}, \ldots$ coincide throughout with the intersection of the perpendiculars at $B_{\tilde{\omega}}$ and $C_{\tilde{\omega}}$ to l and m respectively. Also, the cardinal points $\mathcal{A}_n'$ lie, properly spaced, upon the line $\mathcal{A}_1\mathcal{A}_2$.

It is evident, from the relations

$$w_n'=\sum_{r=1}^{n}\sigma_{n-1,r-1}w_r, \quad \text{and its dual} \quad \omega_n=\sum_{r=1}^{n}\sigma_{n-1,r-1}\omega_r',$$

that, if either set of cardinal points be collinear the dual set also is collinear, the two lines so arising being parallel, and, further, since $\mathcal{A}_1$ and $\mathcal{A}_1'$ coincide, that these two lines themselves coincide.

But, in general, this collinearity arises for isolated values only of the parameter ϕ. We may ask — is it possible for such a set of cardinal points to remain collinear, for varying values of ϕ, and, if so, in what circumstances?

To examine this possibility, we write

$$w_{n+1}(\phi) = k_n(\phi)\, w_n(\phi), \quad w'_{n+1}(\phi) = k'_n(\phi)\, w'_n(\phi),$$

where $k_n(\phi)$ and $k'_n(\phi)$ are real, and, in general, functions of ϕ, depending also upon n. And we may show that if all of these quantities $k_n(\phi)$, $k'_n(\phi)$ are independent of ϕ they are also independent of n, and conversely. Further, if in this case we write

$$k_n(\phi) = k, \quad \text{and} \quad k'_n(\phi) = k',$$

for all n, then

$$k + k' = 1.$$

A more general treatment, of which this is a special case, is given in the following paragraphs.

For all values of ϕ the cardinal points are then spaced, in opposite directions, upon the same line, in two geometrical progressions, having common ratios k and k' respectively. And we refer to such a configuration as a *linear* set of cardinal points.

Consider a linear set of cardinal points, upon the line $l(\phi)$, a function of ϕ; we may ask — how does this line move, in the two planes, as ϕ varies?

We write

$$Z = \alpha_1 z_1 + \alpha_2 z_2, \quad \alpha_1 + \alpha_2 = 1,$$

where α_1, α_2 are real constants, so that the point Z lies upon the line $l(\phi)$.

Further, differentiating μ times with respect to ϕ, we have

$$Z^{(\mu)} = \partial_0^{\mu} k^{\mu-1} w_1 (1 + \alpha_2 k) = 0$$

for all μ, provided that

$$\alpha_1/(k+1) = \alpha_2/(-1) = 1/k.$$

In this case, the point $Z(P_p)$ is fixed in the plane p, and

$$Z = z_1 - w_1/k.$$

Dually, there is a point $Z'(P'_{\tilde{\omega}})$, of the line $l(\phi)$, fixed in ϖ where

$$Z' = \zeta'_1 - \omega'_1/k',$$

referred to the ζ-axes.

Then

$$A_1 P_p = -w_1/k, \quad \text{and} \quad A'_1 P'_{\tilde{\omega}} = -\omega'_1/k' = \omega_1/k',$$

the first being referred to the z-axes, and the second to the ζ-axes. Thus referred to the z-axes we have

$$P_p P'_{\tilde{\omega}} = w_1/k k'.$$

Also, $w_1(\phi)$, as a function of ϕ, is given by

$$w_1(\phi) = \sum_{n=1}^{\infty} w_1^{(n)}(0)\,\phi^n/n! = \sum_{n=1}^{\infty} \partial_0^n\, w_{n+1}(0)\,\phi^n/n! = \sum_{n=1}^{\infty} \partial_0^n\, k^n \phi^n\, w_1(0)/n!.$$

Or,

$$w_1(\phi) = w_1(0)\, \exp\,(\partial_0 k\phi),$$

and so is of constant modulus; as also is $w'_1(\phi)$. It follows that the centrodes, the paths of $A_1(A'_1)$, in the two planes, are circles, of centres P_p, $P'_{\tilde{\omega}}$, and radii $|w_1|/k$, $|\omega'_1|/k$ respectively. Or, the ratio of the radii is k'/k.

This, of course, was to be anticipated, from the linear configuration of the cardinal points; for such a set of points essentially implies permanent kinematical symmetry, of a certain kind, about a moving line.

In passing, we notice two particular cases of the linear set of cardinal points. If $k=0$, $k'=1$, there is a circle of ϖ rolling upon a straight line of p; here A_n, $n>2$, coincide — with the centre of the rolling circle — while the A'_n are equispaced along A_1A_2, A_1 being the point of contact. Again, if $k=-1$, $k'=2$, we have *elliptical* displacement of ϖ relative to p, and, dually, *cardioid* displacement of p relative to ϖ; then A_{2n} coincide, and A_{2n-1} coincide, for all values of n, while A'_n are spaced along the line A_1A_2 in geometrical progression, with common ratio 2.

5. The Spiral Set. For a set of cardinal points in general position, together with the dual set, we may write

$$w_{n+1}(\phi) = \varrho_n(\phi)\, w_n(\phi), \quad \text{and} \quad w'_{n+1}(\phi) = \varrho'_n(\phi)\, w'_n(\phi),$$

where the complex quantities $\varrho_n(\phi)$, $\varrho'_n(\phi)$ are functions of ϕ, and depend upon n, and we assume that none of these quantities vanishes.

It is the case that if either set of $\varrho_n(\phi)$, $\varrho'_n(\phi)$ be independent of ϕ, for all values of n, then they are all independent of n; and, conversely, if either set be independent of n, for a given value of ϕ, then they are all independent of ϕ.

For, if we write

$$R_n = \varrho_{n+1} - \varrho_n,$$

then from §§ 1, 2,

$$\varrho_n^{(1)} = \partial_0\, \varrho_n R_n,$$

and, generally,

$$\varrho_n^{(\mu)} = \partial_0^{\mu} \sum_{\nu=n}^{n+\mu-1} \theta(\mu, \nu) R_\nu,$$

where the coefficients $\theta(\mu, \nu)$ are homogeneous, of degree μ, in the ϱ-quantities; and, in particular

$$\theta(\mu, n+\mu-1) = \prod_{\nu=n}^{n+\mu-1} \varrho_\nu \neq 0.$$

Then, if $R_n, R_{n+1}, \ldots$ vanish, so also do $\varrho_n^{(1)}, \varrho_n^{(2)}, \ldots$; and if $\varrho_n^{(1)}, \varrho_n^{(2)}, \ldots$ vanish, so also do $R_n, R_{n+1}, \ldots$ And dually, in similar fashion, for the functions $\varrho_n'(\phi)$.

In this case we have

$$\varrho_n(\phi) = \varrho, \quad \varrho_n'(\phi) = \varrho'$$

for all values of n, ϱ and ϱ' being complex constants. And

$$w_{n+1}(\phi) = \varrho^n w_1(\phi), \quad \text{and} \quad w_{n+1}'(\phi) = \varrho'^n w_1'(\phi).$$

Since, from §§ 1, 2,

$$w_2' = \sigma_{1,0} w_1 + \sigma_{1,1} w_2, \quad \text{and} \quad w_1 + w_1' = 0,$$

we have

$$\varrho + \varrho' = 1.$$

The various quantities $w_n(\phi)$, $w_n'(\phi)$, $\omega_n(\phi)$, $\omega_n'(\phi)$ depend then upon the parameter ϕ only through their dependence upon $w_1(\phi)$, $w_1'(\phi)$, $\omega_1(\phi)$, $\omega_1'(\phi)$, and, for these, we have

$$w_1(\phi) = w_1(0) \exp(\partial_0 \varrho \phi), \quad w_1'(\phi) = w_1'(0) \exp(\partial_0' \varrho' \phi),$$

together with the dual expressions for $\omega_1(\phi)$ and $\omega_1'(\phi)$ respectively.

We write

$$\varrho = k \exp(i\alpha), \quad \varrho' = k' \exp(i\alpha').$$

If $\alpha = 0$, then also $\alpha' = 0$, and we have the linear set of § 4, with $k + k' = 1$. Moreover, the moduli of the functions $w_1(\phi)$, $w_1'(\phi)$, $\omega_1(\phi)$, $\omega_1'(\phi)$ are then constant; and so the configuration of the cardinal points is conserved in the sense that the set remains linear, for varying ϕ, and also the separations of the points A_n, A_{n+1} and A_n', A_{n+1}' remain constant.

But if $\alpha \neq 0$, then also $\alpha' \neq 0$, and we have a certain conservation of *shape* only; for the modulus of each element $A_n A_{n+1}$ has a factor $\exp(-k\phi \sin\alpha)$, and of each dual element $A_n' A_{n+1}'$ a factor $\exp(-k'\phi \sin\alpha')$.

The cardinal point $\mathcal{A}_n(z_n, \zeta_n)$ of such a set, associated with the complex constant ϱ, is given by

$$z_n = z_1 + w_1(1-\varrho^{n-1})/\varrho', \quad \text{or} \quad \zeta_n = \zeta_1 + w_1(1-\varrho^{n-1})/\varrho',$$

so that, for all values of n, this point lies upon the curve $S(\phi)$, given by

$$z(\phi, t) = z_1(\phi) + (1-\varrho^t)\, w_1(\phi)/\varrho', \quad \text{or} \quad \zeta(\phi, t) = \zeta_1(\phi) + (1-\varrho^t)\, \omega_1(\phi)/\varrho',$$

where t is a variable real parameter; that is, a logarithmic spiral, the pole $P_{\tilde{\omega}}$ of which is

$$z_1(\phi) + w_1(\phi)/\varrho', \quad \text{or} \quad \zeta_1(\phi) + \omega_1(\phi)/\varrho',$$

and, from the formulae of §§ 1, 2, is fixed in the plane ϖ.

In the usual notation, r being the radius vector at angle θ with the axis Ox, we may write the equation of this spiral in the form

$$r = a(\phi)\exp\{(\theta \log \alpha)/k\},$$

where

$$a(\phi) = a(0)\exp(-b\phi),$$

and

$$b = k(\alpha \sin \alpha + \log k \cos \alpha)/\alpha.$$

Thus the angle of the spiral, the constant inclination of the curve to the radius from the pole, is $\tan^{-1}(\alpha/\log k)$, and is independent of ϕ.

Dually, the associated cardinal points $\mathcal{A}'_n(z'_n, \zeta'_n)$, associated with the complex constant ϱ', lie upon a logarithmic spiral $S'(\phi)$, given by

$$z'(\phi, t') = z'_1(\phi) + (1-\varrho'^{t'})\, w'_1(\phi)/\varrho, \quad \text{or} \quad \zeta'(\phi, t') = \zeta'_1(\phi) + (1-\varrho'^{t'})\, \omega'_1(\phi)/\varrho,$$

t' being a variable real parameter. The pole P_p of this spiral is

$$z'_1(\phi) + w'_1(\phi)/\varrho, \quad \text{or} \quad \zeta'_1(\phi) + \omega'_1(\phi)/\varrho,$$

and is fixed in the z-plane. The angle of the spiral $S'(\phi)$ is $\tan^{-1}(\alpha'/\log k')$, and is similarly independent of ϕ.

It is evident, by writing $t = t' = -1$, that the pole of each of these spirals $S(\phi)$, $S'(\phi)$ lies upon the dual spiral; and also each curve passes through the point $\mathcal{A}_1(\mathcal{A}'_1)$.

We refer to such configurations as *spiral* sets of cardinal points.

If we write $\alpha = 0$, so that $\alpha' = 0$, and then $\varrho = k$ and $\varrho' = k'$, the poles of these two spirals coincide with the points P_p, $P_{\tilde{\omega}}$ of § 4, fixed in their respective planes, and associated with the linear set of cardinal points.

6. The Centrodes. The relative motion, or infinitesimal displacement, of the planes p and ϖ is uniquely characterised by the configuration of either of the dual sets of cardinal points A_n and A'_n. It is of interest to examine the associated centrodes — the two paths, in p and ϖ respectively, of the coincident points A_1, A'_1.

We consider a spiral set of cardinal points A_n, together with its dual set A'_n, the associated complex constants being ϱ and ϱ' respectively. Then from the formulae of §§ 1, 2, the path, in the plane p, of the point $A_1\,(A'_1)\,(z_1, \zeta_1)$ is the curve s, given by

$$z_1(\phi) = z_1(0) - w_1(0)/\varrho + \{w_1(0)/\varrho\} \exp(\partial_0 \varrho \phi),$$

for variation of the parameter ϕ; that is, a logarithmic spiral, of pole

$$z_1(0) - w_1(0)/\varrho,$$

namely, the point P_p, the pole of the spiral $S'(\phi)$, upon which lie the points A'_n. And the constant angle, between the curve s and the radius from the pole P_p, is $\pi/2 + \alpha$. This logarithmic spiral s is then the p-centrode.

Dually, as the ϖ-centrode, we have a logarithmic spiral s', given by

$$\zeta_1(\phi) = \zeta_1(0) - \omega'_1(0)/\varrho' + \{\omega'_1(0)/\varrho'\} \exp(\partial'_0 \varrho' \phi),$$

the pole being the point

$$\zeta_1(0) - \omega'_1(0)/\varrho',$$

coinciding with the pole $P_{\tilde{\omega}}$ of the spiral $S(\phi)$, upon which the points A_n lie. And the corresponding constant angle is $\pi/2 + \alpha'$.

If $\alpha = 0$, so that $\alpha' = 0$, the spirals s and s' become circles, of centres P_p, $P_{\tilde{\omega}}$ respectively, while the spirals $S(\phi)$, $S'(\phi)$ become coincident straight lines $l(\phi)$, passing through P_p, $P_{\tilde{\omega}}$; we have the linear sets of § 4.

If $k = 1, \alpha \neq 0$, the spiral $S(\phi)$, upon which the A_n lie, becomes a circle, of centre $P_{\tilde{\omega}}$ fixed in the plane ϖ. We have then a *circular* set of cardinal points A_n, the dual set A'_n remaining a spiral set, the associated angle being $(\pi + \alpha)/2$.

7. The Orthogonal Set. If, for a set of cardinal points A_n in general position, we write, as in § 5,

$$w_{n+1}(\phi) = \varrho_n(\phi)\, w_n(\phi),$$

we have

$$\varrho_n^{(\mu)}(\phi) = \partial_0^\mu \sum_{\nu=n}^{n+\mu-1} \theta(\mu, \nu)\, R_\nu,$$

where

$$R_n = \varrho_{n+1} - \varrho_n,$$

and $\theta(\mu, \nu)$ is homogeneous, of degree μ, in the various quantities $\varrho_n(\phi)$. And this is a quite general result.

If now $\varrho_n(\phi)$ be a pure imaginary, for all n and a single value of ϕ, it follows from the preceding that $\varrho_n^{(\mu)}(\phi)$ is a pure imaginary for that value of ϕ, and for all n; and so $\varrho_n(\phi)$ is a pure imaginary for all values of ϕ, and for all values of n.

The implication of this is that $A_n A_{n+1}$ is perpendicular to $A_{n+1} A_{n+2}$, for all values of n; and, further, that this property is conserved as the parameter ϕ varies. We refer to such a configuration of the A_n as constituting an *orthogonal* set of cardinal points, and it is evident that orthogonality is a property which is conserved. It is also evident, from the formulae of §§ 1, 2, that the dual set is not then orthogonal.

The corresponding p-centrode is then a straight line. For if we write

$$w_{n+1}(\phi) = \partial_0 k_n(\phi)\, w_n(\phi),$$

$k_n(\phi)$ being a real function of ϕ, we have

$$z_1^{(\mu)}(\phi) = \partial_0^\mu w_\mu(\phi) = \partial_0 K_\mu(\phi)\, w_1(\phi),$$

$K_\mu(\phi)$ being real; and, for varying ϕ,

$$z_1(\phi) = z_1(0) + \partial_0 w_1(0) f(\phi),$$

where $f(\phi)$ is a real function of ϕ. It follows that the path of $A_1(A_1')$ in the plane p, is a straight line perpendicular to $A_1 A_2$.

Thus the kinematical implication of orthogonality of a set of cardinal points is that one of the usual centrodes is a straight line, the other being an arbitrary curve determined by the moduli of the quantities $A_n A_{n+1}$, each of which is an arbitrary function of ϕ.

A special case of this arises if the orthogonal set be also a spiral set; if, in the notation of § 5, $\alpha = \pi/2$. Then the centrode s has associated angle π, and is a straight line.

Reference.

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See also: Math. Gazette. May 1952. No. 316. Vol. XXXVI. pp. 111—119.