ON THE INTERVALS BETWEEN NUMBERS THAT ARE SUMS OF TWO SQUARES

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1. Introduction

The purpose of the present memoir is to prove that, if $s_1, s_2, \dots, s_n, \dots$ denote in ascending order the numbers (not counted according to multiplicity of representations) that can be expressed as a sum of two squares, then

$$\sum_{s_{n+1} \leqslant x} (s_{n+1} - s_n)^{\gamma} = O(x \log^{\frac{1}{2}(\gamma-1)} x)$$

for $0 \leqslant \gamma < 5/3$. This result, which for the stated range of γ is certainly best possible on account of Landau's asymptotic formula

$$\sum_{s_n \leqslant x} 1 \sim \frac{Cx}{\sqrt{\log x}},$$

may be contrasted with the inequality

$$s_{n+1} - s_n = O(s_n^{\frac{1}{4}})$$

due to Bambah and Chowla [1] and with the conjecture that

$$s_{n+1} - s_n = O(s_n^{\varepsilon}).$$

Comparison also may be made with the conjecture

$$\sum_{p_{n+1} \leqslant x} (p_{n+1} - p_n)^2 = O(x \log x)$$

due to Erdös [2] and with the result

$$\sum_{p_{n+1} \leqslant x} \frac{(p_{n+1} - p_n)^2}{p_n} = O(\log^3 x)$$

that has been obtained by A. Selberg [5] on the Riemann hypothesis.

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The method depends on the application of an idea already used by the author in connection with a similar problem concerning the numbers prime to n [4], the principle involved being that there is a connection between the size of the moments of $s_{n+1}-s_n$ and that of the dispersion of sums of the form

$$\sum_{m\leqslant n<m+h} f(n)$$

for given h and variable m, where $f(n)$ is a function that vanishes unless n be a sum of two squares. Furthermore by a transformation the dispersion in turn can be estimated provided asymptotic formulae with sufficiently accurate error terms can be obtained for various sums involving $f(n)$, including in particular a sum of the form

$$\sum_{n\leqslant x} f(n)\, f(n+k).$$

The choice of function $f(n)$, however, for the problem considered here presents a major difficulty in relation to the latter sum, since neither of the two most natural functions $r(n)$ or $r_1(n)$ is altogether satisfactory, defining as usual $r(n)$ to be the number of representations of n as a sum of two squares and $r_1(n)$ to be 1 or 0 according as n be or not be expressible as a sum of two squares. On the one hand, although there is an asymptotic formula due to Estermann for the sum when $f(n)=r(n)$, the use of $r(n)$ only permits the contribution of the larger intervals $s_{n+1}-s_n$ to be assessed satisfactorily on account of the well-known phenomenon that the weighting inherent in $r(n)$ tends to emphasis numbers with an abnormally large number of prime factors. On the other hand the use of $r_1(n)$ is impossible because it would require an asymptotic formula (not merely an upper or lower bound) for the sum

$$\sum_{n\leqslant x} r_1(n)\, r_1(n+k),$$

the determination and proof of such a formula apparently involving much the same difficulties as are encountered in connection with the sum

$$\sum_{n\leqslant x} \Lambda(n)\Lambda(n+k).$$

To obviate the difficulties relating to the smaller intervals a new function $\varrho(n)$ is defined by affecting $r(n)$ with a dampening factor $t(n)$ which has the overall effect of reducing the order of magnitude of the function but which at the same time allows the corresponding asymptotic formulae to be determined. Defined in such a way that it mimics as far as possible the function $2^{-\omega^*(n)}$, where $\omega^*(n)$ is the number of prime factors of n congruent to 1, modulo 4, the function $t(n)$ is not necessarily always positive but gives rise by its multiplication with $r(n)$ to a function $f(n)$ that has the necessary feature that it vanishes when n is not a sum of two squares. The asymptotic formulae, which require some delicate and intricate analysis for their proof, then furnish an estimate for the dispersion that is satisfactory in respect of the smaller intervals.

Finally it may be remarked that the introduction of the function $t(n)$ forms the genesis of a principle which has other applications in the theory of numbers and of which it is hoped to give an account in due course.

2. Notation

The letters $\delta, h, k, l, \lambda, \mu, n$, and ϱ are positive integers; α, β, and m are integers; p is a positive prime number; the letter d with or without a subscript or an asterisk is throughout a positive square-free integer (possibly 1) composed entirely of prime factors p such that $p \equiv 1$, mod 4.

The letter x denotes an integer variable to be regarded as tending to infinity, all appropriate inequalities that are true for sufficiently large x being therefore assumed to hold; y, y_1 are real numbers not less than 1; v is a function of x to be defined later; $s = \sigma + it$.

The positive highest common factor and lowest common multiple of λ, μ are denoted by (λ, μ) and $[\lambda, \mu]$, respectively; $\sigma_a(h)$ is the sum of the a^{th} powers of the divisors of h; $\omega(n)$ is the number of distinct prime factors of n; $[u]$ is, where appropriate, the integral part of u: $\chi(n)$ is the non-principal character, modulo 4.

The letter ε indicates an arbitrarily small positive constant while η indicates some positive constant appropriate to the context. The equation $f = O(|g|)$ denotes an inequality of the form $|f| \leqslant C|g|$ that is true for all values of the variables consistent with stated conditions, where C is a positive constant that depends at most on either ε or γ.

3. The fundamental inequality

Let $f(n)$ be a real function of the positive integer n and let $F(m, h, x)$ and $F(m, h)$ be defined for any integer m and any positive integer h by

$$F(m, h, x) = \sum_{\substack{m \leqslant n < m+h \\ n \leqslant y}} f(n) \quad \text{and} \quad F(m, h) = \sum_{m \leqslant n < m+h} f(n).$$

Then
$$\sum_{-h+1 < m \leqslant x} F^2(m, h, x) = h \sum_{n \leqslant x} f^2(n) + 2 \sum_{k<h} (h-k) \sum_{n \leqslant x-k} f(n) f(n-k). \tag{1}$$

Also, for a suitable absolutely bounded number $\Delta = \Delta(x)$ to be specified according to the context later, we have

$$\begin{aligned}\sum_{-h+1<m\leqslant x}(F(m,h,x)-\Delta h)^2 &= \sum_{-h+1<m\leqslant x}F^2(m,h,x)-2\Delta h\sum_{-h+1<m\leqslant x}F(m,h,x)\\ &+\Delta^2h^2(x+h-1)=\sum_{-h+1<m\leqslant x}F^2(m,h,x)-2\Delta h^2\sum_{n\leqslant x}f(n)+\Delta^2h^2x+O(h^3).\end{aligned} \tag{2}$$

Therefore, combining (1) and (2), we obtain

$$\begin{aligned}\sum_{0<m\leqslant x-h+1}(F(m,h)-\Delta h)^2 &= \sum_{0<m\leqslant x-h+1}(F(m,h,x)-\Delta h)^2\\ &\leqslant h\sum_{n\leqslant x}f^2(n)+2\sum_{k<h}(h-k)\sum_{n\leqslant x-k}f(n)f(n+k)-2\Delta h^2\sum_{n\leqslant x}f(n)+\Delta^2h^2x+O(h^3).\end{aligned} \tag{3}$$

The properties of the distribution of the intervals between consecutive members of the sequence will be inferred from this inequality through the use with appropriate values of Δ of two functions $f(n)$ both of which will have the feature that $f(n)=0$ unless n be a sum of two squares. Since upper bounds for the sinister side of (3) are required for the application of the final method, we proceed in the next sections to estimate the dexter side for the functions $f(n)$ to be chosen. Throughout it will be assumed that $h<x^{\frac{1}{2}}$.

The case $f(n)=r(n)$

In this section we take $f(n)$ to be $r(n)$, Δ to be π, and denote the corresponding sinister side of (3) by $R(x,h)$. The estimations in this case depend on the following two lemmata, the first of which is too familiar to require comment and the second of which is due to Estermann [3], the error term in Estermann's formula being stated here in a form that takes account of Weil's subsequent work on the Kloosterman sum.

LEMMA 1. *We have*

(i) $\sum_{n\leqslant y}r(n)=\pi y+O(y^{\frac{1}{2}})$

(ii) $\sum_{n\leqslant y}r^2(n)=O(y\log 2y)$.

LEMMA 2. *We have, for* $0<k<y$,

$$\sum_{n\leqslant y}r(n)\,r(n+k)=8\gamma(k)\,y+O(y^{\frac{5}{6}+\varepsilon}),$$

where $\gamma(k)$ *is the multiplicative function of* k *determined by the conditions*

$$\gamma(2^\alpha)=\sum_{\delta|2^\alpha}\frac{(-1)^{2^\alpha/\delta}}{\delta}\quad(\alpha>0)$$

and $$\gamma(p^\alpha)=\sigma_{-1}(p^\alpha)\quad \textit{for}\ \ p\neq 2.$$

In view of Lemma 1 it suffices to estimate

$$\sum_{k<h}(h-k)\sum_{n\leqslant x-k} r(n)\,r(n+k),$$

which, by Lemma 2 and the condition $h<x^{\frac{1}{2}}$, is equal to

$$8x\sum_{k<h}(h-k)\,\gamma(k)+O(h^2x^{\frac{5}{6}+\varepsilon}). \tag{4}$$

In order to evaluate the sum occurring in (4) we consider a similar sum in which the factor $h-k$ is absent. We have

$$\sum_{k\leqslant y}\gamma(k)=\sum_{\substack{2^\alpha l\leqslant y\\(l,2)=1}}\gamma(2^\alpha)\,\sigma_{-1}(l)=\sum_{2^\alpha\leqslant y}\gamma(2^\alpha)\sum_{\substack{l\leqslant y/2^\alpha\\(l,2)=1}}\sigma_{-1}(l). \tag{5}$$

Also

$$\sum_{\substack{l\leqslant y_1\\(l,2)=1}}\sigma_{-1}(l)=\sum_{\substack{\lambda\mu\leqslant y_1\\(\lambda\mu,2)=1}}\frac{1}{\lambda}=\sum_{\substack{\lambda\leqslant y_1\\(\lambda,2)=1}}\frac{1}{\lambda}\sum_{\substack{\mu\leqslant y_1/\lambda\\(\mu,2)=1}}1=\frac{1}{2}y_1\sum_{\substack{\lambda\leqslant y_1\\(\lambda,2)=1}}\frac{1}{\lambda^2}+O\left(\sum_{\lambda\leqslant y_1}\frac{1}{\lambda}\right)$$

$$=\frac{1}{2}y_1\sum_{\substack{\lambda=1\\(\lambda,2)=1}}^{\infty}\frac{1}{\lambda^2}+O\left(y_1\sum_{\lambda>y_1}\frac{1}{\lambda^2}\right)+O\left(\sum_{\lambda\leqslant y_1}\frac{1}{\lambda}\right)=\frac{1}{16}\pi^2y_1+O\{\log(y_1+2)\}.$$

Therefore by this and (5) we have

$$\begin{aligned}\sum_{k\leqslant y}\gamma(k)&=\frac{1}{16}\pi^2y\sum_{2^\alpha\leqslant y}\frac{\gamma(2^\alpha)}{2^\alpha}+O\left(\log(y+2)\sum_{2^\alpha\leqslant y}\gamma(2^\alpha)\right)=\frac{1}{16}\pi^2y\sum_{\alpha=0}^{\infty}\frac{\gamma(2^\alpha)}{2^\alpha}+O\{\log^2(y+2)\}\\&=\frac{1}{16}\pi^2y\left(1+\sum_{\alpha=1}^{\infty}(2^{1-\alpha}-3\cdot2^{-\alpha})\right)+O\{\log^2(y+2)\}=\frac{1}{8}\pi^2y+O\{\log^2(y+2)\},\end{aligned} \tag{6}$$

the equation holding trivially in the range $0\leqslant y<1$ if the condition $y\geqslant1$ be temporarily relaxed. Finally, integrating (6) through the range $0\leqslant y\leqslant h$, we infer that

$$\sum_{k<h}(h-k)\,\gamma(k)=\frac{1}{16}\pi^2h^2+O(h\log^2 2h)$$

and hence from (4) that

$$\sum_{k<h}(h-k)\sum_{n\leqslant x-k}r(n)\,r(n+k)=\tfrac{1}{2}\pi^2h^2x+O(hx\log^2 2h)+O(h^2x^{\frac{5}{6}+\varepsilon}) \tag{7}$$

The estimate for $R(x,h)$ follows immediately from (3), (7), and Lemma 1. We deduce that

$$\begin{aligned}R(x,h)&\leqslant\pi^2h^2x-2\pi^2h^2x+\pi^2h^2x+O(hx\log x)+O(hx\log^2 2h)+O(h^2x^{\frac{5}{6}+\varepsilon})\\&\quad+O(h^2x^{\frac{1}{2}})+O(h^3)=O(hx\log x)+O(hx\log^2 2h)+O(h^2x^{5/6+\varepsilon})\end{aligned} \tag{8}$$

on substituting π for Δ.

5. The case $f(n) = \varrho(n)$: the definition of $\varrho(n)$ and preliminary lemmata

In order to define $\varrho(n)$ and to facilitate some subsequent estimations involving it we introduce the function

$$\psi(s, \delta) = \prod_{\substack{p \mid \delta \\ p \equiv 1, \bmod 4}} \left(2 - \frac{1}{p^s}\right)$$

and let

$$\psi(\delta) = \psi(1, \delta).$$

We then introduce the number $v = x^{1/20}$ and write

$$\varrho(n) = \varrho_x(n) = t(n)\, r(n),$$

where

$$t(n) = t_x(n) = \sum_{\substack{d \mid n \\ d \leqslant v}} \frac{\mu(d)}{\psi(d)} \left(1 - \frac{\log d}{\log v}\right),$$

the numbers d in accordance with Section 2 being restricted to square-free numbers composed entirely of prime factors p such that $p \equiv 1 \bmod 4$. We accordingly now take $f(n)$ to be $\varrho(n)$, Δ to be $A_1 \log^{-\frac{1}{3}} v$ where A_1 is a suitable positive constant to be specified later, and denote the corresponding sinister side of (3) by $\mathrm{P}(x, h)$. The apparently more natural choice of $f(n)$ as the function that would have been obtained had $\psi(\delta)$ been replaced by $2^{\omega(\delta)}$ in the definition would have served equally well except that certain additional complications in the work would have appeared.

As in the first case the estimations depend on formulae for $\sum f(n)$, $\sum f^2(n)$, and $\sum f(n) f(n+k)$. In this case, however, these are not already available and require for their determination and proof the following preliminary lemmata. The proofs of the latter are given in outline only, since for all but the last the method is along familiar lines as exemplified by [6] while for the last considerations of space preclude the inclusion of a detailed proof.

LEMMA 3. *We have, for any given d,*

$$\sum_{dn \leqslant y} r(dn) = \frac{\psi(d)}{d} \pi y + O(y^{\frac{2}{3}}).$$

We consider the generating function

$$f_d(s) = \sum_{n=1}^{\infty} \frac{r(dn)}{(dn)^s},$$

and denote $f_1(s)$ by $f(s)$. Then, for $\sigma > 1$ in the first place,

$$\frac{f_d(s)}{f(s)}=\prod_{p|d}\left(\sum_{\alpha=1}^{\infty}\frac{r(p^\alpha)}{p^{\alpha s}}\right)\left(\sum_{\alpha=0}^{\infty}\frac{r(p^\alpha)}{p^{\alpha s}}\right)^{-1}=\prod_{p|d}\left(\sum_{\alpha=1}^{\infty}\frac{\alpha+1}{p^{\alpha s}}\right)\left(\sum_{\alpha=0}^{\infty}\frac{\alpha+1}{p^{\alpha s}}\right)^{-1}$$

$$=\prod_{p|d}\left\{1-\left(1-\frac{1}{p^s}\right)^2\right\}=\frac{1}{d^s}\prod_{p|d}\left(2-\frac{1}{p^s}\right)=\frac{\psi(s,d)}{d^s}.$$

Therefore, since $f(s)=4\,\zeta(s)\,L(s)$ for all s, where $L(s)$ is the Dirichlet's L function formed with the character $\chi(n)$, we have

$$f_d(s)=\frac{4}{d^s}\,\psi(s,d)\,\zeta(s)\,L(s)$$

without restriction on s. The result then follows by using the formula

$$\sum_{dn\leqslant y} r(dn)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f_d(s)\,\frac{y^s}{s}\,ds \qquad (c>1),$$

valid when y/d is not an integer, in conjunction with mean value theorems for $\zeta(s)$ and $L(s)$, the error term given not being the best possible but being sufficient for the applications hereafter.

LEMMA 4. *Let $u(n)$ be the coefficient of n^{-s} in the expansion of*

$$\left(1-\frac{1}{2^s}\right)^{-1}\prod_{p\equiv 1,\,\mathrm{mod}\,4}\left(1-\frac{4}{p^s}+\frac{2}{p^{2s}}\right)^{-1}\prod_{p\equiv 3,\,\mathrm{mod}\,4}\left(1-\frac{1}{p^{2s}}\right)^{-1}$$

as a Dirichlet's series for $\sigma>1$. Then

(i) $16\,u(n)\geqslant r^2(n)$*:*

(ii) *for any given d,*

$$\sum_{dn\leqslant y} u(dn)=\frac{2^{\omega(d)}\psi(d)}{d}\,y(a\log y+b+a\theta(d))+O(x^{5/6}),$$

where a and b are constants and

$$\theta(d)=\left(\frac{\psi'(s,d)}{\psi(s,d)}\right)_{s=1}-\log d.$$

Since we have, for $\sigma>1$, from Euler's identity

$$\frac{1}{16}\sum_{n=1}^{\infty}\frac{r^2(n)}{n^s}=\left(1-\frac{1}{2^s}\right)^{-1}\prod_{p\equiv 1,\,\mathrm{mod}\,4}\left(\sum_{\alpha=0}^{\infty}\frac{(\alpha+1)^2}{p^{\alpha s}}\right)\prod_{p\equiv 3,\,\mathrm{mod}\,4}\left(1-\frac{1}{p^{2s}}\right)^{-1},$$

the generating functions for $u(n)$ and $r^2(n)$ when expressed as infinite products differ only in respect of the factors relating to primes congruent to 1, modulo 4, the product of the common factors relating to the primes incongruent to 1, modulo 4, being a Dirichlet's series with positive coefficients. Therefore to prove part (i) of the lemma it is enough to show that, if a_α be the coefficient of $p^{-\alpha s}$ in the expansion of $(1-4p^{-s}+2p^{-2s})^{-1}$ as a power series, then $a_\alpha \geqslant (\alpha+1)^2$. The latter fact can be inferred by a simple inductive argument involving the recurrence relations $a_0=b_0=1$, $a_1=b_1=4$, $a_\alpha=4a_{\alpha-1}-2a_{\alpha-2}$, $b_\alpha=2b_{\alpha-1}-b_{\alpha-2}$ $(\alpha\geqslant 2)$, where $b_\alpha=(\alpha+1)^2$.

For part (ii) we use the generating function

$$g_d(s)=\sum_{n=1}^{\infty}\frac{u(dn)}{(dn)^s}$$

and write $g_1(s)$ as $g(s)$. Then, for $\sigma>1$ initially, we have

$$\frac{g_d(s)}{g(s)}=\prod_{p\mid d}\left\{\left(1-\frac{4}{p^s}+\frac{2}{p^{2s}}\right)^{-1}-1\right\}\left\{1-\frac{4}{p^s}+\frac{2}{p^{2s}}\right\}=\prod_{p\mid d}\left(\frac{4}{p^s}-\frac{2}{p^{2s}}\right)=\frac{2^{\omega(d)}\psi(d,s)}{d^s}.$$

Also
$$\prod_{p\equiv 1,\,\mathrm{mod}\,4}\left(1-\frac{4}{p^s}+\frac{2}{p^{2s}}\right)=K_1(s)\prod_{p\equiv 1,\,\mathrm{mod}\,4}\left(1-\frac{1}{p^s}\right)^{-4},$$

where $K_1(s)$ is bounded and regular for $\sigma>(\log(2+\sqrt{2})/\log 5)+\eta$ and therefore, in particular, for $\sigma\geqslant 4/5$. Therefore, since

$$\prod_{p\equiv 1,\,\mathrm{mod}\,4}\left(1-\frac{1}{p^s}\right)^{-2}=\zeta(s)\,L(s)\left(1-\frac{1}{2^s}\right)\prod_{p\equiv 3,\,\mathrm{mod}\,4}\left(1-\frac{1}{p^{2s}}\right)$$

for $\sigma>1$, we deduce that

$$g_d(s)=\frac{2^{\omega(d)}\psi(d,s)}{d^s}\,\zeta^2(s)\,L^2(s)\,K_2(s),$$

where $K_2(s)$ is bounded and regular for $\sigma\geqslant 4/5$. The result then follows much as in the previous lemma by using Perron's formula in conjunction with mean value theorems for $\zeta(s)$ and $L(s)$, the point $s=1$ being a double pole of the integrand $g_d(s)\,x^s/s$.

LEMMA 5. *We have*

$$\sum_{d\leqslant y}\frac{\mu(d)}{d}\log\frac{y}{d}=\frac{4A\log^{\frac{1}{2}}y}{\pi}+O\left(\frac{1}{\log^{\frac{1}{2}}2y}\right),$$

where
$$A=\prod_{p\equiv 3,\,\mathrm{mod}\,4}\left(1-\frac{1}{p^2}\right)^{-\frac{1}{2}}.$$

Here the generating function is, for $\sigma>1$,

$$h(s)=\sum_{d=1}^{\infty}\frac{\mu(d)}{d^s}=\prod_{p\equiv 1,\ \mathrm{mod}\ 4}\left(1-\frac{1}{p^s}\right)=\frac{K_3(s)}{\{\zeta(s)\,L(s)\}^{\frac{1}{2}}},$$

where $K_3(s)$ given by

$$K_3^2(s)=\left(1-\frac{1}{2^s}\right)^{-1}\prod_{p\equiv 3,\ \mathrm{mod}\ 4}\left(1-\frac{1}{p^{2s}}\right)^{-1}$$

is regular and bounded for $\sigma\geqslant 3/4$ and where the positive determination of square roots is taken for $s>1$. Defining $h(s)$ by analytic continuation in appropriate zero-free regions of $\zeta(s)$ and $L(s)$, we apply the formula

$$\sum_{d\leqslant y}\frac{\mu(d)}{d}\log\frac{y}{d}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}h(s)\,\frac{y^{s-1}}{(s-1)^2}\,ds \qquad (c>1)$$

and deform the contour according to the methods used in [6]. We conclude that

$$\sum_{d\leqslant y}\frac{\mu(d)}{d}\log\frac{y}{d}=\frac{1}{2\pi i}\,\frac{K_3(1)}{L^{\frac{1}{2}}(1)}\int_C\frac{y^{s-1}\,ds}{(s-1)^{\frac{3}{2}}}+O\left(\frac{1}{\log^{\frac{1}{2}}2y}\right),$$

where C is a contour encompassing $s=1$ of the type used in Hankel's integral for the gamma function. Therefore

$$\sum_{d\leqslant y}\frac{\mu(d)}{d}\log\frac{y}{d}=-\frac{1}{2\pi}\,\frac{\sqrt{2}\,A}{(\pi/4)^{\frac{1}{2}}}\,2\sin\left(-\frac{3\pi}{2}\right)\Gamma\left(-\frac{1}{2}\right)\log^{\frac{1}{2}}y+O\left(\frac{1}{\log^{\frac{1}{2}}2y}\right),$$

which on simplification yields the lemma.

Lemma 6. *Let $\Psi(d,\varrho)=\psi\{(d,\varrho^*)\}$, where $\varrho^*=\varrho/(d,\varrho)$. Then we have*

$$\sum_{d\leqslant y}\frac{\mu(d)\,(d,\varrho)}{d\Psi(d,\varrho)}\log\frac{y}{d}=\frac{4A\,C(\varrho)\log^{\frac{1}{2}}y}{\pi}+O\left(\frac{\varrho^{\frac{1}{2}}}{\log^{\frac{1}{2}}2y}\right),$$

where
$$C(\varrho)=\begin{cases}\dfrac{1}{\psi(\varrho)}, & \text{if } p|\varrho \text{ and } p\equiv 1,\ \mathrm{mod}\ 4 \text{ implies } p^2|\varrho,\\ 0, & \text{otherwise.}\end{cases}$$

The proof is an obvious generalisation of that of the previous lemma, the generating function being now $h_\varrho(s)=h(s)\,C(s,\varrho)$, where

$$C(s,\varrho)=\prod_{\substack{p^2|\varrho\\ p\equiv 1,\ \mathrm{mod}\ 4}}\left(1-\frac{p}{p^{s-1}(2p-1)}\right)\prod_{\substack{p|\varrho\\ p\nmid\varrho^2\\ p\equiv 1,\ \mathrm{mod}\ 4}}\left(1-\frac{p}{p^s}\right)\prod_{\substack{p|\varrho\\ p\equiv 1,\ \mathrm{mod}\ 4}}\left(1-\frac{1}{p^s}\right)^{-1}.$$

Since $C(\varrho)=C(1,\varrho)$ the main term is obtained immediately while the error term is estimated by giving close attention to the effect $C(s, \varrho)$ has on the residual contours used in the previous proof.

LEMMA 7. *We have, for any given d^*, that*

(i) $$\sum_{\substack{d \leqslant y \\ (d, d^*)=1}} \frac{\mu(d)\, 2^{\omega(d)}}{d} \log \frac{y}{d} = O\{\sigma_{-\frac{1}{2}}(d^*)\}$$

(ii) $$\sum_{\substack{d \leqslant y \\ (d, d^*)=1}} \frac{\mu(d)\, 2^{\omega(d)}\, \theta(d)}{d} \log \frac{y}{d} = O\{\log 2y\, \sigma_{-\frac{1}{2}}(d^*)\},$$

where $\theta(d)$ is defined as in Lemma 4.

For part (i) we use the generating function

$$k_{d^*}(s) = \sum_{\substack{d=1 \\ (d, d^*)=1}}^{\infty} \frac{\mu(d)\, 2^{\omega(d)}}{d^s} = \frac{K_4(s)}{\zeta(s)\, L(s)} \prod_{p \mid d^*} \left(1 - \frac{2}{p^s}\right)^{-1},$$

where $K_4(s)$ is regular and bounded for $\sigma > 1 - \eta$.

Part (ii) is most easily proved by observing that, since $\theta(d)$ is an additive function, the sinister side of the proposed formula may be written as

$$\sum_{\substack{p \leqslant y \\ p \nmid d \\ p \equiv 1, \bmod 4}} \theta(p) \sum_{\substack{d \leqslant y \\ (d, d^*)=1 \\ d \equiv 1, \bmod p}} \frac{\mu(d)\, 2^{\omega(d)}}{d} \log \frac{y}{d} = - \sum_{\substack{p \leqslant y \\ p \nmid d^* \\ p \equiv 1, \bmod 4}} \frac{2\theta(p)}{p} \sum_{\substack{d_1 \leqslant y/p \\ (d_1, pd^*)=1}} \frac{\mu(d_1)\, 2^{\omega(d_1)}}{d_1} \log \left(\frac{y/p}{d_1}\right),$$

the latter expression being by part (i) equal to

$$O\left(\sigma_{-\frac{1}{2}}(d^*) \sum_{p \leqslant y} \frac{\log p}{p}\right) = O\{\log 2y\, \sigma_{-\frac{1}{2}}(d^*)\}.$$

LEMMA 8. *We have, for $0 < k < y$ and given d_1, d_2, that*

$$\sum_{\substack{n \leqslant y \\ n \equiv 0, \bmod d_1 \\ n+k \equiv 0, \bmod d_2}} r(n)\, r(n+k) = \pi^2 \gamma_1(k)\, \Gamma(k, d_1, d_2)\, y + O(d_1^{\frac{1}{2}} d_2^{\frac{1}{2}} y^{\frac{5}{6}+\varepsilon}),$$

where $\gamma_1(k) = \gamma(2^\alpha)$ for $2^\alpha \| k$ and where, if $\Phi_k(\varrho)$ denote Ramanujan's sum $\sum_{\delta \mid k, \delta \mid \varrho} \mu(\varrho/\delta)\, \delta$, then

$$\Gamma(k, d_1, d_2) = \frac{1}{d_1 d_2} \sum_{\substack{\varrho=1 \\ (\varrho, 2)=1}}^{\infty} \frac{\Phi_k(\varrho)}{\varrho^2} \frac{(d_1, \varrho)\, (d_2, \varrho)\, \psi(d_1)\, \psi(d_2)}{\Psi(d_1, \varrho)\, \Psi(d_2, \varrho)}.$$

This generalisation of Lemma 2 can be established by following Estermann's method closely although the proof is long owing to the additional arithmetical complications caused by the presence of d_1, d_2. Alternatively there are a number of other approaches which do not, however, lessen to any extent the length of the proof, Whatever method be adopted we are led to a main term involving a series (over appropriate ranges of summation) of the form

$$\sum_{\substack{l_1, l_2 \\ ([d_1, l_1], [d_2, l_2]) | k}} \frac{\chi(l_1)\,\chi(l_2)}{[d_1, l_1]\,[d_2, l_2]} \,([d_1, l_1], [d_2, l_2])$$

which can be expressed in terms of $\Gamma(k, d_1, d_2)$ by means of the relation

$$\sum_{\varrho | \delta} \Phi_k(\varrho) = \begin{cases} \delta, & \text{if } \delta | k, \\ 0, & \text{otherwise,} \end{cases}$$

the special constitution of d_1, d_2 as products of primes being relevant to the argument.

We are now in a position to consider the sums involving $\varrho(n)$.

6. Estimation of $\sum \varrho(n)$ and $\sum \varrho^2(n)$

The estimation of $\sum \varrho(n)$ is quickly effected. We have, using Lemma 3 and then Lemma 5, that

$$\begin{aligned}\sum_{n \leqslant x} \varrho(n) &= \sum_{n \leqslant x} t(n)\, r(n) = \sum_{n \leqslant x} r(n) \sum_{\substack{d | n \\ d \leqslant v}} \frac{\mu(d)}{\psi(d)} \left(1 - \frac{\log d}{\log v}\right) = \sum_{d \leqslant v} \frac{\mu(d)}{\psi(d)} \left(1 - \frac{\log d}{\log v}\right) \sum_{\substack{n \leqslant x \\ n \equiv 0, \bmod d}} r(n) \\ &= \frac{\pi x}{\log v} \sum_{d \leqslant v} \frac{\mu(d)}{d} \log \frac{v}{d} + O\left(x^{\frac{2}{3}} \sum_{d \leqslant v} 1\right) = \frac{4Ax}{\log^{\frac{1}{2}} v} + O\left(\frac{x}{\log^{\frac{3}{2}} x}\right) + O(x^{\frac{2}{3}} v) \\ &= \frac{4Ax}{\log^{\frac{1}{2}} v} + O\left(\frac{x}{\log^{\frac{3}{2}} x}\right). \end{aligned} \tag{9}$$

The estimation of $\sum \varrho^2(n)$ on the other hand takes longer. We have, by both parts of Lemma 4 (¹),

$$\begin{aligned}\sum_{n \leqslant x} \varrho^2(n) &= \sum_{n \leqslant x} t^2(n)\, r^2(n) \leqslant 16 \sum_{n \leqslant x} t^2(n)\, u(n) \\ &= 16 \sum_{n \leqslant x} u(n) \sum_{\substack{d_1 | n,\, d_2 | n \\ d_1,\, d_2 \leqslant v}} \frac{\mu(d_1)\,\mu(d_2)}{\psi(d_1)\,\psi(d_2)} \left(1 - \frac{\log d_1}{\log v}\right) \left(1 - \frac{\log d_2}{\log v}\right) \\ &= \frac{16}{\log^2 v} \sum_{d_1,\, d_2 \leqslant v} \frac{\mu(d_1)\,\mu(d_2)}{\psi(d_1)\,\psi(d_2)} \log \frac{v}{d_1} \log \frac{v}{d_2} \sum_{\substack{n \leqslant x \\ n \equiv 0, \bmod [d_1, d_2]}} u(n) \end{aligned}$$

(¹) As will be seen the replacement of $r^2(n)$ by $u(n)$ reduces the complexity of the subsequent calculations.

$$= \frac{16x}{\log^2 v} \sum_{d_1, d_2 \leqslant v} \frac{\mu(d_1)\mu(d_2)\, 2^{\omega\{[d_1,d_2]\}}\psi\{[d_1,d_2]\}}{\psi(d_1)\,\psi(d_2)\,[d_1,d_2]}$$

$$\times (a \log x + b + \theta\{[d_1,d_2]\}) \log \frac{v}{d_1} \log \frac{v}{d_2} + O\left(x^{5/6} \sum_{d_1, d_2 \leqslant v} 1\right)$$

$$= \frac{x}{\log^2 v} \sum_{d_1, d_2 \leqslant v} \frac{\mu(d_1)\mu(d_2)\, 2^{\omega(d_1)}2^{\omega(d_2)}\,(d_1 d_2)}{2^{\omega\{(d_1,d_2)\}}\psi\{(d_1,d_2)\}\, d_1 d_2} (a_1 \log x + b_1 + 2a_1\theta(d_1)$$

$$- a_1\theta\{(d_1,d_2)\}) \log \frac{v}{d_1} \log \frac{v}{d_2} + O\left(\frac{x}{\log^{\frac{1}{3}} x}\right) = \frac{x}{\log^2 v}\,\Sigma_v + O\left(\frac{x}{\log^{\frac{1}{3}} x}\right), \text{ say,} \tag{10}$$

the penultimate line following from symmetry, where a_1, b_1 are suitable constants.

Next, by the Möbius inversion formula,

$$\frac{(d_1,d_2)}{2^{\omega\{(d_1,d_2)\}}\psi\{(d_1,d_2)\}} = \sum_{d_3 | d_1,\, d_3 | d_2} \mathrm{H}_1(d_3)$$

and

$$\frac{(d_1,d_2)\,\theta\{(d_1,d_2)\}}{2^{\omega\{(d_1,d_2)\}}\psi\{(d_1,d_2)\}} = \sum_{d_3 | d_1,\, d_3 | d_2} \mathrm{H}_2(d_3),$$

where

$$\mathrm{H}_1(d_3) = \sum_{\delta | d_3} \mu\left(\frac{d_3}{\delta}\right) \frac{\delta}{2^{\omega(\delta)}\psi(\delta)} = \frac{d_3}{2^{\omega(d_3)}\psi(d_3)} \sum_{\delta | d_3} \frac{\mu(\delta)\, 2^{\omega(\delta)}\psi(\delta)}{\delta}$$

and

$$\mathrm{H}_2(d_3) = \sum_{\delta | d_3} \mu\left(\frac{d_3}{\delta}\right) \frac{\delta\theta(\delta)}{2^{\omega(\delta)}\psi(\delta)} = \theta(d_3)\,\mathrm{H}_1(d_3) - \frac{d_3}{2^{\omega(d_3)}\psi(d_3)} \sum_{\delta | d_3} \frac{\mu(\delta)\, 2^{\omega(\delta)}\psi(\delta)\,\theta(\delta)}{\delta}.$$

Writing

$$\mathrm{H}_1(d_3) = \frac{d_3}{2^{\omega(d_3)}\psi(d_3)}\,\mathrm{H}_3(d_3)$$

and

$$\mathrm{H}_2(d_3) = \frac{\theta(d_3)\, d_3}{2^{\omega(d_3)}\psi(d_3)}\,\mathrm{H}_3(d_3) - \frac{d_3}{2^{\omega(d_3)}\psi(d_3)}\,\mathrm{H}_4(d_3),$$

we therefore have

$$\Sigma_v = \sum_{d_1, d_2 \leqslant v} \frac{\mu(d_1)\mu(d_2)\, 2^{\omega(d_1)}2^{\omega(d_2)}}{d_1 d_2} (a_1 \log x + b_1 + 2a_1\theta(d_1)) \log \frac{v}{d_1} \log \frac{v}{d_2} \sum_{d_3 | d_1,\, d_3 | d_2} \mathrm{H}_1(d_3)$$

$$- a_1 \sum_{d_1, d_2 \leqslant v} \frac{\mu(d_1)\mu(d_2)\, 2^{\omega(d_1)}2^{\omega(d_2)}}{d_1 d_2} \log \frac{v}{d_1} \log \frac{v}{d_2} \sum_{d_3 | d_1,\, d_3 | d_2} \mathrm{H}_2(d_3)$$

$$= \sum_{d_3 \leqslant v} \frac{4^{\omega(d_3)}}{d_3^2}\,\mathrm{H}_1(d_3) \left(\sum_{\substack{(d_1', d_3) = 1 \\ d_1' \leqslant v/d_3}} \frac{\mu(d_1')\, 2^{\omega(d_1')}}{d_1'} (a_1 \log x + b_1 + 2a_1\theta(d_1') + 2a_1\theta(d_3)) \log \frac{v/d_3}{d_1'} \right)$$

$$\times\left(\sum_{\substack{(d_2', d_3)=1\\ d_2'\leqslant v/d_3}} \frac{\mu(d_2')\,2^{\omega(d_2')}}{d_2'} \log \frac{v/d_3}{d_2'}\right) - a_1 \sum_{d_3\leqslant v} \frac{4^{\omega(d_3)}}{d_3^2}\,\mathrm{H}_2(d_3)\left(\sum_{\substack{(d', d_3)=1\\ d'\leqslant v/d_3}} \frac{\mu(d')\,2^{\omega(d')}}{d'} \log \frac{v/d_3}{d'}\right)^2$$

$$= \sum_{d_3\leqslant v} \frac{2^{\omega(d_3)}}{d_3\psi(d_3)}\,\{\mathrm{H}_3(d_3)\,(a_1 \log x + b_1 + a_1\theta(d_3)) + a_1\,\mathrm{H}_4(d_3)\}\,\Sigma^2_{v,d_3,1}$$

$$+ \sum_{d_3\leqslant v} \frac{2^{\omega(d_3)}}{d_3\psi(d_3)}\,\mathrm{H}_3(d_3)\,\Sigma_{v,d_3,1}\,\Sigma_{v,d_3,2}, \tag{11}$$

where
$$\Sigma_{v,d_3,1} \sum_{\substack{(d', d_3)=1\\ d'\leqslant v/d_3}} \frac{\mu(d')\,2^{\omega(d')}}{d'} \log \frac{v/d_3}{d'}$$

and
$$\Sigma_{v,d_3,2} = \sum_{\substack{(d', d_3)=1\\ d'\leqslant v/d_3}} \frac{\mu(d')2^{\omega(d')}\,\theta(d')}{d'} \log \frac{v/d_3}{d'}.$$

We deduce from (11) and Lemma 7 that

$$\Sigma_v = O\left(\log x \sum_{d_3\leqslant v} \frac{2^{\omega(d_3)}\,\sigma^3_{-\frac{1}{2}}(d_3)}{d_3\psi(d_3)}\right) = O\left(\log x \sum_{d_3\leqslant v} \frac{2^{\omega(d_3)}\,\sigma_{-\frac{1}{4}}(d_3)}{d_3\psi(d_3)}\right),$$

since $\mathrm{H}_3(d_3) = O(1)$ and $\mathrm{H}_4(d_3) = O\{\log d_3\,\sigma_{-\frac{1}{2}}(d_3)\}$. Therefore we have

$$\Sigma_v = O\left(\log x \prod_{\substack{p\leqslant v\\ p\equiv 1,\ \mathrm{mod}\ 4}} \left\{1 + \frac{1}{p}\left(1 - \frac{1}{2p}\right)^{-1}\left(1 + \frac{1}{p^{\frac{1}{4}}}\right)\right\}\right) = O\left\{\log x \prod_{\substack{p\leqslant v\\ p\equiv 1,\ \mathrm{mod}\ 4}} \left(1 + \frac{1}{p}\right)\right\} = O(\log^{\frac{3}{2}} x),$$

from which and (10) we conclude that

$$\sum_{n\leqslant x} \varrho^2(n) = O\left(\frac{x}{\log^{\frac{1}{2}} x}\right). \tag{12}$$

7. Estimation of $\sum \varrho(n)\,\varrho(n+k)$

In assessing the sum appearing above we assume that $0 < k < x$. Then

$$\sum_{n\leqslant x} \varrho(n)\,\varrho(n+k) = \sum_{n\leqslant x} t(n)\,t(n+k)\,r(n)\,r(n+k)$$

$$= \sum_{n\leqslant x} r(n)\,r(n+k) \sum_{\substack{d_1|n,\ d_2|n+k\\ d_1, d_2\leqslant v}} \frac{\mu(d_1)\,\mu(d_2)}{\psi(d_1)\,\psi(d_2)}\left(1 - \frac{\log d_1}{\log v}\right)\left(1 - \frac{\log d_2}{\log v}\right)$$

$$= \frac{1}{\log^2 v} \sum_{d_1, d_2\leqslant v} \frac{\mu(d_1)\,\mu(d_2)}{\psi(d_1)\,\psi(d_2)} \log \frac{v}{d_1} \log \frac{v}{d_2} \sum_{\substack{n\leqslant x\\ n\equiv 0,\ \mathrm{mod}\ d_1\\ n+k\equiv 0,\ \mathrm{mod}\ d_2}} r(n)\,r(n+k).$$

Hence, by Lemma 8, we have

$$\sum_{n\leqslant x} \varrho(n)\,\varrho(n+k) = \frac{\pi^2\gamma_1(k)\,x}{\log^2 v} \sum_{d_1,d_2\leqslant v} \frac{\mu(d_1)\,\mu(d_2)}{\psi(d_1)\,\psi(d_2)} \Gamma(k,d_1,d_2) \log\frac{v}{d_1}\log\frac{v}{d_2} + O\left(x^{\frac{5}{6}+\varepsilon} \sum_{d_1,d_2\leqslant v} (d_1 d_2)^{\frac{1}{2}}\right)$$

$$= \frac{\pi^2\gamma_1(k)\,x}{\log^2 v} \sum_{d_1,d_2\leqslant v} \frac{\mu(d_1)\,\mu(d_2)}{\psi(d_1)\,\psi(d_2)} \Gamma(k,d_1,d_2) \log\frac{v}{d_1}\log\frac{v}{d_2} + O\left(\frac{x}{\log^2 x}\right)$$

$$= \frac{\pi^2\gamma_1(k)\,G(k,v)\,x}{\log^2 v} + O\left(\frac{x}{\log^2 x}\right), \text{ say.} \tag{13}$$

Next, by the definition of $\Gamma(k,d_1,d_2)$ in Lemma 8 and then by Lemma 6, we have

$$G(k,v) = \sum_{\substack{\varrho=1\\(\varrho,2)=1}}^{\infty} \frac{\Phi_k(\varrho)}{\varrho^2} \sum_{d_1,d_2\leqslant v} \frac{\mu(d_1)\,\mu(d_2)\,(d_1,\varrho)\,(d_2,\varrho)}{d_1 d_2 \Psi(d_1,\varrho)\,\Psi(d_2,\varrho)} \log\frac{v}{d_1}\log\frac{v}{d_2}$$

$$= \sum_{\substack{\varrho=1\\(\varrho,2)=1}}^{\infty} \frac{\Phi_k(\varrho)}{\varrho^2} \left(\sum_{d\leqslant v} \frac{\mu(d)\,(d,\varrho)}{d\Psi(d,\varrho)} \log\frac{v}{d}\right)^2$$

$$= \frac{16A^2\log v}{\pi^2} \sum_{\substack{\varrho=1\\(\varrho,2)=1}}^{\infty} \frac{\Phi_k(\varrho)\,C^2(\varrho)}{\varrho^2} + O\left(\sum_{\substack{\varrho=1\\(\varrho,2)=1}}^{\infty} \frac{|\Phi_k(\varrho)|}{\varrho^{\frac{3}{2}}}\right)$$

$$= \frac{16A^2\log v}{\pi^2} \Sigma_{k,1} + O(\Sigma_{k,2}), \text{ say.} \tag{14}$$

Considering the easier sum first we have

$$\Sigma_{k,2} = O\left(\sum_{\varrho=1}^{\infty} \frac{\sigma\{(k,\varrho)\}}{\varrho^{\frac{3}{2}}}\right) = O\left(\sum_{\delta\mid k} \delta \sum_{\substack{\varrho=1\\ \varrho\equiv 0,\ \mathrm{mod}\ \delta}}^{\infty} \frac{1}{\varrho^{\frac{3}{2}}}\right) = O\left(\sum_{\delta\mid k} \frac{1}{\delta^{\frac{1}{2}}}\right) = O\{\sigma_{-\frac{1}{2}}(k)\}. \tag{15}$$

To estimate $\Sigma_{k,1}$ we define k_1, k_3 for given $k = \prod p^\beta$ by

$$k_1 = \prod_{p\equiv 1,\ \mathrm{mod}\ 4} p^\beta, \quad k_3 = \prod_{p\equiv 3,\ \mathrm{mod}\ 4} p^\beta,$$

and define ϱ_1, ϱ_2 for general ϱ in a similar manner. Then, since $\Phi_k(\varrho)$ and $C(\varrho)$ are multiplicative and $C(\varrho_3) = 1$, we have

$$\Sigma_{k,1} = \sum_{\varrho_1} \frac{\Phi_k(\varrho_1)\,C^2(\varrho_1)}{\varrho_1^2} \sum_{\varrho_3} \frac{\Phi_k(\varrho_3)}{\varrho_3^2} = \Sigma_{k,1}^{(1)}\,\Sigma_{k,1}^{(3)}, \text{ say.} \tag{16}$$

Next, by the expression of $\Phi_k(\varrho)$ as a divisor sum and the definition of A in Lemma 5,

$$\Sigma^{(3)}_{k,1} = \sum_{\substack{\varrho_3', \varrho_3'' \\ \varrho_3'|k}} \frac{\varrho_3' \mu(\varrho_3'')}{(\varrho_3' \varrho_3'')^2} = \tau(k_3) \prod_{p\equiv 3,\, \text{mod}\, 4} \left(1 - \frac{1}{p^2}\right) = \frac{\tau(k_3)}{A^2}, \tag{17}$$

where $\tau(k_3) = \sigma_{-1}(k_3)$. To consider $\Sigma^{(1)}_{k,1}$ we need the fact that if $\varrho^\beta \| k$ then

$$\Phi_k(p^\alpha) = \begin{cases} \phi(p^\alpha), & \text{if } \alpha \leqslant \beta, \\ -p^\beta, & \text{if } \alpha = \beta + 1, \\ 0, & \text{if } \alpha > \beta + 1. \end{cases}$$

Then

$$\Sigma^{(1)}_{k,1} = \prod_{p\equiv 1,\, \text{mod}\, 4} \left(1 + \sum_{\alpha=2}^{\infty} \frac{\Phi_k(p^\alpha)\, C^2(p^\alpha)}{p^{2\alpha}}\right) = \prod_{p|k_1} \left(1 + \frac{1}{(2p-1)^2} \sum_{\alpha=2}^{\beta+1} \frac{\Phi_k(p^\alpha)}{p^{2\alpha-2}}\right)$$

$$= \prod_{p|k_1} \left(1 + \frac{1}{(2p-1)^2} \left\{1 - \frac{1}{p^{\beta-1}} - \frac{1}{p^\beta}\right\}\right) = \tau(k_1), \text{ say.} \tag{18}$$

Thus, by (16), (17), and (18), we infer that

$$\Sigma_{k,1} = \frac{\tau(k_1)\, \tau(k_3)}{A^2}. \tag{19}$$

Finally, by (13), (14), (15), and (19), we obtain

$$\sum_{n \leqslant x} \varrho(n)\, \varrho(n+k) = \frac{16\, \tau(k)\, x}{\log v} + O\left(\frac{\sigma_{-\frac{1}{2}}(k)\, x}{\log^2 x}\right), \tag{20}$$

where $\tau(k) = \gamma_1(k)\, \tau(k_1)\, \tau(k_3)$.

8. The case $f(n) = \varrho(n)$ completed: estimation of $\mathrm{P}(x, h)$

In order to be able to estimate $\mathrm{P}(x, h)$ it remains to consider, for $0 < h < x^{\frac{1}{2}}$,

$$\sum_{k<h} (h-k) \sum_{n \leqslant x-k} \varrho(n)\, \varrho(n+k),$$

which by (20) is equal to

$$\frac{16\, x}{\log v} \sum_{k<h} (h-k)\, \tau(k) + O\left(\frac{hx}{\log^2 x} \sum_{k<h} \sigma_{-\frac{1}{2}}(k)\right) = \frac{16\, x}{\log v} \sum_{k<h} (h-k)\, \tau(k) + O\left(\frac{h^2 x}{\log^2 x}\right). \tag{21}$$

To evaluate the latter sum consider for $\sigma > 1$ the generating function

$$f(s) = \sum_{k=1}^{\infty} \frac{\tau(k)}{k^s} = \prod_p \left(1 + \sum_{\alpha=1}^{\infty} \frac{\tau(p^\alpha)}{p^{\alpha s}}\right),$$

the application of Euler's theorem being valid since $\tau(k)$ is multiplicative and the series is absolutely convergent. In the infinite product the factor when $p=2$ is

$$1+\sum_{\alpha=1}^{\infty}\frac{\gamma(2^{\alpha})}{2^{\alpha s}}=1+\sum_{\alpha=1}^{\infty}\frac{(2-3\cdot 2^{-\alpha})}{2^{\alpha s}}=\left(1-\frac{1}{2^s}+\frac{1}{4^s}\right)\left(1-\frac{1}{2^{s+1}}\right)^{-1}\left(1-\frac{1}{2^s}\right)^{-1}.$$

When $p\equiv 3$, mod 4, the factor is

$$\sum_{\alpha=0}^{\infty}\frac{\sigma_{-1}(p^{\alpha})}{p^{\alpha s}}=\sum_{\alpha=0}^{\infty}\frac{p^{-\alpha s}-p^{-\alpha(s+1)-1}}{1-p^{-1}}=\left(1-\frac{1}{p^{s+1}}\right)^{-1}\left(1-\frac{1}{p^s}\right)^{-1},$$

and when $p\equiv 1$, mod 4, it is

$$1+\sum_{\alpha=1}^{\infty}\frac{\tau(p^{\alpha})}{p^{\alpha s}}=\sum_{\alpha=0}^{\infty}\frac{1}{p^{\alpha s}}+\frac{1}{(2p-1)^2}\sum_{\alpha=1}^{\infty}\left(\frac{1}{p^{\alpha s}}-\frac{p+1}{p^{\alpha(s+1)}}\right)$$

$$=\left(1-\frac{1}{p^s}\right)^{-1}\left(1+\frac{1}{(2p-1)^2p^s}\left\{\frac{1}{p^s}-\frac{1}{p}\right\}\left\{1-\frac{1}{p^{s+1}}\right\}^{-1}\right)$$

after some simplification. Therefore $f(s)=K_5(s)\,\zeta(s)$, where

$$K_5(s)=\left(1-\frac{1}{2^s}+\frac{1}{4^s}\right)\left(1-\frac{1}{2^{s+1}}\right)^{-1}\prod_{p\equiv 3,\ \mathrm{mod}\ 4}\left(1-\frac{1}{p^{s+1}}\right)^{-1}$$

$$\times\prod_{p\equiv 1,\ \mathrm{mod}\ 4}\left(1+\frac{1}{(2p-1)^2p^s}\left\{\frac{1}{p^s}-\frac{1}{p}\right\}\left\{1-\frac{1}{p^{s+1}}\right\}^{-1}\right)$$

is regular and bounded for $\sigma>\eta$. Hence, using the formula

$$\sum_{k<h}(h-k)\,\tau(k)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}f(s)\,\frac{h^{s+1}}{s(s+1)}\,ds\qquad(c>1),$$

we deduce that

$$\sum_{k<h}(h-k)\,\tau(k)=\tfrac{1}{2}K_5(1)\,h^2+O(h^{1+\varepsilon})=\tfrac{1}{2}A^2h^2+O(h^{1+\varepsilon}),\tag{22}$$

since the integral is absolutely convergent for $c>0$ $(c\neq 1)$.

By (21) and (22) we have now

$$\sum_{k<h}(h-k)\sum_{n\leqslant x-k}\varrho(n)\,\varrho(n+k)=\frac{8A^2h^2x}{\log v}+O\left(\frac{h^{1+\varepsilon}x}{\log x}\right)+O\left(\frac{h^2x}{\log^2 x}\right).$$

Substituting this together with (9) and (12) in (3) with $\varrho(n)=f(n)$ and $\Delta=4A\log^{-\frac{1}{2}}v$, we obtain

$$\mathrm{P}(x,h) \leqslant \frac{16A^2h^2x}{\log v} - \frac{32A^2h^2x}{\log v} + \frac{16A^2h^2x}{\log v} + O\left(\frac{hx}{\log^{\frac{1}{2}} x}\right) + O\left(\frac{h^{1+\varepsilon}x}{\log x}\right)$$

$$+ O\left(\frac{h^2x}{\log^2 x}\right) + O\left(\frac{h^2x}{\log^2 x}\right) + O(h^3) = O\left(\frac{hx}{\log^{\frac{1}{2}} x}\right) + O\left(\frac{h^2x}{\log^2 x}\right). \tag{23}$$

which is the estimate required.

9. The distribution of the intervals

The theorem on the intervals will follow from (8) and (23) according to the procedure used in [4]. Adopting a notation similar to that in the latter paper, we let $N(x)$ be the number of intervals $s_{n+1} - s_n$ of length l for which $s_{n+1} \leqslant x$ and then let

$$S_l^{(t)}(x) = N_l(x) + 2^t N_{l+1}(x) + 3^t N_{l+2}(x) + \dots .$$

We let also γ be a real constant such that $\gamma < 5/3$ and suppose for the time being, for ease in exposition, that $\gamma > 1$ also. Then firstly, since $s_{n+1} - s_n = O(x^{\frac{1}{4}})$ for $s_{n+1} \leqslant x$ by [1], we have

$$\sum_l N_l(x)\, l^\gamma = \sum_{l<u_1} N_l(x)\, l^\gamma + \sum_{u_1 \leqslant l < u_2} N_l(x)\, l^\gamma = \Sigma_1 + \Sigma_2, \text{ say,} \tag{24}$$

where $u_1 = [\log^{\frac{1}{2}} x]$ and $u_2 = [Bx^{\frac{1}{4}}]$ for some sufficiently large constant B. Next

$$\Sigma_1 \leqslant u_1^\gamma \sum_l N_l(x) = O\left(\frac{u_1^\gamma x}{\log^{\frac{1}{2}} x}\right) = O(x \log^{\frac{1}{2}(\gamma-1)} x). \tag{25}$$

Furthermore, by partial summation,

$$\Sigma_2 = \sum_{u_1 \leqslant l < u_2} (S_l^{(0)}(x) - S_{l+1}^{(0)}(x))\, l^\gamma = O\left(u_1^\gamma S_{u_1}^{(0)}(x) + \sum_{u_1 \leqslant l < u_2} S_l^{(0)}(x)\, l^{\gamma-1}\right)$$

$$= O\left(u_1^\gamma S_{u_1}^{(0)}(x) + \sum_{u_1 \leqslant l < u_2} (S_l^{(1)}(x) - S_{l+1}^{(1)}(x))\, l^{\gamma-1}\right)$$

$$= O(u_1^\gamma S_{u_1}^{(0)}(x)) + O(u_1^{\gamma-1} S_{u_1}^{(1)}(x)) + O\left(\sum_{u_1 \leqslant l < u_2} S_l^{(1)} l^{\gamma-2}\right)$$

$$= O(u_1^\gamma x \log^{-\frac{1}{2}} x) + O(u_1^{\gamma-1} x) + O\left(\sum_{u_1 \leqslant l < u_2} S_l^{(1)} l^{\gamma-2}\right) = O(x \log^{\frac{1}{2}(\gamma-1)} x) + O(\Sigma_3), \text{ say,} \tag{26}$$

which completes the first stage of the estimation.

The sum Σ_3 is written as

$$\Sigma_3 = \sum_{u_1 \leqslant l < u_3} + \sum_{u_3 \leqslant l < u_2} = \Sigma_4 + \Sigma_5, \text{ say,} \tag{27}$$

and then Σ_4 and Σ_5 are assessed through the estimates for $\mathrm{P}(x,h)$ and $R(x,h)$, respectively. Setting $h = l - 1$ in (3) we have

$$\frac{16\,A^2(l-1)^2}{\log v}\,S_l^{(1)}(x) \leqslant \mathrm{P}(x, l-1),$$

since regardless of the sign of $t(n)$ we have $\varrho(n)=0$ when $r(n)=0$. Therefore, by (23),

$$S_l^{(1)}(x) = O\left(\frac{x \log^{\frac{1}{2}} x}{l}\right) + O\left(\frac{x}{\log x}\right). \tag{28}$$

Similarly we infer from (3) that

$$\pi^2(l-1)^2\,S_l^{(1)}(x) \leqslant R(x, l-1)$$

and hence from (8) that

$$S_l^{(1)}(x) = O\left(\frac{x \log x}{l}\right) + O\left(\frac{x \log^2 2l}{l}\right) + O(x^{\frac{5}{6}+\varepsilon}). \tag{29}$$

From (28) we have

$$\Sigma_4 = \left(Ox \log^{\frac{1}{2}} x \sum_{l \geqslant u_1} l^{\gamma-3}\right) + O\left(\frac{x}{\log x} \sum_{l < u_3} l^{\gamma-2}\right) = O(x \log^{\frac{1}{2}(\gamma-1)} x) + O(x u_3^{\gamma-1} \log^{-1} x). \tag{30}$$

Also, from (29),

$$\Sigma_5 = O\left(x \log x \sum_{l \geqslant u_3} l^{\gamma-3}\right) + O\left(x \sum_{l=1}^{\infty} l^{\gamma-3} \log 2l\right) + O\left(x^{\frac{5}{6}+\varepsilon} \sum_{l < u_2} l^{\gamma-2}\right)$$
$$= O(x u_3^{\gamma-2} \log x) + O(x) + O(x^{\frac{5}{6}+\frac{1}{4}\ \gamma-1)+\varepsilon}) = O(x u_3^{\gamma-2} \log x) + O(x),$$

since $\gamma < 5/3$. Then, by this, (24), (25), (26), (27), and (30), it follows that

$$\sum_l N_l(x)\, l^{\gamma} = O(x \log^{\frac{1}{2}(\gamma-1)} x) + O(x u_3^{\gamma-1} \log^{-1} x) + O(x u_3^{\gamma-2} \log x),$$

Hence, by choosing u_3 so that $x u_3^{\gamma-1} \log^{-1} x = x u_3^{\gamma-2} \log x$ with the consequence that both $u_3 = \log^2 x$ and the condition $u_1 < u_2 < u_3$ is satisfied, we have

$$\sum_l N_l(x)\, l^{\gamma} = O(x \log^{\frac{1}{2}(\gamma-1)} x) + O(x \log^{2\gamma-3} x) = O(x \log^{\frac{1}{2}(\gamma-1)} x),$$

since $\gamma < 5/3$. It is appropriate at this point to remark that the upper limit $5/3$ for γ is of crucial importance for two independent reasons in the above argument.

We therefore infer at once the following theorem for the case $\gamma > 1$, the extension to the case $0 \leqslant \gamma \leqslant 1$ being made through the obvious special cases $\gamma = 0$ and $\gamma = 1$ and an application of Hölder's inequality.

THEOREM. *Let $s_1, s_2, \ldots, s_n, \ldots$ be in ascending order the numbers that are equal to a sum of two squares. Then, for $0 \leqslant \gamma < 5/3$, we have as $x \to \infty$*

$$\sum_{s_{n+1} \leqslant x} (s_{n+1} - s_n)^{\gamma} = O(x \log^{\frac{1}{2}(\gamma-1)} x).$$

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