# NEAR INCLUSIONS OF $C^*$-ALGEBRAS

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## 1. Introduction

For $C^*$-subalgebras $A$ and $B$ of a $C^*$-algebra $C$ we study the relation $A \overset{\gamma}{\subset} B$, which means that for any $a$ in $A$, there exists an operator $b$ in $B$ such that $\|a-b\| \leqslant \gamma\|a\|$.

The main reason why we have investigated those relations, is that we think, that if $\gamma$ is small enough, $B$ must have a subalgebra which shares some of its properties with $A$, and in turn we hope that we can get information on the space of $C^*$-subalgebras of a given $C^*$-algebra.

Our methods yield positive answers in several cases, and we prove under some conditions on $A$ and $B$ that there exists a unitary operator $u$ on a underlying Hilbert space such that $u$ is close to the identity and $uAu^*$ is contained in $B$, (Th. 4.1, Cor. 4.2, Th. 4.3, Th. 5.3). The theorems in section 4 are, generally speaking, obtained in the situation where $A$ and $B$ are von Neumann algebras on a Hilbert space and one of them is injective.

Theorem 5.3 tells that $B$ contains such a twisted copy of $A$, if $A$ is finite-dimensional and $\gamma$ is less than $10^{-4}$. In particular one should remark that the result is independent of the dimension of $A$.

Having the result of section 5 we are able to show in section 6 that if $A$ is the norm closure of an increasing sequence of finite dimensional $C^*$-algebras (AF for short), $A$ and $B$ satisfy $A \overset{\gamma}{\subset} B$, $B \overset{\gamma}{\subset} A$ and $\gamma$ is less than $10^{-9}$, then $B$ is also AF. This implies that $B$ is unitarily equivalent to $A$ in these cases.

At the end of section 6 we study the relations $A \overset{\gamma}{\subset} B$, $B \overset{\gamma}{\subset} A$ for other types of $C^*$-algebras, and we find that if $A$ is nuclear and $\gamma$ is less than $10^{-2}$ then $B$ is also nuclear and the dual spaces $A^*$ and $B^*$ are isomorphic via a completely positive isometry.

The proofs of the results in the sections 4 and 5 are made in three steps.

Suppose $A \overset{\gamma}{\subset} B$, then the first step is to find a completely positive linear map of $A$ into $B$ which is close to the identity on $A$. In the case where $B$ is an injective von Neumann

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algebra one can get this map simply by restricting a projection from $B(H)$ with image $B$ to $A$.

In the cases where neither $A$ nor $B$ is injective it is in general impossible for us even to find a linear embedding of $A$ into $B$. On the other hand when $A$ is finite-dimensional and $B$ arbitrary we get the desired map via the results in section 3. In that paragraph we do prove that for any nuclear $C^*$-algebra $D$ the relation $A \overset{\gamma}{\subset} B$ implies $A \otimes D \overset{6\gamma}{\subset} B \otimes D$. This tells that it is possible, simultaneously, to approximate several elements in $A$ with elements from $B$ in such a way, that certain linear and algebraic relations between the elements from $A$ are nearly fulfilled by those from $B$. Having this we can construct a linear completely positive map of $A$ into $B$ which is close to the identity on $A$.

The second step is to perturb this completely positive map such that the perturbed map is a star-homomorphism of $A$ into $B$. A technique yielding such a result was developed in [6]. The third and final step is to show that such a star-homomorphism is implemented by a unitary close to the identity i.e. the homomorphism is given by $a \to uau^*$. Questions of this type were discussed in [6] and [7], and it follows that in the situation considered here, we are able to find such a unitary. Therefore we get that $uAu^*$ is contained in $B$ for some unitary $u$ close to the identity and we are done.

In order to be able to perform the second and third step, the analysis from [6] and [7] show, that it is important that the algebra $A$ has the property that any operator in $C$ which nearly commutes with all elements in $A_1$ is close to the commutant of $A$ in $C$. In section 2 we recapitulate these concepts in detail, and we show how the results in [4], [8] and [15] can be used to extend the validity of the results in [6] and [7].

## 2. Preliminaries

In their article [18] Kadison and Kastler defined the distance between two von Neumann algebras as the Hausdorff distance between the respective unitballs. In the articles [5], [6], [7] we used this notion too, but since then we have found it more natural and easier to deal with the distance concept introduced below. The metrics are of course equivalent.

*2.1. Definition.* Let $E$ and $F$ be subspaces of a normed space $G$ and let $\gamma > 0$.

If for any $e$ with $\|e\| \leqslant 1$ there exists an $f$ in $F$ such that $\|e-f\| \leqslant \gamma$, then $E$ is said to be $\gamma$ contained in $F$ and we write $E \overset{\gamma}{\subseteq} F$. If $E \overset{\gamma_0}{\subseteq} F$ for some $\gamma_0 < \gamma$ we write $E \overset{\gamma}{\subset} F$. The distance between $E$ and $F$ is the infimum over all $\gamma > 0$ for which $E \overset{\gamma}{\subseteq} F$ and $F \overset{\gamma}{\subseteq} E$. The distance between $E$ and $F$ is denoted by $\|E-F\|$.

Let $H$ be a Hilbert space; the algebra of all bounded operators is denoted by $B(H)$, vectors by small greek letters, operators by small latin letters, von Neumann algebras by the letters $M$ and $N$ and general $C^*$-algebras by the letters $A$, $B$, and $C$. For an operator $x$ in $B(H)$, ad $(x)$ denotes the derivation on $B(H)$ implemented by $x$ i.e. $\operatorname{ad}(x)(m) = [x, m] = xm - mx$. If $u$ is any unitary operator in $B(H)$ or more generally in a $C^*$-algebra, Ad $(u)$ is defined as the automorphism implemented by $u$, i.e. $\operatorname{Ad}(u)(m) = umu^*$.

Let $M$ be a von Neumann algebra on a Hilbert space $H$, and let $x$ be a bounded operator on $H$. If $x$ is close to the commutant $M'$ of $M$, we get easily that $\|\operatorname{ad}(x) | M\|$ is small, but on the other hand if $\|\operatorname{ad}(x) | M\|$ is small we proved in [7], that the distance from $x$ to $M'$ is small provided $M$ is not non injective and of type $\mathrm{II}_1$. The definition below reflects that we do not know whether a general result is valid.

*2.2. Definition.* Let $A$ be a $C^*$-algebra and let $k$ be a positive real; $A$ is said to have property $D_k$ if for any representation $\pi$ of $A$ on a Hilbert space $H$ and any operator $x$ in $B(H)$

$$\inf \{\|x - m\| \mid m \in \pi(A)'\} \leqslant k \|\operatorname{ad}(x) | \pi(A)\|.$$

*2.3. Definition.* For any $k$, $0 \leqslant k \leqslant 1$ we define $\delta(k) = k2^{\frac{1}{2}}(1 + (1 - k^2)^{\frac{1}{2}})^{-\frac{1}{2}}$.

During the last years the injectivity concept in the category of $C^*$-algebras and completely positive maps, has been investigated very much ([3], [4], [8], [9], [15], [25]). We benefit from this, since Remark 6 of [15] implies, that injective von Neumann algebras do have the property $P$ of Schwartz, so we obtain the following:

2.4. THEOREM. *If $M$ is an injective von Neumann algebra on a Hilbert space $H$, then for any $x$ in $B(H)$*

$$\tfrac{1}{2}\|\operatorname{ad}(x) | M\| \leqslant d(x, M') \leqslant \|\operatorname{ad}(x) | M\|$$

*Proof.* [7, Theorem 2.3].

2.5. THEOREM. *If $M$ is an injective von Neumann algebra on a Hilbert space $H$, $0 \leqslant k < 1$ and $\alpha$ is star homomorphism of $M$ into $B(H)$, such that for any $m$ in $M$, $\|\alpha(m) - m\| \leqslant k\|m\|$, then there exists a unitary $u$ in $(M \cup \alpha(M))''$ such that $\alpha = \operatorname{Ad}(u)$ and $\|I - u\| \leqslant \delta(k)$.*

*Proof.* [6, Proposition 4.2].

A $C^*$-algebra $A$ is said to be nuclear if any of the following equivalent conditions is fulfilled ([4], [12]).

1. For any finite number $a_1, \ldots, a_s$ of operators in $A$ and any $\varepsilon > 0$ there exists a full matrix algebra $M_n$ and completely positive maps $\psi: A \to M_n$ and $\varphi: M_n \to A$ such that $\|a_i - \varphi\psi(a_i)\| < \varepsilon$ and $\|\varphi\| \leqslant 1$, $\|\psi\| \leqslant 1$.

2. For each representation $\pi$ of $A$, $\overline{\pi(A)}$ is injective.
3. The bidual $A^{**}$ is an injective von Neumann algebra.

From 2.4 and Kaplansky's density theorem we then get.

2.6. PROPOSITION. *Any nuclear $C^*$-algebra has property $D_1$.*

We call a $C^*$-algebra approximately finite-dimensional AF for short, if it contains a dense subalgebra, which is the union of an increasing sequence of finite-dimensional $C^*$-algebras.

Finally we remark that type I $C^*$-algebras and AF $C^*$-algebras are nuclear.

Before closing this section we mention

2.7. PROPOSITION. *If a unital $C^*$-algebra $A$ contains two isometries $v$ and $w$ such that $vv^* + ww^* \leqslant I$ then $A$ has property $D_{3/2}$.*

*Proof.* The von Neumann algebra generated by any non degenerate representation of $A$ must be properly infinite, and the proposition follows from [7, Theorem 2.4].

## 3. Tensorproducts of inclusions

Suppose $A$ and $B$ are $C^*$-subalgebras of a $C^*$-algebra $C$.

If $A \overset{\gamma}{\subseteq} B$ and $D$ is an arbitrary $C^*$-algebra, we want to investigate the relations between the subalgebras $A \otimes D$ and $B \otimes D$ of $C \otimes D$. (The sign $\otimes$ means minimal $C^*$-tensorproduct whereas $\overline{\otimes}$ means spatial von Neumann algebra tensorproduct.)

Suppose that $A$ can be twisted into $B$ by a unitary close to the identity, then one easily deduces that $A \otimes D$ is nearly contained in $B \otimes D$.

On the other hand if $A \otimes D$ is nearly contained in $B \otimes D$ for a "big" algebra $D$, we do have the hypothesis, that there will exist a completely positive map $\varphi$ of $A$ into $B$ which is close to the identity map on $A$.

In the proof of Theorem 5.2 we actually verify this hypothesis in a special case.

3.1. THEOREM. *Let $C$ be a $C^*$-algebra with $C^*$-subalgebras $A \overset{\gamma}{\subset} B$, and let $D$ be a nuclear $C^*$-algebra. If $A$ has property $D_k$ then $A \otimes D \overset{6k\gamma}{\subset} B \otimes D$.*

*Proof.* Let $\pi$ be a representation of $C$ on a Hilbert space $K$ and let $H$ be an infinite-dimensional Hilbert space then

$$\pi(A) \otimes \mathbf{C}_H \overset{\gamma}{\subset} \pi(B) \otimes \mathbf{C}_H.$$

Since $A$ has property $D_k$, we find that

$$\pi(B)'\overline{\otimes}B(H)\overset{2k\gamma}{\subset}\pi(A)'\overline{\otimes}B(H),$$

because for any $x$ in $\pi(B)'\overline{\otimes}B(H)$ and any $a$ in $A$, $b$ in $B$ we get, when we define $\tilde{\pi}(c)=\pi(c)\otimes I$;

$$\|[x,\tilde{\pi}(a)]\| = \|[x,\tilde{\pi}(a)-\tilde{\pi}(b)]\| \leqslant 2\|x\|\,\|\tilde{\pi}(a-b)\|.$$

Now $\pi(B)'\overline{\otimes}B(H)\otimes \mathbf{C}_H$ is properly infinite, so Proposition 2.7 shows that this algebra has property $D_{3/2}$. We can then repeat the argument with 3/2 instead of $k$ and get

$$(\pi(A)\overline{\otimes}B(H))''\overset{6k\gamma}{\subset}(\pi(B)\overline{\otimes}B(H))''. \tag{1}$$

Any finite-dimensional $C^*$-algebra $M$ can be represented on $H$ such that $I_M=I_{B(H)}$, moreover there exists a normal projection of norm one from $B(H)$ onto $M$, so the relation (1) can be projected into

$$\pi(A)''\otimes M\overset{6k\gamma}{\subset}\pi(B)''\otimes M. \tag{2}$$

Let us continue to consider a finite-dimensional $C^*$-algebra $M$, and let $\varphi$ be a continuous functional of norm one on $C\otimes M$ which vanishes on $B\otimes M$.

Let $(\pi, H_u)$ denote the universal representation of $C\otimes M$, then $\varphi$ has a unique extension to an ultraweakly continuous functional $\tilde{\varphi}$ on $\pi(C\otimes M)''$, $\tilde{\varphi}$ vanishes on $\pi(B\otimes M)''$, and therefore the restriction of $\tilde{\varphi}$ to $\pi(A\otimes M)''$ has norm less than $6k\gamma$. This in turn implies, that the restriction of $\varphi$ to $A\otimes M$ has norm less than $6k\gamma$, so from Hahn-Banach's theorem we may conclude, that whenever $M$ is a finite-dimensional $C^*$-algebra

$$A\otimes M\overset{6k\gamma}{\subset}B\otimes M. \tag{3}$$

Let $x$ be an operator of norm less than one in $A\otimes D$, then to any $\varepsilon>0$ there exists operators $a_1, \ldots, a_n$ in $A$ and $y_1, \ldots, y_n$ in $D$ such that $\|x-\sum_{i=1}^n a_i\otimes y_i\|<\varepsilon$.

To the operators $y_1, \ldots, y_n$ we can find a finite-dimensional algebra $M$ and completely positive contractions $\varrho: D\to M$, $\varphi: M\to D$ such that $\|y_i-\varphi(\varrho(y_i))\|<\varepsilon(\sum_{i=1}^n\|a_i\|)^{-1}$. The completely positive maps $\mathrm{id}\otimes\varrho: C\otimes D\to C\otimes M$ and $\mathrm{id}\otimes\varphi: C\otimes M\to C\otimes D$ maps $A\otimes D$ into $A\otimes M$ and $B\otimes M$ into $B\otimes D$. By (3) we conclude that there exists $z_0$ in $B\otimes M$ such that

$$\left\|\sum_{i=1}^n a_i\otimes\varrho(y_i)-z_0\right\|\leqslant 6k\gamma(1+\varepsilon).$$

When we define $z = \mathrm{id} \otimes \varphi(z_0)$, we get that $z$ belongs to $B \otimes D$ and

$$\|x-z\| \leqslant \left\| x - \sum_{i=1}^{n} a_i \otimes y_i \right\| + \left\| \sum_{i=1}^{n} a_i \otimes (\varphi\varrho(y_i) - y_i) \right\| + \left\| \mathrm{id} \otimes \varphi \left( \sum_{i=1}^{n} a_i \otimes \varrho(y_i) - z_0 \right) \right\| \leqslant 6k\gamma(1+\varepsilon) + 2\varepsilon.$$

The theorem follows, since we do assume $\|\varphi\| \leqslant 1$.

3.2. THEOREM. *Suppose that* $A \overset{\gamma}{\subset} B$ *are* $C^*$*-subalgebras of a* $C^*$*-algebra* $C$. *If* $D$ *is an abelian* $C^*$*-algebra, then* $A \otimes D \overset{\gamma}{\subset} B \otimes D$.

*Proof*. Choose $\varepsilon > 0$ such that $A \overset{\gamma-4\varepsilon}{\subset} B$, and let $T$ denote the spectrum of $D$. The algebras $A \otimes D$ and $B \otimes D$ are then isomorphic to the algebras of continuous functions on $T$ with values in $A$ (resp. $B$) which vanish at infinity, and both algebras can of course be considered as subalgebras of $C_0(T, C)$, the algebra of continuous functions on $T$ with values in $C$ which vanish at infinity.

Suppose $x = x(t) \in C_0(T, A)$ and $\|x\| = \sup_t \|x(t)\| \leqslant 1$, then there exists a compact subset $K$ of $T$ such that $\|x(t)\| \leqslant \varepsilon$ for $t$ in $T \setminus K$. Let $O_1, ..., O_n$ be a finite covering of $K$ with open sets in $T$ such that for any $s, t$ in $O_i$ we have $\|x(s) - x(t)\| \leqslant \varepsilon$. We want now to use a partition of the unit, on $K$, subordinate to this covering. Let $\{\psi_j | j = 1, ..., m\}$ be such a partition consisting of non-negative continuous functions with compact support such that each $\psi_j$ has its support in some $O_i$ and

$$1_K \leqslant \sum_{j=1}^{m} \psi_j \leqslant 1_T.$$

We can now construct a $y$ in $C_0(T, B)$, close to $x$ by first choosing $t_j$ in the support of $\psi_j$, and secondly operators $y_j$ in $B$ such that $\|x(t_j) - y_j\| \leqslant \gamma - 4\varepsilon$. A simple calculation shows that the operator $y$ in $C_0(T, B)$ defined by $y = \sum_{j=1}^{m} \psi_j y_j$ satisfies $\sup_t \|x(t) - y(t)\| \leqslant \gamma - \varepsilon$, and the theorem follows.

## 4. Inclusions with one injective von Neumann algebra

In this paragraph we study the relation $M \overset{\gamma}{\subset} N$ for von Neumann algebras $M$ and $N$, We show—for sufficiently small $\gamma$'s—that if $M$ has property $D_k$ and $N$ is injective, or if $M$ is injective and $N$ arbitrary then $M$ can be twisted into $N$ via a unitary close to the identity. As a corollary of this we find, as Raeburn and Taylor did [22], that the set of injective von Neumann algebras on a Hilbert space is open and closed.

The proofs follow the ideas sketched in the introduction.

In the case where $N$ is injective and hence has a projection of norm one onto itself. we get immediately a completely positive map from $M$ into $N$. By restricting this projection to $M$, we get a situation similar to these discussed in [6].

In the case where $M$ is injective; $M$ has property $D_1$ and we get $N'\overline{\otimes} B(K) \overset{2\gamma}{\subset} M'\overline{\otimes} B(K)$. Now $M'\overline{\otimes} B(K)$ is injective and we can use the previous result for this case too.

4.1. THEOREM. *Let $A$ be a unital $C^*$-algebra with property $D_k$ acting on a Hilbert space $H$ and $N$ an injective von Neumann algebra on $H$.*

*If $A \overset{\gamma}{\subset} N$ then there exists a star homomorphism $\Phi$ of $A$ into $N$ such that $\|(\Phi-\mathrm{id})|A\| \leqslant (2+6k)\gamma$. If $\gamma<(6k^2+2k)^{-1}$ then there exists a unitary $u$ in $B(H)$, such that $\Phi(a)=uau^*$ and $\|I-u\| \leqslant (9k^2+3k)\gamma$.*

*Proof.* If $k\gamma \geqslant \frac{1}{2}$ then $\Phi$ is chosen to be zero, if $k\gamma<\frac{1}{2}$ then let $\varrho$ be a projection of norm one from $B(H)$ onto $N$ and let $(\pi, K, p)$ be chosen such that $\pi$ is a representation of $B(H)$ on $K$ and for any $x$ in $B(H)$; $\varrho(x)=p\pi(x)|H$ ([24], [6, Theorem 3.1]). Since $\varrho|N$ is a star isomorphism it follows that $p$ commutes with $\pi(N)$. Let $a\in A$ and choose $n\in N$ such that $\|a-n\| \leqslant \gamma\|a\|$, then one finds

$$\|\pi(a)p-p\pi(a)\| = \tfrac{1}{2}\|\pi(a-n)(2p-I)-(2p-I)\pi(a-n)\| \leqslant \gamma\|a\|.$$

Therefore there exists an operator $x$ on $K$ in $\pi(A)'$ such that $\|p-x\| \leqslant k\gamma$.

According to Arveson's commutation result [1, Theorem 1.3] we know that $\pi$ and $K$ can be chosen such that the commutant $[p\cup\pi(B(H))]'$ is isomorphic to the commutant $N'$ of $N$ in $B(H)$. Hence $N'$ and $[p\cup\pi(B(H))]''$ are both injective [25]. Let $\varphi$ be a projection of norm one from $B(K)$ onto $[p\cup\pi(B(H))]''$. Then $\varphi$ maps $x$ into $\pi(A)'$ because $\varphi$ is a module map, in fact one gets for $x$ in $\pi(A)'$ and $a$ in $A$, $\pi(a)\varphi(x)=\varphi(\pi(a)x)=\varphi(x\pi(a))=\varphi(x)\pi(a)$. It is clear that $\varphi(p)=p$ so that for $y=\varphi(x)$ we get $\|y-p\| \leqslant k\gamma$ and $y\in[p\cup\pi(B(H))]''\cap\pi(A)'$. When we now continue as at the end of the proof of Lemma 3.3 of [6] with $t^{\frac{1}{2}}$ replaced by $k\gamma$, we find a projection $q$ in $\pi(A)'\cap[p\cup\pi(B(H))]''$ and a unitary $v$ in $[p\cup\pi(B(H))]''$ such that $v^*pv=q$, $\|p-q\|<2k\gamma$ and $\|I-v\| \leqslant \delta(2k\gamma) \leqslant 3k\gamma$. The map $\Phi$ of $A$ into $N$ given by

$$a\to\pi(a)\to v\pi(a)qv^*\to v\pi(a)qv^*|H = v\pi(a)v^*|H,$$

is a star homomorphism of $A$ into $N$, because $p[p\cup\pi(B(H))]''|H=N$.

For each $a$ in $A_1$ there exists $n$ in $N$ such that $\|a-n\|<\gamma$; hence we get

$$\begin{aligned}\|\Phi(a)-a\| &\leqslant \|p(v\pi(a)v^*-\pi(n))p\|+\|a-n\| \\ &\leqslant \gamma+\|\pi(a)-\pi(n)\|+\|v\pi(a)v^*-\pi(a)\| \leqslant (2+6k)\gamma.\end{aligned}$$

If $\gamma<(6k^2+2k)^{-1}$ then $(2+6k)\gamma<k^{-1}$ and one finds that the argument given in the proof of [7, Proposition 3.2] applies. This means, that there exists a unitary $u$ in $B(H)$ such that $\Phi(a)=uau^*$ and $\|1-u\|\leqslant\delta((2+6k)\gamma k)\leqslant(3k+9k^2)\gamma$.

The following corollaries 4.2 (a), (b), (c), (d) follow from Theorem 4.1 and the remarks made in section 2. The last statement 4.2 (e) is commented upon below.

4.2. COROLLARY. (a) *Let* $A\overset{\gamma}{\subset}N$ *be as above. If* $A$ *is nuclear and* $\gamma<\frac{1}{8}$ *then there exists a unitary* $u$ *in* $(A\cup N)''$ *such that* $uAu^*\subseteq N$, $\|uau^*-a\|\leqslant 8\gamma\|a\|$ *and* $\|I-u\|\leqslant 12\gamma$.

(b) *If* $M\overset{\gamma}{\subset}N$, $M$ *and* $N$ *injective von Neumann algebras on a Hilbert space* $H$, *and* $\gamma<\frac{1}{8}$ *then there exists a unitary* $u$ *in* $(M\cup N)''$ *such that* $uMu^*\subseteq N$ *and* $\|I-u\|\leqslant 12\gamma$.

(c) *If* $\|M-N\|<\frac{1}{8}$, $M$ *and* $N$ *are injective then there exists a unitary* $u$ *in* $(M\cup N)''$ *such that* $uMu^*=N$ *and* $\|I-u\|\leqslant 12\gamma$.

(d) *Let* $A\overset{\gamma}{\subset}N$ *be as above, if* $A$ *is a properly infinite von Neumann algebra* $0<\gamma\leqslant\frac{2}{33}$, *then there exists a unitary* $u$ *in* $B(H)$ *such that,* $uAu^*\subseteq N$ *and* $\|I-u\|\leqslant 25\gamma$.

(e) *Let* $A\overset{\gamma}{\subset}B$ *be finite-dimensional* $C^*$*-subalgebras of a unital* $C^*$*-algebra* $C$. *Suppose all three have the same unit and that* $\gamma<\frac{1}{8}$ *then there exists a unitary* $u$ *in* $C$ *such that,* $uAu^*\subseteq B$, $\|uau^*-a\|\leqslant 8\gamma\|a\|$, $\|I-u\|\leqslant 12\gamma$.

*Proof*. Ad. e. The proof of Theorem 4.1 yields a starhomomorphism $\Phi$ of $A$ into $B$ such that $\|\Phi(a)-a\|\leqslant 8\gamma\|a\|$.

Since the unitary group in $A$ is compact it is easy to see that the proof of [7, Proposition 4.2] works in this case too. We can therefore find an operator $x$ in $C$ such that $x\Phi(a)=ax$ and $\|I-x\|\leqslant 8\gamma$. This inequality implies that $x^*x$ is invertible, and hence that the unitary part in the polar decomposition of $x$ belongs to $C$. The collorary follows.

We will now turn to the case where an injective algebra is nearly contained in an arbitrary von Neumann algebra.

4.3. THEOREM. *Let* $N\overset{\gamma}{\subset}M$ *be an injective and an arbitrary von Neumann algebra on a Hilbert space* $H$. *Suppose* $0\leqslant\gamma<10^{-2}$, *then there exists a unitary* $v$ *in* $(N\cup M)''$ *such that* $\|I-v\|\leqslant 150\gamma$, $vNv^*\subseteq M$ *and* $\|vnv^*-n\|\leqslant 100\gamma\|n\|$ *for any* $n$.

*Proof*. Since $N$ has property $D_1$ we can argue as in the beginning of the proof of Theorem 3.1 in order to get $M'\mathbin{\overline{\otimes}}B(K)\overset{2\gamma}{\subset}N'\mathbin{\overline{\otimes}}B(K)$. Corollary 4.2 (d) shows that there is a unitary $u$ in $B(H)\otimes B(K)$ such that $\|I-u\|\leqslant 50\gamma$ and $u^*(N\otimes\mathbf{C})u\subseteq(M\otimes\mathbf{C})$.

By Theorem 2.4 there is a unitary $v$ in $(N\cup M)''$ such that $\|I-v\|\leqslant\delta(100\gamma)\leqslant 150\gamma$, $vNv^*\subseteq M$ and $vnv^*=u^*nu$.

4.4. COROLLARY. *If* $\|M-N\|<\gamma<101^{-1}$ *and* $N$ *is an injective von Neumann algebra then there exists a unitary* $v$ *in* $(M\cup N)''$ *such that* $vNv^*=M$.

*Proof.* By 4.3 there is a unitary $v$ in $(M\cup N)''$ such that $\|I-v\|\leqslant 150\gamma$ and $\|vnv^*-n\|\leqslant 100\gamma\|n\|$ for any $n$ in $N$. Hence we get $M\overset{101\gamma}{\subset} vNv^*\subseteq M$ and by a standard argument which is given in [6], we get $M=vNv^*$, and the corollary follows.

Especially we have reproved the result due to Raeburn and Taylor, that the set of injective von Neumann algebras is open.

## 5. Inclusions with finite-dimensional $C^*$-algebras

Suppose $C$ is a $C^*$-algebra which contains the $C^*$-algebras $A$ and $F$, suppose moreover that $F$ is a finite-dimensional factor and that $\{e_{ij}\mid i,j=1,\dots,n\}$ are matrix units for $F$, then in [16] Glimm proved; to any $\varepsilon>0$ there exists a $\delta(n,\varepsilon)$, such that if $A$ contains operators $x_{ij}$ satisfying $\|x_{ij}-e_{ij}\|\leqslant\delta(n,\varepsilon)$ then $A$ also contains matrix units $f_{ij}$ such that $\|f_{ij}-e_{ij}\|\leqslant\varepsilon$. In other words if a set of matrix units for $F$ is close enough to $A$, then $A$ contains a copy of $F$.

As indicated the constant $\delta(n,\varepsilon)$ is very much dependent upon $n$.

If one considers the relation $F\overset{\gamma}{\subset}A$, meaning that any element in the unitball of $F$ is within distance $\gamma$ to $A$, then we give a proof independent of the dimension of $F$, which shows that $A$ contains a copy of $F$.

Since a set of matrix units is also a basis, it is possible to deduce Glimm's result from the one of our's.

We start with the case, where $F$ is abelian say with minimal projections $p_1,\dots,p_k$. The idea is then to show, that there exist natural numbers $n_1,\dots,n_k$ such that the images of the function $f(z)=p_1z^{n_1}+p_2z^{n_2}+\dots p_kz^{n_k}$, $z\in\mathbf{T}=\{z\in\mathbf{C}\mid |z|=1\}$ is $\varepsilon$ dense in the set of unitaries in the algebra $F$. We then find a $g$ in $C(\mathbf{T},A)$ with power series expansion $g(z)=a_1z^{n_1}+\dots+a_kz^{n_k}$ such that $a_i\geqslant 0$ and $g$ is close to $f$, then the map $\Phi(\sum\lambda_ip_i)=\sum\lambda_ia_i$ is a completely positive map of $F$ into $A$ close to the identity on $F$. The details follow in 5.1 and 5.2 below.

This abelian result combind with elementary technique give the general finite-dimensional algebra result.

5.1. LEMMA. *Let* $k\in\mathbf{N}$ *and* $\varepsilon>0$, *then there exist positive integers* $n_1,\dots,n_k$ *such that for any* $(\gamma_1,\dots,\gamma_k)\in\mathbf{T}^k$ *there is a* $\zeta$ *in* $\mathbf{T}$ *for which*

$$\sum_{i=1}^{k}|\gamma_i-\zeta^{n_i}|<\varepsilon.$$

*Proof.* It is possible to get a proof via a simple induction argument, but it is also known from the theory of lacunary series, that one can find integers $n_1, ..., n_k$ such that the functions $z^{n_1}, z^{n_2}, ..., z^{n_k}$ on $\mathbf{T}$ satisfy any wanted degree of independence.

5.2. PROPOSITION. *Let $F$ be a finite-dimensional abelian $C^*$-subalgebra and $B$ a $C^*$-subalgebra of a $C^*$-algebra $C$. If for some $\gamma \leqslant 10^{-3}$, $F \overset{\gamma}{\subset} B$, then there exists a partial isometry $v$ in $C$ such that $v^*v = I_F$ and*

$$vFv^* \subseteq B; \ \|vfv^* - f\| \leqslant 15\gamma^{\frac{1}{2}}\|f\|; \ \|I_F - v\| \leqslant 37\gamma^{\frac{1}{2}}.$$

*Proof.* We follow the method sketched above and construct first a completely positive map of $F$ into $B$. Then by some technique taken from [6] we perturb the positive map slightly such that the perturbed map becomes a star homomorphism. Finally we show that this map is given by $f \to vfv^*$ for some partial isometry having the properties above.

Let $p_1, ..., p_k$ be the minimal projections in $F$, $\varepsilon > 0$ such that $F \overset{\gamma-\varepsilon}{\subset} B$ and $n_1, ..., n_k$ positive integers for which the statement in Lemma 5.1 is fulfilled with respect to $\varepsilon$.

By Theorem 3.2 there exists a continuous function $f$ on $\mathbf{T}$ with values in $B$ such that

$$\sup\left\{\left\| f(z) - \sum_{i=1}^{k} p_i z^{n_i} \right\| \,\middle|\, z \in \mathbf{T}\right\} < \gamma - \varepsilon.$$

Since the inequality is sharp and the trigonometric polynomials are dense in $C(\mathbf{T}, \mathbf{C})$ we may assume that $f$ has the form $f(z) = \sum_{i=-m}^{m} b_i z^i$.

We let $\tau_\theta$ denote the translation operator $\tau_\theta h(z) = h(\theta^{-1}z)$ and define $g$ by $g(z) = \sum_{i=1}^{k} p_i z^{n_i}$. For any $\theta$ in $\mathbf{T}$

$$\|[(\tau_\theta f)^* f - (\tau_\theta g)^* g](0)\| \leqslant \|(\tau_\theta(f-g))^* g\| + \|(\tau_\theta(f))^*(f-g)\| \leqslant 2\gamma - 2\varepsilon.$$

When written out this inequality becomes

$$\left\| \left(\sum_{i=-m}^{m} b_i^* b_i \theta^i\right) - \left(\sum_{j=1}^{k} p_j \theta^{n_j}\right) \right\| < 2\gamma - 2\varepsilon \quad \text{for any } \theta \text{ in } \mathbf{T}.$$

In order to get rid of excessive terms we estimate

$$\gamma^2 \geqslant (f^* - g^*)(f-g)(0) = \sum_{i \neq n_j} b_i^* b_i + \sum_{j=1}^{k} (b_{n_j} - p_j)^* (b_{n_j} - p_j).$$

Therefore for any $\xi, \eta$ in $H$

$$|(\sum_{i \neq n_j} b_i^* b_i \theta^i \xi | \eta)| \leqslant (\sum_{i \neq n_j} \|b_i \xi\|^2)^{1/2} (\sum_{i \neq n} \|b_i \eta\|^2)^{1/2} \leqslant \gamma^2 \|\xi\| \, \|\eta\|,$$

and we have proved that for any $\theta$ in $\mathbf{T}$

$$\left\| \left( \sum_{j=1}^{k} p_j \theta^{n_j} \right) - \left( \sum_{j=1}^{k} b_{n_j}^* b_{n_j} \theta^{n_j} \right) \right\| < 2\gamma - 2\varepsilon + \gamma^2.$$

Define a completely positive map $\Phi$ of $F$ into $B$ by $\Phi(p_j) = b_{n_j}^* b_{n_j}$, then Lemma 5.1 and the arguments above show that for any unitary $u$ in $F$, $\|\Phi(u) - u\| < 2\gamma - \varepsilon + \gamma^2$. Since the unitball in $F$ is the convex hull of the unitaries we get $\|\Phi(f) - f\| \leq (2\gamma + \gamma^2)\|f\|$ for all $f$ in $F$.

Let $q$ be the spectral projection for $\Phi(I_F)$ corresponding to the interval $[1/2, 3/2]$, then an argument similar to the one given in [5, Lemma 2.1] shows that $\|q - \Phi(I_F)\| \leq (2\gamma + \gamma^2)$.

Let $b$ denote the inverse to $q\Phi(I_F)$ in $B_q$ then the map $\Gamma$ of $F$ into $B_q$ defined by $\Gamma(f) = b^{\frac{1}{2}}\Phi(f)b^{\frac{1}{2}}$ satisfies $\Gamma(I_F) = q$ and $\|\Gamma(u^*)\Gamma(u) - q\| \leq 12{,}05\gamma$ for all unitaries $u$ in $F$, (see [6, Theorem 3.4] for a similar argument). Since the group of unitaries in $F$ is compact, the methods from [6, Lemma 3.3] can be used at the "$C^*$-level" and we find that there exists a star homomorphism $\Psi$ of $F$ into $B$ such that $\|\Gamma - \Psi\| \leq 14\gamma^{\frac{1}{2}}$.

If we do examine the constructions of $\Gamma$ and $\Phi$ we can easily prove that for any $f$ in $F$, $\|\Psi(f) - f\| \leq 15\gamma^{\frac{1}{2}}$.

We want now to suppose that $F$ and $C$ have the same unit. If this is not the case or if $C$ has not got a unit we do simply adjoin one and define a star homomorphism $\tilde{\Psi}$ of $\tilde{F} = \mathbf{C}I \oplus F$ into $\mathbf{C}I \oplus B$ by $\tilde{\Psi}(\lambda + f) = \lambda + \Psi(f)$. Now $\tilde{\Psi}$ satisfies $\tilde{\Psi}(I) = I$ and for each $\tilde{f}$ in $\tilde{F}$, $\|\tilde{\Psi}(\tilde{f}) - \tilde{f}\| \leq 30\gamma^{\frac{1}{2}}\|\tilde{f}\|$.

The group of unitaries in $\tilde{F}$ is compact, and also [6, Proposition 4.2] works at the "$C^*$-level". Hence we find that there exists a unitary $u$ in $\tilde{C}$ implementing $\tilde{\Psi}$ such that $\|I - u\| \leq 37\gamma^{\frac{1}{2}}$. The theorem follows when we define $v = uI_F$.

Having this abelian result the general result for a finite-dimensional $C^*$-algebra $A$ is proved by first to twist a maximal abelian subalgebra of $A$ into $B$ and then secondly to show, that in this situation a set of matrix units for the perturbed algebra can easily be twisted into $B$ via a unitary close to the identity.

5.3. Theorem. *Let $A$ be a finite-dimensional $C^*$-subalgebra and $B$ a $C^*$-subalgebra of a $C^*$-algebra $C$.*

*Suppose $0 \leq \gamma \leq 10^{-4}$ and $A \overset{\gamma}{\subset} B$, then there exists a partial isometry $v$ in $C$ such that $\|I_A - v\| \leq 120\gamma^{\frac{1}{2}}$ and $vAv^* \subseteq B$.*

*Proof.* Let $F$ be a maximal abelian $C^*$-subalgebra of $A$ and let $u$ be a partial isometry in $C$ such that $\|I_F - u\| \leq 37\gamma^{\frac{1}{2}}$ and $uFu^* \subseteq B$.

We may assume that the minimal projections in $F$ are the self-adjoint elements in a set of matrix units for $A$. Since $A$ has the form $A=M_{n_1}\oplus\ldots\oplus M_{n_m}$, where $M_{n_k}$ is a full matrix-algebra of dimension $n_k^2$ we may enummerate the matrix units by $f_{ij}^k$ where $1\leqslant k\leqslant m$, $1\leqslant i,j\leqslant n_k$.

Choose $x_{i1}^k$ in $B$ such that $\|f_{i1}^k-x_{i1}^k\|\leqslant\gamma$, and define $g_{ii}^k=uf_{ii}^k u^*$, then

$$\|g_{ii}^k x_{i1}^k g_{11}^k-uf_{i1}^k u^*\|\leqslant\|g_{ii}^k(x_{i1}^k-uf_{i1}^k u^*)g_{11}^k\|\leqslant 75\gamma^{\frac{1}{2}}.$$

When we define $g_{i1}^k$ as the partial isometry part of the polar decomposition of $g_{ii}^k x_{i1}^k g_{11}^k$ and $a_i^k$ as the positive part we obtain;

$$\|uf_{1i}^k u^* g_{i1}^k a_i^k-g_{11}^k\|\leqslant 75\gamma^{\frac{1}{2}}.$$

Lemma 2.7 in [5] implies that the isometry part $uf_{1i}^k u^* g_{i1}^k$ of the operator satisfies $\|uf_{1i}^k u^* g_{i1}^k-g_{11}^k\|\leqslant\delta(75\gamma^{\frac{1}{2}})$.

This relation shows that

$$\|g_{i1}^k-uf_{i1}^k u^*\|\leqslant\delta(75\gamma^{\frac{1}{2}})\leqslant 83\gamma^{\frac{1}{2}}$$

and since $83\gamma^{\frac{1}{2}}<1$, $g_{i1}^k(g_{i1}^k)^*=g_{ii}^k$; $(g_{i1}^k)^*g_{i1}^k=g_{11}^k$. We may then define matrix units $g_{ij}^k$ by $g_{ij}^k=g_{i1}^k(g_{j1}^k)^*$ and we have got a system of matrix units in $B$ which is close to the system $f_{ij}^k$. This is verified by constructing a partial isometry close to $I_A$ which twists $f_{ij}^k$ into $g_{ij}^k$. Let $w=\sum_{k=1}^m\sum_{i=1}^{n_k} g_{i1}^k uf_{1i}^k u^*$ then $wg_{ii}^k=g_{ii}^k w$, so

$$\|w-\sum_k\sum_i f_{ii}^k\|\leqslant 83\gamma^{1/2}.$$

Let $v=wu$ then $v\in C$, $vAv^*\subseteq B$ and

$$\|I_A-v\|\leqslant\|wu-u\|+\|u-I_A\|\leqslant 83\gamma^{\frac{1}{2}}+37\gamma^{\frac{1}{2}}\leqslant 120\gamma^{\frac{1}{2}}.$$

## 6. Perturbations of nuclear $C^*$-algebras

In the article [6], we did prove that two commutative $C^*$-algebras and two ideal or dual $C^*$-algebras ($C^*$-algebras of compact operators) are unitarily equivalent, when closer than $10^{-1}$ and $600^{-1}$ respectively [6, Th. 5.1, Th. 5.3].

We do prove a result of this type for AF $C^*$-algebras below.

John Phillips and Ian Raeburn have proved, that close AF $C^*$-algebras are unitarily equivalent, by an application of the dimension group theory [20], [14]. Our approach is different except for the last steps, which are based upon arguments due to Powers and

Bratteli. We use in the first part the results from section 6 together with some twisting arguments which have been used by Glimm [16], Dixmier [11], and Bratteli [2].

In the last part of the section we study close nuclear $C^*$-algebras and show, that the set of nuclear $C^*$-algebras is open, and further any two sufficiently close nuclear $C^*$-algebras have isomorphic duals and biduals.

6.1. THEOREM. *Let $A$ and $B$ be $C^*$-subalgebras of a $C^*$-algebra $C$. If $A$ is* AF *and $\|A-B\|<10^{-9}$ then $B$ is* AF.

*Proof.* If $A$, $B$ and $C$ do not have a common unit, we adjoin a unit $I$ to $C$ and obtain $\|\tilde{A}-\tilde{B}\|<2\cdot 10^{-9}$ inside $\tilde{C}$. We do therefore assume in the following computations that $\|A-B\|<2\cdot 10^{-9}$ and the algebras have a common unit. Suppose $A=\mathrm{cl}\,(\bigcup_{n=1}^{\infty} A_n)$ where $(A_n)_{n\in\mathbf{N}}$ is an increasing sequence of finite-dimensional $C^*$-algebras, all containing the identity in $C$. Since $A$ is separable and $\|A-B\|<\frac{1}{2}$ it is easy to check that $B$ is separable. Let $(b_i)_{i\in\mathbf{N}}$ be a dense sequence in the unitball of $B$, we want then to show, that there exists an increasing sequence $B_i$ of finite-dimensional $C^*$-subalgebras of $B$ such that for any $i$ in $\mathbf{N}$; span $\{b_k \mid 1\leqslant k\leqslant i\} \overset{\frac{1}{i}}{\subset} B_i$. We do make the proof by induction and copy arguments due to Glimm [16, Th. 1.13].

To start the induction suppose $b_1=0$ and $B_1=0$. Let $V=\mathrm{span}\,(\{b_k \mid 1\leqslant k\leqslant i+1\}\cup B_i)$ and let $n$ in $\mathbf{N}$ be chosen such that $V$ is $2\cdot 10^{-9}=\gamma$ contained in $A_n$. Find a unitary $u$ in $C$ such that $\|I-u\|\leqslant 120\gamma^{\frac{1}{2}}$ and $uA_nu^*\subseteq B$, (Th. 5.3). It is easy to see that $B_i$ is $240\gamma^{\frac{1}{2}}$ contained in $uA_nu^*$, so by Corollary 4.2 (e) there exists a unitary $w$ in $B$ such that $wuA_nu^*w^*$ contains $B_i$ and $\|I-w\|\leqslant 2\,880\gamma^{\frac{1}{2}}$.

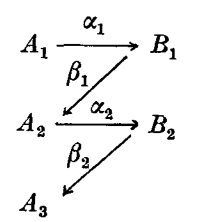
Now $V$ is contained $\gamma+240\gamma^{\frac{1}{2}}+2\cdot 2\,880\gamma^{\frac{1}{2}}\leqslant 0,3$ in $B_{i+1}=wuA_nu^*w^*$ and the theorem follows.

6.2. THEOREM. *If $A$ and $B$ are* AF *$C^*$-subalgebras of a $C^*$-algebra $C$ and $\|A-B\|<1/16$, then $A$ and $B$ are isomorphic.*

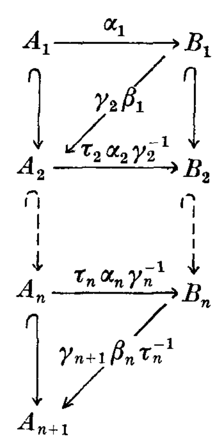
*Proof.* Let $\|A-B\|<\gamma<1/16$.

The proof is based upon Theorem 5.3 and a modified version of Brattelis isomorphism argument given in [2]. By [2, Theorem 2.2] it is possible to find increasing sequences $(A_n)_{n\in\mathbf{N}}$, $(B_n)_{n\in\mathbf{N}}$ of finite-dimensional $C^*$-subalgebras of $A$ and $B$ such that their unions are dense in $A$ and $B$ and for each $n$ in $\mathbf{N}$; $A_n \overset{\gamma}{\subset} B_n$ and $B_n \overset{\gamma}{\subset} A_{n+1}$. Corollary 4.2 implies that there exists homomorphisms $\alpha_n$ of $A_n$ into $B_n$ and $\beta_n$ of $B_n$ into $A_{n+1}$ such that $\|\alpha_n-\mathrm{id}\mid A_n\|<8\gamma$ and $\|\beta_n-\mathrm{id}\mid B_n\|<8\gamma$.

We have now got a diagram,




and we want to show, that there exists inner automorphisms $\tau_n$ on $B_n$ and $\gamma_n$ on $A_n$ such that the diagram below commutes.




The existence of $\gamma_2$ is clear since $16\gamma < 1$ so $\beta_1\alpha_1$ is implemented by a unitary in $A_2$. Suppose now that we have found $\gamma_2, \tau_2, \dots, \gamma_n, \tau_n$ such that the diagram commutes. Then $\gamma_n$ is implemented by a unitary $v$ in $A_n$, hence $\gamma_n$ can be extended to $A_{n+1}$ when defining $\tilde{\gamma}_n = \mathrm{Ad}\,(v)$ the map $\beta_n\alpha_n$ can be extended to an inner automorphism Ad $(u)$ of $A_{n+1}$ because $\|\beta_n\alpha_n - \mathrm{id}\ |A_n\| < 1$. Let us then define $\gamma_{n+1}$ as Ad $(vu^*)$, and the theorem follows.

6.3. COROLLARY. *Let $A$ and $B$ be* AF *$C^*$-algebras on a Hilbert space $H$. If $\|A-B\| < 1/16$ then $A$ and $B$ are unitarily equivalent.*

*Proof.* The proof is due to Phillips and Raeburn [20] and Corollary 4.2, the idea being that by 4.2 we can find a unitary $u$ in $(A \cup B)''$ such that $\bar{A} = u\bar{B}u^*$ (bar denotes weak closure).

Let $\alpha$ be an isomorphism from $A$ onto $B$ obtained as in 6.2 then Ad $(u)\circ\alpha$ has the property that projections in $A$ which are equivalent in $\bar{A}$ are mapped into equivalent projections in $\bar{A}$ by Ad $(u)\circ\alpha$. To see this one must use that $\alpha$ is constructed from inner automorphism in $A$, $B$ and homomorphisms, which are close to the identity. Phillips and Raeburn then use Brattelis and Powers arguments to show that Ad $(u)\circ\alpha$ is an inner automorphism of $\bar{A}$, and the result follows.

6.4. COROLLARY. *Let $A$, $B$ and $C$ be as in the theorem. For any finite-dimensional $C^*$-subalgebra $A_0$ of $A$ there exists an isomorphism $\alpha$ of $A$ onto $B$ such that for any $a$ in $A_0$, $\|\alpha(a)-a\|\leq 8\|A-B\|\,\|a\|$.*

*Proof.* Choose $A_1$ such that $A_0\subseteq A_1$.

We will now discuss perturbations of nuclear $C^*$-algebras.

6.5. THEOREM. *Let $A$ be a nuclear $C^*$-subalgebra of $C^*$-algebra $C$. If $B$ is a $C^*$-subalgebra of $C$ and $\|A-B\|<\gamma<10^{-2}$, then $B$ is nuclear, $B^{**}$ is as von Neumann algebra isomorphic to $A^{**}$ and $A^*$ is isomorphic to $B^*$ through a completely positive isometry.*

*Proof.* Let $\pi$ be the universal representation of $C$ on a Hilbert space $H$. By [18, Lemma 5] $\|\overline{\pi(A)}-\overline{\pi(B)}\|<10^{-2}$ (bar denotes here weak closure). The nuclearity of $A$ implies that $\overline{\pi(A)}$ is an injective von Neumann algebra (not necessarily containing the identity on $H$).

Corollary 4.4 implies that $\overline{\pi(B)}$ is injective and isomorphic to $\overline{\pi(A)}$ through an inner automorphism Ad $(v)$ on $\overline{\pi(C)}$.

Since any representation $\varrho$ of $A$ or $B$ can be extended to a representation of $C$ [10, Prop. 2.10.2] we find that $\overline{\pi(A)}$ and $\overline{\pi(B)}$ are isomorphic to the second duals of $A$ and $B$ [10, Cor. 12.1.3]. The second dual of $B$ is then injective, hence $B$ is nuclear and the rest of the theorem follows from the remarks above by transposition.

We will now go back to the near inclusion situation $A\overset{\gamma}{\subset}B$.

If $A$ is a non separable $C^*$-algebra we will say, that $A$ is AF if any finite number of elements in $A$ can be approximated arbitrarily well with elements from a finite-dimensional $C^*$-subalgebra of $A$.

The following proposition is then an immediate consequence of Theorem 5.3.

6.6. PROPOSITION. *Let $A\overset{\gamma}{\subset}B$ be $C^*$-subalgebras of a $C^*$-algebra $C$. If $\gamma<10^{-4}$ and $A$ is AF (separable or not), then to any finite-dimensional subspace $F$ of $A$ and any $\varepsilon>0$ there exists a partial isometry $v$ in $C$ such that*

$$vFv^*\overset{\varepsilon}{\subset}B \quad \textit{and} \quad \|F-vFv^*\|\leq 240\gamma^{\frac{1}{2}}.$$

*For any $f$ in $F$*

$$\|vfv^*\| \geqslant (1-\varepsilon)\,\|f\|.$$

*If $C$ has a unit, $v$ can be chosen unitary with $\|I-v\| \leqslant 120\gamma^{\frac{1}{2}}$.*

6.7. Proposition. *Let $A \overset{\gamma}{\subset} B$ be $C^*$-subalgebras of a $C^*$-algebra $C$. Suppose $A$ has approximately inner flip and that $A$, $B$ and $C$ have a common unit, then to any finite dimensional subspace $F$ of $A$ there exists a completely positive map $\Phi$ of $A$ into $B$ such that for any $f$ in $F$, $\|\Phi(f)-f\| \leqslant (36\gamma^2+12\gamma)\|f\|$.*

*Proof.* Choose $\varepsilon > 0$ such that $A \overset{\gamma-\varepsilon}{\subset} B$ and find $(f_1, ..., f_n)$ in the unitball of $F$ such that any $f$ in this unitball is inside an $\varepsilon$ ball with center in some $f_i$.

By [13, Proposition 2.8] $A$ is nuclear and therefore by Theorem 3.1

$$A \otimes A \overset{6(\gamma-\varepsilon)}{\subset} B \otimes A.$$

Choose a unitary $v$ in $A \otimes A$ such that for any $i=1, ..., n$, $\|v(f_i \otimes I)v^* - I \otimes f_i\| < \varepsilon$, and find $x$ in $B \otimes A$ such that $\|v-x\| < 6(\gamma-\varepsilon)$. Let $\varphi$ be a state on $A$ then the slice map [25, § 1] $R_\varphi$: $C \otimes A \to C \otimes \mathbf{C}$ maps $B \otimes A$ onto $B \otimes \mathbf{C}$ and $A \otimes A$ onto $A \otimes \mathbf{C}$, we therefore obtain for $f_i$, $\|R_\varphi(x^*(I \otimes f_i)x) - f_i \otimes I\| \leqslant \|x^*(I \otimes f_i)x - f_i \otimes I\| \leqslant \varepsilon + \|x^*(I \otimes f_i)\,x - v^*(I \otimes f_i)v\| \leqslant \varepsilon + (1+6(\gamma-\varepsilon))\,6(\gamma-\varepsilon) + 6(\gamma-\varepsilon)$. Define $\Phi$ by $\Phi(a) \otimes I = R_\varphi(x^*(I \otimes a)\,x)$.

It is rather easy to see that this method when applied to a finite-dimensional full matrix algebra, say of type $I_n$, yields a result of the type discussed in section 5. In fact one can prove.

6.8. Corollary. *Let $A \overset{\gamma}{\subset} B$ be $C^*$-subalgebras of a $C^*$-algebra $C$. Suppose $A$ is finite-dimensional factor of type $I_n$.*

*If $\gamma < 2 \cdot 10^{-4}$ then there exists a partial isometry $v$ in $C$ such that $vAv^* \subseteq B$ and $\|I_A - v\| \leqslant 57\gamma^{\frac{1}{2}}$.*

*If $A$, $B$ and $C$ have a common unit $I$ and $\gamma < 10^{-3}$, then there exists a unitary $u$ in $C$ such that $uAu^* \subseteq B$ and $\|I-u\| \leqslant 28\gamma^{\frac{1}{2}}$.*

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