LOCALIZATION THEOREM IN K-THEORY FOR SINGULAR VARIETIES [1]

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0. Introduction

0.1. The preceding paper [1] constructs a map $L.: K_0^{eq}(X) \to K_0^{abs}(|X|) \otimes \Lambda$ for any equivariant quasi-projective X with projective fixed point scheme $|X|$. The construction of $L.$ involves an imbedding in a nonsingular variety; it is proved that $L.$ is independent of the imbedding and is a covariant natural transformation. A fixed point formula results by mapping X to a point. Here we give a direct proof of the following stronger result.

LOCALIZATION THEOREM. *Let i: $|X| \to X$ be the inclusion of the fixed point subvariety. Then the induced map i_* from $K_0^{eq}(|X|) \otimes \Lambda$ to $K_0^{eq}(X) \otimes \Lambda$ is an isomorphism.*

In § 1 we recall the construction of $L.$ for a fixed imbedding of X in a nonsingular variety, and we show that $L. \circ i_*$ is the identity endomorphism of $K_0^{abs}(|X|) \otimes \Lambda$. We prove in § 2 that i_* is surjective. Thus since $L.$ and i_* are inverse isomorphisms, $L.$ is independent of the imbedding. Since i_* is clearly covariant, the covariance of $L.$ follows. One recovers the Lefschetz–Riemann–Roch formula of [1], since the other properties of $L.$ listed in that theorem are consequences of the corresponding properties of i_*.

This localization theorem is an analogue of localization theorems in topological K-theory (cf. [1], 0.8), and Nielsen's result [4] in algebraic K-theory for nonsingular varieties.

0.2. As in [1], "equivariant varieties" are quasi-projective schemes with an endomorphism of finite order prime to the characteristic of the ground field k, such that the fixed point

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(1) Appendix to Lefschetz–Riemann–Roch for singular varieties by P. Baum, W. Fulton and G. Quart.

scheme is projective; Λ is the localization of $Z[k]$ at the multiplicative set generated by $\{([1]-[\lambda]) \mid \lambda \text{ is a root of unity and } \lambda \neq 1\}$. Other notations are as in [1].

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1. Injectivity of i_*

Here we make explicit without relative K-groups the map $L.$ of [1] for a fixed imbedding $X \subset M$.

Let α: $X \to M$ be a fixed closed embedding where M is a nonsingular variety. Let $\mathcal{F}$ be an equivariant coherent sheaf on X and suppose $\mathcal{L}.'$ and $\mathcal{L}.''$ are two complexes of equivariant locally free sheaves on M that resolve $\alpha_* \mathcal{F}$. There is a third resolution $\mathcal{L}.'''$ of $\alpha_* \mathcal{F}$ by locally free equivariant sheaves on M which maps surjectively to $\mathcal{L}.'$ and $\mathcal{L}.''$ and induces the identity map on the zeroth degree equivariant homology sheaves; this follows from the fact that every coherent equivariant sheaf is the image of a locally free sheaf (cf. [3] p. 261) and the *dominating* sequence argument of Borel–Serre ([2], p. 107). The kernel of each of the surjections is an acyclic complex of equivariant locally free sheaves on M. Thus, for each q, $\mathcal{H}_q(\mathcal{L}.'|_{|M|}) = \mathcal{H}_q(\mathcal{L}.'''|_{|M|}) = \mathcal{H}_q(\mathcal{L}.''|_{|M|})$ where $\mathcal{H}_q(\mathcal{L}.^{()}|_{|M|})$ is the qth equivariant homology sheaf of $\mathcal{L}.^{()}$ restricted to $|M|$. Whence, $\mathcal{H}_q(\mathcal{L}.|_{|M|})$ is an equivariant sheaf on $|M|$ with support on $|X|$ determined up to canonical isomorphism. Also, if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of equivariant coherent sheaves on X, there is an exact sequence of complexes of equivariant locally free sheaves $0 \to \xi.' \to \xi.'' \to \xi.''' \to 0$ on M with each $\xi.$ exact in degree >0 and the map on the zeroth degree homology sheaves is $0 \to \alpha_* \mathcal{F}_1 \to \alpha_* \mathcal{F}_2 \to \alpha_* \mathcal{F}_3 \to 0$. After restricting $0 \to \xi.' \to \xi.'' \to \xi.''' \to 0$ to $|M|$, we obtain the usual long exact sequence in the $\mathcal{H}_q$'s:

$$\dots \to \mathcal{H}_q(\xi.'|_{|M|}) \to \mathcal{H}_q(\xi.''|_{|M|}) \to \mathcal{H}_q(\xi.'''|_{|M|}) \to \mathcal{H}_{q-1}(\xi.'|_{|M|}) \to \dots.$$

We define I^M: $K_0^{\mathrm{eq}}(X) \to K_0^{\mathrm{eq}}(|X|)$ by the formula $I^M([\mathcal{F}]) = \sum (-1)^q [\mathcal{H}_q(\mathcal{L}.|_M)]$. Here we have identified the Grothendieck group of equivariant coherent sheaves on $|M|$ with support on $|X|$ with $K_0^{\mathrm{eq}}(|X|)$.

Let $\lambda_M \in K^0_{\mathrm{eq}}(|M|) \otimes \Lambda$ be the alternating sum of exterior powers of the conormal bundle of $|M|$ in M; λ_M is a unit in $K^0_{\mathrm{eq}}(|M|) \otimes \Lambda$ (cf. [1], 0.5).

We define as in [1], the Lefschetz–Riemann–Roch map $L.$: $K_0^{\mathrm{eq}}(X) \to K_0^{\mathrm{eq}}(|X|) \otimes \Lambda$ by $L. = |\alpha|^*(\lambda_M^{-1}) \frown I^M$.

The fact that i_* is injective after localization at S follows immediately from the fact that $|\alpha|^*(\lambda_M)$ is invertible in $K^0_{\mathrm{eq}}(|X|) \otimes \Lambda$ and the following lemma.

LEMMA 1. *$I^M \circ i_*$ is equal to multiplication by $|\alpha|^*(\lambda_M)$ as an endomorphism of $K_0^{eq}(|X|)$.*

Proof. Let $\mathcal{F}$ be an equivariant coherent sheaf on $|X|$ and a resolution of $i_*\mathcal{F}$ by locally free equivariant sheaves on M; $\mathcal{L}.$ is also a resolution of $|\alpha|_*\mathcal{F}$ on M. Since $\mathcal{F}$ is a sheaf on $|X|$, then

$$\Lambda^q(N)\otimes_{O_{|M|}}\mathcal{F}\approx(\Lambda^q(N)\otimes_{O_{|M|}}O_{|X|})\otimes_{O_{|X|}}\mathcal{F}\approx|\alpha|^*(\Lambda^q(N))\otimes_{O_{|X|}}\mathcal{F},$$

where $\Lambda^q(N)$ is the qth exterior power of the conormal bundle of $|M|$ in M equipped with its canonical endomorphism. Thus it suffices to show if $\mathcal{F}$ is an equivariant coherent sheaf on $|M|$, then $\mathcal{H}_q(\mathcal{L}.|_{|M|})$ is isomorphic to $\Lambda^q(N)\otimes_{O_{|M|}}\mathcal{F}$. At the level of coherent sheaves on $|M|$, we have a canonical isomorphism from $\mathcal{H}_q(\mathcal{L}.|_{|M|})$ to $\Lambda^q(N)\otimes_{O_{|M|}}\mathcal{F}$ ([2], Prop. 12). We must see that this isomorphism is compatible with the endomorphisms of these coherent sheaves. Since this isomorphism is compatible when $\mathcal{F}$ is a locally free equivariant sheaf on $|M|$ ([4], p. 91) and every coherent equivariant sheaf on $|M|$ has a resolution by locally free equivariant sheaves on $|M|$, it suffices to apply a dimension shifting argument: if $0\to\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{F}_3\to 0$ is an exact sequence of coherent equivariant sheaves on $|M|$, then the isomorphism of [2] is compatible for $\mathcal{F}_3$ if it is compatible for $\mathcal{F}_1$ and $\mathcal{F}_2$. After writing out the long exact sequence in the $\mathcal{H}_q$'s obtained from the short exact sequence of sheaves on $|M|$ and applying the compatibility hypothesis to $\mathcal{F}_1$ and $\mathcal{F}_2$, we obtain the compatibility for $\mathcal{F}_3$.

We record the following facts concerning Poincaré duality for later use:

Remark 1. Suppose M is nonsingular. The duality map from $K^0_{eq}(M)$ to $K_0^{eq}(M)$ which takes the class of a locally free sheaf to its class as a coherent sheaf is an isomorphism; if $\mathcal{F}$ is a coherent sheaf and $\mathcal{L}.$ is a resolution by locally free equivariant sheaves on M, then $[\mathcal{F}]\mapsto\sum(-1)^i[\mathcal{L}_i]$ is the inverse map. If $y\in K_0^{eq}(M)$, we denote its "dual" element in $K^0_{eq}(M)$ by $y^\#$. In section 3 we give an example of a singular variety for which the duality map is S-torsion.

Remark 2. Suppose M is nonsingular and imbed M in itself by the identity. The map I^M from $K_0^{eq}(M)$ to $K_0^{eq}(|M|)$ is compatible with the restriction map i^* from $K^0_{eq}(M)$ to $K^0_{eq}(|M|)$. Whence, the above lemma asserts if $y\in K_0^{eq}(M)$, we have

$$i^*((i_*(y))^\#)=\lambda_M\frown y^\# \tag{1}$$

in $K^0_{eq}(|M|)$. In particular, if $\mathcal{L}.$ is a resolution on M of the structure sheaf of $|M|$ with the identity endomorphism, then $i^*(\sum(-1)^i[\mathcal{L}_i])=\lambda_M$, a unit in $K^0_{eq}(|M|)\otimes\Lambda$.

Remark 3. Since λ_M is invertible after localization at S, then the formula (1) asserts that the composition

$$K_0^{\mathrm{eq}}(|M|) \xrightarrow{i_*} K_0^{\mathrm{eq}}(M) \approx K^0_{\mathrm{eq}}(M) \xrightarrow{i^*} K^0_{\mathrm{eq}}|M|$$

is an isomorphism after localization at S (the middle map is the duality isomorphism).

§ 2. Surjectivity of i_*

We now reduce the surjectivity of i_* after localization to a computation on projective space.

If p is a diagonal linear automorphism on projective n-space $\mathbf{P}$, then $K^0_{\mathrm{eq}}(\mathbf{P})$ and $K^0_{\mathrm{eq}}(|\mathbf{P}|)$ are free $Z[k]$-modules of rank $(n+1)$ ([1], 2.3). Since the composition $i^* \circ i_*$ is an isomorphism of $K^0_{\mathrm{eq}}(|\mathbf{P}|)$ after localization by Remark 3 of § 1, we conclude that i^* from $K^0_{\mathrm{eq}}(\mathbf{P})$ to $K^0_{\mathrm{eq}}(|\mathbf{P}|)$ is an isomorphism after localization at S. If $\mathcal{L}.$ is a resolution on $\mathbf{P}$ of the structure sheaf of $|\mathbf{P}|$ by locally free equivariant sheaves on $\mathbf{P}$, then $i^* (\sum (-1)^i [\mathcal{L}_i])$ is a unit in $K^0(|\mathbf{P}|) \otimes \Lambda$ (Remark 2 of § 1) and so $\sum (-1)^i [\mathcal{L}_i]$ is a unit in $K^0_{\mathrm{eq}}(\mathbf{P}^n) \otimes \Lambda$.

To prove for an arbitrary equivariant variety X that i_* is surjective after localization, we fix an embedding $\alpha\colon X \to \mathbf{P}$ where p is a diagonal linear automorphism of $\mathbf{P}$ (cf. [1]). Let $\varphi = \alpha^* (\sum (-1)^i [\mathcal{L}_i])$ in $K^0_{\mathrm{eq}}(X)$ where $\mathcal{L}.$ is a resolution of the structure sheaf of $|\mathbf{P}|$ by locally free equivariant sheaves on $\mathbf{P}$.

LEMMA 2. *φ satisfies properties A and B:*

(A) *φ is a unit in $K^0_{\mathrm{eq}}(X) \otimes \Lambda$*

(B) *if $\Psi \in K_0^{\mathrm{eq}}(X)$, then $\varphi \frown \Psi$ is in the image of i_*.*

Proof. Since $\alpha^*\colon K^0_{\mathrm{eq}}(\mathbf{P}) \to K^0_{\mathrm{eq}}(X)$ is a ring homomorphism and $(\sum (-1)^i [\mathcal{L}_i])$ is a unit in $K^0_{\mathrm{eq}}(\mathbf{P})$, then $\alpha^* (\sum (-1)^i [\mathcal{L}_i]) = \varphi$ is a unit in $K^0_{\mathrm{eq}}(X)$. If Ψ is a coherent equivariant sheaf on X, then

$$\varphi \frown \Psi = \sum (-1)^q (\alpha^*(\mathcal{L}_q) \otimes \Psi) = \sum (-1)^q \mathcal{H}_q(\alpha^*(\mathcal{L}.) \otimes \Psi),$$

where $\mathcal{H}_q(\alpha^*(\mathcal{L}.) \otimes \Psi)$ is the qth equivariant homology sheaf on X of the complex $\alpha^*(\mathcal{L}.) \otimes \Psi$. Since $\mathcal{L}.$ is exact on $\mathbf{P}$ off $|\mathbf{P}|$ and the support of the tensor product of two complexes is contained in the intersection of their supports, then $\mathcal{H}_q(\alpha^*(\mathcal{L}.) \otimes \Psi)$ has support on $|X|$. Since i_* maps $K_0^{\mathrm{eq}}(|X|)$ surjectively (in fact, isomorphically) to the Grothendieck group of equivariant coherent sheaves on X with support on $|X|$, then $\varphi \frown \Psi$ is in the image of i_*.

Let φ in $K^0_{\mathrm{eq}}(X)$ satisfy (A) and (B) and let $\Psi \in K_0^{\mathrm{eq}}(X)$. There is γ in $K_0^{\mathrm{eq}}(|X|)$ such that

$\varphi \frown \Psi^{\cdot} = i_*(\gamma)$. After localizing this relation and all the K-groups in sight, $\Psi^{\cdot} = \varphi^{-1} \frown i_*(\gamma) = i_*(i^*(\varphi^{-1}) \frown \gamma)$ by the projection formula. Whence, i_* is surjective after localization at S, which completes the proof.

§ 3. Computations

Remark 1. If all the local contributions on the fixed points of a singular variety are zero, i.e., $L.(O_X) = 0$ in $K_0(|X|)_S$, then the module property of $L.$ implies that $L.(\mathcal{E}) = 0$ for any locally free sheaf $\mathcal{E}$ on X. Since $L.$ is an isomorphism after localization, then the duality map $K^0_{eq}(X) \to K^{eq}_0(X)$ is S-torsion. For example, the curve $x^4 + y^4 + x^2z^2 = 0$ with the action $[x, y, z] \to [-x, iy, z]$ has this property; here the characteristic of k is not 2 and i is the square root of -1.

Remark 2. If X is nonsingular, then $L.$ lifts to the Grothendieck group of equivariant locally free sheaves and $L.$: $K^0_{eq}(X) \to K^0_{eq}(|X|)$ is given by $L.(\Psi^{\cdot}) = \lambda_X^{-1} \frown i^*(\Psi^{\cdot})$. To see this from the localization theorem alone, we need only remark that $i^* (\sum (-1)^i [\mathcal{L}_i])$ equals λ_X where $\mathcal{L}.$ is a resolution of the structure sheaf of $|X|$ by locally free equivariant sheaves on X (see Remark 2 of § 1).

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