EICHLER INTEGRALS WITH SINGULARITIES

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Introduction

Let Γ be a non-elementary Kleinian group with region of discontinuity Ω and let $q \geqslant 2$ be an integer. We shall show that there exist Eichler integrals of degree q (this term, as all others used here, will be defined below) with preassigned singularities at finitely many non-equivalent points of Ω and with preassigned parabolic singularities at finitely many non-equivalent cusps, and that these integrals have certain pleasing properties. Our results are a modest improvement of those of Ahlfors [3], who constructed Eichler integrals with preassigned poles at preassigned ordinary points in Ω. The method, however, may be of interest since it clarifies the connection between Eichler integrals with poles and generalized Beltrami coefficients (as defined in Bers [5]). That such a connection must exist becomes obvious, at least for a finitely generated group Γ, by comparing recent results of Ahlfors [3] with those of Kra [6].

1. Preliminaries

We are given a Kleinian group Γ, that is, a group of Möbius transformations $\gamma(z) = (az+b)/(cz+d)$ which acts discontinuously on some open set of the Riemann sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. The largest open set Ω for which this is true is called the *region of discontinuity* of Γ; the complement $\Lambda = \hat{\mathbf{C}} - \Omega$ is nowhere dense and is called the *limit set* of Γ. We assume that Γ is *non-elementary*, that is, that Λ is infinite. The Poincaré metric $\lambda(z)|dz|$ in Ω is defined by the condition: for every component Δ of Ω, and for every universal holomorphic covering h: $U \to \Delta$ of Δ by the upper half-plane U, one has $\lambda(h(\zeta))|h'(\zeta)| = 2/|\zeta - \bar{\zeta}|$ for $h(\zeta) \in \Delta$. It is known that $\lambda(\gamma(z))|\gamma'(z)| = \lambda(z)$ for $\gamma \in \Gamma$.

The stabilizer in Γ of a point $z_0 \in \Omega$ is either the identity (then z_0 is called an *ordinary point*), or a finite cyclic group (then z_0 is called an *elliptic vertex*).

A *cusp* for Γ is a set $C \subset \Omega$ with the following properties. The boundary of C consists of 3 circular (or straight) arcs α_1, α_2, α_3 and there is a parabolic element $\gamma_0 \in \Gamma$ which maps α_1 onto α_2 and generates the stabilizer of the intersection point z_0 of α_1 and α_2 (this point is called the *vertex* of the cusp). Also, C consists of all points in one of the components of $\hat{\mathbf{C}} - \{\alpha_1 \cup \alpha_2 \cup \alpha_3\}$ and of all inner points of the arcs α_1 and α_2, and no two interior points in C are equivalent under Γ. It follows that the image of C under the canonical projection $\Omega \to \Omega/\Gamma$ is conformal to a punctured disc. Two cusps, C_1 and C_2, are called *equivalent* if there is a cusp C_3 such that the image of C_3 under the canonical projection is contained in the intersection of the images of C_1 and C_2.

Note that a fixed point of a parabolic element of Γ need not be the vertex of a cusp, and can be the vertex of at most two non-equivalent cusps.

Let $q \geqslant 2$ be an integer chosen once and for all. If $f(z)$ is a function defined on a set $\Sigma \subset \mathbf{C}$ and α a Möbius transformation, one defines: $(f\alpha)(z) = f(\alpha(z))\,\alpha'(z)^{1-q}$, for $z \in \alpha^{-1}(\Sigma)$. Let Π denote the vector space of polynomials of degree at most $2q-2$; if $p \in \Pi$ then $p\alpha \in \Pi$. A *cocycle* (on Γ with coefficients in Π) is a mapping χ: $\Gamma \to \Pi$ such that (writing χ_α for the image of α under χ) we have $\chi_{\alpha\circ\beta} = \chi_\alpha \beta + \chi_\beta$. A cocycle χ is a *coboundary* if there is a fixed $p \in \Pi$ with $\chi_\gamma = p\gamma - p$ for all $\gamma \in \Gamma$. A cocyle χ is called *parabolic* (or Ω-parabolic) if for every parabolic subgroup Γ_D of Γ belonging to a cusp, $\chi \,|\, \Gamma_D$ is a coboundary. We call χ *strongly parabolic* if the same is true for all parabolic subgroups of Γ.

A function f defined on a Γ invariant set Σ will be called an *automorphic integral* if for every $\gamma \in G$ there is a $\chi_\gamma \in \Pi$ such that $f\gamma - f = \chi_\gamma \,|\, \Sigma$. If so, χ is a cocycle, we call it the *period* of f. We shall be concerned with two types of automorphic integrals: potentials of Beltrami coefficients and Eichler integrals.

2. Automorphic forms, potentials, Eichler integrals

A function $\varphi(z)$ defined and holomorphic in Ω, except perhaps for isolated singularities, is called an *automorphic form* (of weight $-2q$) if $\varphi(\gamma(z))\gamma'(z)^q = \varphi(z)$ for all $\gamma \in \Gamma$. If $|\lambda^{-q}\varphi|$ is bounded, φ is called a *bounded form*; in this case φ has no singularities. If $|\lambda^{2-q}\varphi|$ is integrable over a fundamental domain of Γ, then φ is called *integrable*; in this case φ has no singularities except, perhaps, simple poles.

An automorphic form φ is said to satisfy the *cusp condition* in a cusp C if $\varphi \,|\, C$ approaches 0 as z approaches the vertex of C. A bounded form always satisfies this condition. So does an integrable one, provided that it has only finitely many poles in C.

We shall often use a distinguished automorphic form, defined as follows. Let $A_1, \ldots, A_{2q-1}$ be distinct points in $\hat{\mathbf{C}}$, $z \in \mathbf{C}$ a point such that $z \neq A_j$. Set

$$f_{A_1,\dots,A_{2q-1}}(z,\zeta)=f(z,\zeta)=\frac{1}{2\pi i}\sum_{\gamma\in\Gamma}\prod_{j=1}^{2l-1}\frac{z-A_j}{\gamma(\zeta)-A_j}\,\frac{\gamma'(\zeta)^q}{\gamma(\zeta)-z}$$

(where we agree, *once and for all*, to omit any term of the form $z-\infty$ or $\gamma(\zeta)-\infty$). This Poincaré series converges because (in view of the inequalities on λ stated, for instance in Ahlfors [1])

$$\iint_\Omega \frac{\lambda(\zeta)^{2-q}\,|d\zeta\wedge d\bar\zeta|}{|(\zeta-A_1)\dots(\zeta-A_{2q-1})\,(\zeta-z)|}<\infty.$$

For a fixed z, $f(z,\zeta)$ is an integrable automorphic form. Its simple poles are the points $\gamma(A_j)$ and $\gamma(z)$, $\gamma\in\Gamma$, located in Ω.

A measurable function $\mu(z)$, $z\in\Omega$, is called a (*generalized*) *Beltrami coefficient* (of order q) if $\mu(\gamma(z))\gamma'(z)^{1-q}\overline{\gamma'(z)}=\mu(z)$ for all $\gamma\in\Gamma$ and $\lambda^{q-2}\mu$ is bounded. For instance, if ψ is a bounded automorphic form, $\lambda^{2-2q}\bar\psi$ is a Beltrami coefficient.

If μ is a Beltrami coefficient and φ an integrable automorphic form, then $\varphi(z)\mu(z)\,dx\,dy$ is invariant under Γ, so that one may define

$$\langle\varphi,\mu\rangle=\iint_{\Omega/\Gamma}\varphi(z)\,\mu(z)\,dz\wedge d\bar z.$$

For a fixed μ, this is a continuous linear functional on integrable automorphic forms (with respect to the L_1 norm of $|\lambda^{2-q}\varphi|$ over a fundamental domain), and every such functional can be so represented. Two Beltrami coefficients, μ and ν, are called *equivalent* if $\langle\varphi,\mu\rangle=\langle\varphi,\nu\rangle$ for every integrable holomorphic automorphic form φ (not necessarily for integrable forms with poles). It is known ([5], [7]) that every Beltrami coefficient is equivalent to a unique Beltrami coefficient of the form $\lambda^{2-2q}\bar\psi$ where ψ is a bounded automorphic form.

We recall that $i\langle\varphi,\lambda^{2-2q}\bar\psi\rangle/2=(\varphi,\psi)$ is the familiar Petersson *scalar product*.

Let μ be a (generalized) Beltrami coefficient. A *potential* F of μ is a continuous function $F(z)$, $z\in\mathbf{C}$, such that $F(z)=O(|z|^{2q-2})$, $z\to\infty$, and $\bar\partial F=\partial F/\partial\bar z$, in the sense of distribution theory, is a measurable function such that $\bar\partial F|\Omega=\mu$ and $\bar\partial F|\Lambda=0$ a.e. If F_1 and F_2 are potentials of μ, then $F_1-F_2\in\Pi$. If F is a potential of μ, so is $F+p$, $p\in\Pi$. If $\gamma\in\Gamma$, one computes easily that $\bar\partial(F\gamma-F)=0$, so that $F\gamma-F\in\Pi$. Thus F is an automorphic integral. The cohomology class of the period of F depends only on μ.

The existence of potentials has been established in [5]. We shall recall this construction and make it more precise: the function

$$F(z)=\langle f(z,\cdot),\mu\rangle$$

is a potential of μ which vanishes at all finite points A_j and is $o(|z|^{2q-2})$, $z\to\infty$, if one

$A_j = \infty$. By abuse of language we shall say, in this latter case, that "the potential F vanishes at ∞". A simple calculation shows that one also has

$$F(z) = \frac{1}{2\pi i} \iint_{\Omega} \prod_{j=1}^{2q-1} \frac{z - A_j}{\zeta - A_j} \frac{\mu(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}.$$

If all A_j lie in Λ, then, for $z \in \Lambda$, $f(z, \zeta)$ is holomorphic for $\zeta \in \Omega$. In this case the period of F depends only on the equivalence class of μ. The same is therefore true for the cohomology class of the period of F, in all cases.

An *Eichler integral* (of degree q) for Γ is an automorphic integral $E(z)$, which is holomorphic in Ω, except for isolated singularities. If E is an Eichler integral, so is $E+p$, $p \in \Pi$. We have that $\partial^{2q-1}(E\gamma - E) = 0$ for $\gamma \in \Gamma$, where $\partial = d/dz$. Since $\partial^{2q-1}E\gamma = (\partial^q E\gamma)(\gamma')^q$ (Bols' identity, cf. [2]) one concludes that $\partial^{2q-1}E$ is an automorphic form. E is called regular in a cusp C if $\partial^{2q-1}E$ satisfies the cusp condition in C. E is called *parabolic*, or *strongly parabolic*, if its period is.

3. Principal parts

In order to describe the possible singularities of Eichler integrals, we introduce the following terminology. Let z_0 be either a point in Ω (case 1) or the vertex of a cusp C (case 2), and let γ_0 be a generator of the stabilizer of z_0 in Γ; $\gamma_0 = \mathrm{id}$ if $z_0 \in \Omega$ is not an elliptic vertex. Let $D \subset \Omega$ be an open set such that $\gamma_0(D) = D$, and either $\{z_0\} \cup D$ is a disc or half plane (in case 1) or D is a disc or half plane and $D \cap C$ is a cusp (in case 2). Let h be a holomorphic function defined in D such that there is a polynomial $p \in \Pi$ with $h\gamma_0 - h = p \mid D$. Then we say that the pair (D, h) represents the *principal part* of an Eichler integral at z_0 (in case 1) or at C (in case 2). Two representatives, (D_1, h_1) and (D_2, h_2) are called equivalent (at z_0 or at C) if, setting $D = D_1 \cap D_2$, one has: in case 1, that $h_1 \mid D - h_2 \mid D$ has a removable singularity at z_0, or, in case 2, that $\partial^{2q-1}h_1 \mid D \cap C - \partial^{2q-1}h_2 \mid D \cap C$ approaches 0 at z_0. A principal part H (at z_0 or at C) is the equivalence class of representatives. A principal part H at C is called *parabolic* if it contains a pair (D, h) with $h\gamma_0 = h$.

Let E be an Eichler integral. If $z_0 \in \Omega$, let D_0 be a sufficiently small disc containing z_0, with $\gamma(D_0) = D_0$ for $\gamma \in \Gamma$, $\gamma(z_0) = z_0$, and set $D = D_0 - \{z_0\}$. Then $(D, E \mid D)$ represents a principal part, called the principal part of E, at z_0. If C is a cusp with vertex z_0, let D be a disc containing z_0 on its boundary, such that $\gamma(D) = D$ for $\gamma \in \Gamma$, $\gamma(z_0) = z_0$, and such that $D \cap C$ is a cusp. Then $(D, E \mid D)$ represents a principal part, called the principal part of E, at C. If E is parabolic, the principal part H represented by $(D, E \mid D)$ is also parabolic, since it can be represented by $(D, E \mid D - p \mid D)$ where $p \in \Pi$ is chosen so that $E\gamma - E = p\gamma - p$ for γ in the stabilizer of z_0 in Γ.

Let H be a principal part at $z_0 \in \Omega$. We associate with it a *linear functional* l on the space of automorphic forms φ which are regular at z_0. The definition reads:

$$l(\varphi)=\frac{1}{m}\int_\sigma \varphi(z)\,h(z)\,dz$$

where m is the order of the stabilizer of z_0 in Γ, (D, h) is a representative of H chosen so that $\varphi(z)$ is holomorphic for $z \in \{z_0\} \cup D$, and σ is a simple closed curve in D with winding number 1 with respect to z_0.

The restriction of l to the Banach space of holomorphic integrable automorphic forms φ is *continuous*. Indeed, for such φ one can compute $l(\varphi)$ using a fixed representative (D, h) of H and a fixed smooth curve σ. One may also assume that there is an open set G with compact closure, and m fundamental regions of Γ, $w_1, w_2, \ldots, w_m$, such that

$$\sigma \subset G \subset (w_1 \cup w_2 \cup \ldots \cup w_m).$$

Let M denote the maximum of $|h(z)|$ for $z \in \sigma$, k the length of σ, r the distance from σ to the boundary of G, c a (positive) lower bound for $\lambda(z)$ in G, and $\|\varphi\|$ the norm of φ in the Banach space considered. We have

$$\begin{aligned}|l(\varphi)| &\leqslant \frac{M}{m}\int_\sigma |\varphi(z)|\,|dz| \leqslant \frac{M}{m}\int_\sigma \left\{\frac{1}{\pi r^2}\iint_{|z-\zeta|<r}|\varphi(z+\zeta)|\,d\xi\,d\eta\right\}|dz| \\ &\leqslant \frac{Mk}{m\pi r^2}\iint_G |\varphi(z)|\,dx\,dy \leqslant \frac{Mkc^{2-q}}{m\pi r^2}\iint_G |\varphi(z)|\,\lambda(z)^{2-q}\,dx\,dy \\ &\leqslant \frac{Mkc^{2-q}}{m\pi r^2}\iint_{w_1\cup\ldots\cup w_m}|\varphi(z)|\,\lambda(z)^{2-q}\,dx\,dy = \frac{Mkc^{2-q}\|\varphi\|}{\pi r^2},\end{aligned}$$

as asserted.

Now let H be a parabolic principal part at a cusp C with vertex z_0. We associate with it a linear functional l on the space of automorphic forms φ which satisfy the cusp condition in C. The definition reads:

$$l(\varphi)=\int_\sigma \varphi(z)\,h(z)\,dz$$

where (D, h) is a representative of H chosen so that $\varphi(z)$ is holomorphic for $z \in D$, and $h\gamma_0 = h$ where γ_0 is a generator of the stabilizer of z_0, and σ is a curve in D leading from a point $z_1 \in D$ to the point $\gamma_0(z_1)$. (The reader will easily verify that this definition is legitimate.)

The restriction of l to the Banach space of holomorphic integrable form is continuous. This can be proved by an argument similar to the one given above.

Now let there be given a system $\mathcal{H}=\{H_1, H_2, \ldots, H_r\}$ of finitely many principal parts at non-equivalent points $z_1, \ldots, z_n$ of Ω and at non-equivalent cusps $C_{n+1}, C_{n+2}, \ldots, C_r$. If E is an Eichler integral which is regular at all points of Ω non-equivalent to $1, \ldots, z_n$, and at all cusps non-equivalent to $C_{n+1}, \ldots, C_r$, and if H_j is the principal part of E at z_j, for $j=1, \ldots, n$, and at C_j for $j=n+1, \ldots, r$, then we call $\mathcal{H}$ a *complete system* of principal parts of E.

If all principal parts at cusps are parabolic, we associate with $\mathcal{H}$ a *linear functional* l on the space of automorphic forms φ which are regular at $z_1, \ldots, z_n$ and satisfy the cusp condition in $C_{n+1}, \ldots, C_r$. The definition reads: $l(\varphi)=l_1(\varphi)+\ldots+l_r(\varphi)$ where l_j is the linear functional associated with H_j. The restriction of l to the Banach space of holomorphic integrable automorphic forms is, of course, continuous. For such a φ one has, therefore, $l(\varphi)=\langle\varphi, \lambda^{2-2q}\bar{\psi}\rangle$ where ψ is a uniquely determined bounded automorphic form. We call it the *form associated with* $\mathcal{H}$.

4. Statement of the theorem

We can now state our result.

THEOREM. *Let $\mathcal{H}$ be a given finite system of principal parts, at non-equivalent points and cusps. Let all parts defined at cusps be parabolic. Let l be the linear functional associated to $\mathcal{H}$, and ψ the associated bounded automorphic form. Also, let $A_1, \ldots, A_{2q-1}$ be 2q–1 distinct points in Λ, $f_{A_1,\ldots,A_{2q-1}}(z,\zeta)=f(z,\zeta)$ the corresponding automorphic form, F the potential of $\lambda^{2-2q}\bar{\psi}$ which vanishes at $A_1, \ldots, A_{2q-1}$, and set, for $z\in\Omega$, z not equivalent to a point occurring in $\mathcal{H}$,*

$$E(z) = -l(f(z, \cdot)).$$

Then E is a strongly parabolic Eichler integral, with $\mathcal{H}$ as a complete system of principal parts. The period of E is that of F, and if Δ is a component of Ω such that E is regular at all points and cusps in Δ, then $E|\Delta=F|\Delta$.

5. A counter example

The following example shows that the parabolicity condition in the theorem is *essential*.

Let Γ be the principal congruence subgroup modulo 2 of the elliptic modular group; it consists of all mappings $\gamma(z)=(az+b)/(cz+d)$ with $a, b, c, d\in\mathbf{Z}$, $ad-bc=1$, b and c even. This is a free group on two generators, so that there are $2(2q-1)$ linearly indeneqdent cocycles, $2q-1$ of which are coboundaries. The limit set of Γ is $R\cup\{\infty\}$ and, as is well known, there are $2q-4$ linearly independent bounded automorphic forms. Therefore there are $2q-4$ strongly parabolic cocycles no linear combination of which is a coboundary (cf. for instance, Bers [5]). Thus there can be at most 3 linearly independent non-parabolic cocycles, modulo parabolic ones.

On the other hand, there are 6 non-equivalent cusps (2 at $z=\infty$, 2 at $z=1$, and 2 at $z=2$). If one could prescribe arbitrarily non-parabolic principal parts of an Eichler integral, the codimension of the parabolic cocycles in the space of all cocyles would be at least 6.

6. Conjugation

It is useful to state explicitly how the objects which we study behave under conjugation.

Let α be a Möbius transformation and set $\hat{\Gamma}=\alpha^{-1}\hat{\Gamma}\alpha$. Then $\hat{\Gamma}$ is a Kleinian group with region of discontinuity $\alpha^{-1}(\Omega)$ and limit set $\alpha^{-1}(\Lambda)$.

If φ is an automorphic form for Γ, we define $\hat{\varphi}(\zeta)=\varphi(\alpha(\zeta))\alpha'(\zeta)^q$. Then $\hat{\varphi}$ is an automorphic form for Γ. If φ is bounded, so is $\hat{\varphi}$, with the same bound. If φ is integrable, so is $\hat{\varphi}$, with the same norm. If φ is regular at z_0, $\hat{\varphi}$ is regular at $\alpha^{-1}(z_0)$. If φ satisfies the cusp condition in C, $\hat{\varphi}$ satisfies the cusp condition in $\alpha^{-1}(C)$. Also, if ψ is an automorphic form for Γ, and $\hat{\psi}$ the corresponding form for $\hat{\Gamma}$, then $(\varphi,\psi)=(\hat{\varphi},\hat{\psi})$ whenever one of the scalar products exists. If $\varphi(\zeta)=f_{A_1,\dots,A_{2q-1}}(z,\zeta)$, then $\hat{\varphi}(\zeta)=\hat{f}_{\hat{A}_1,\dots,\hat{A}_{2q-1}}(\hat{z},\zeta)$ where $\hat{f}$ is the Poincaré series for the group $\hat{\Gamma}$, $\hat{A}_j=\alpha^{-1}(A_j)$ and $\hat{z}=\alpha^{-1}(z)$.

If μ is a (generalized) Beltrami coefficient for Γ, set $\hat{\mu}(\zeta)=\mu(\alpha(\zeta))\alpha'(\zeta)^{1-q}\overline{\alpha'(\zeta)}$. Then $\hat{\mu}$ is a Beltrami coefficient for Γ, and $\langle\varphi,\mu\rangle=\langle\hat{\varphi},\hat{\mu}\rangle$ for every integrable automorphic form φ. If F is a potential of μ, $\hat{F}=F\alpha$ is a potential of $\hat{\mu}$. If F vanishes at A_j, $\hat{F}$ vanishes at $\hat{A}_j$.

If E is an Eichler integral for Γ, $\hat{E}=E\alpha$ is one for $\hat{\Gamma}$. If H is a principal part for Γ, at a point z_0 or at a cusp C, defined by (D, h), then the pair $(\alpha^{-1}(D), h\alpha)$, defines a principal part $\hat{H}$ for $\hat{\Gamma}$, at $\alpha^{-1}(z_0)$ or at $\alpha^{-1}(C)$. If H is parabolic, so is $\hat{H}$. If H is a principal part of E, then $\hat{H}$ is one of $\hat{E}$.

For every $\gamma\in\Gamma$, set $\hat{\gamma}=\alpha^{-1}\circ\gamma\circ\alpha$. If χ is a cocycle on Γ, set $\hat{\chi}_{\hat{\gamma}}=\chi_\gamma\alpha$. Then $\hat{\chi}$ is a cocycle on $\hat{\Gamma}$. If χ is a coboundary, or parabolic, or strongly parabolic, so is $\hat{\chi}$. If χ is the period of F, or of E, then $\hat{\chi}$ is the period of $\hat{F}$, or of $\hat{E}$.

The proofs of all these assertions are trivial.

We use the remarks just made to show that the period of a potential F of a Beltrami differential is *strongly parabolic,* as observed by Gardiner (unpublished) and Kra [6]. Let $\gamma_0\in\Gamma$ be parabolic. We lose no generality in assuming that $\gamma_0(z)=z+1$; this can be achieved by conjugation. We lose no generality in assuming that F "vanishes at ∞", that is, satisfies $F(z)=o(|z|^{2q-2})$, $z\to\infty$; this can be achieved by subtracting from F an element of Π.

Now the polynomial $F\gamma_0 - F$ equals $F(z+1)-F(z)$, hence it has degree at most $2q-3$, hence it is of the form $p(z+1)-p(z)$ for some $p \in \Pi$. Thus $F\gamma_0 - F = p\gamma_0 - p$, q.e.d.

7. Proof of the theorem

It is clearly sufficient to prove the theorem for $r=1$, that is, for $\mathcal{H}$ containing a single principal part H, defined either at a point $z_0 \in \Omega$ (case 1) or at a cusp C with vertex z_0 (case 2). By a conjugation we can achieve that, in case 1, $z_0 = 0$ and the stabilizer of 0 in Γ is generated by $\gamma_0(z) = e^{2\pi i/m} z$, and that, in case 2, $z_0 = \infty$, the cusp C is the half strip $0 < x < 1$, $y > 0$, and the stabilizer of ∞ in Γ is generated by $\gamma_0(z) = z+1$. In case 1, we may assume that H is defined by (D, h) where D is the set $0 < |z| < \varepsilon_0$ and

$$h(z) = \sum_{h=1}^{\infty} a_n z^{-n}.$$

The condition $h\gamma_0 - h \in \Pi$ is satisfied trivially if $m=1$. For $m>1$ it becomes $h\gamma_0 = h$, or

$$a_n = 0 \quad \text{for } n \neq -q+1 \ (\text{mod } m).$$

We assume also that ε_0 is so small that the disc $|z| \leqslant \varepsilon_0$ does not meet any of its images under elements of Γ distinct from powers of γ_0. In case 2, we may assume that H is defined by (D, h) where D is a half-plane, $y > 1/\varepsilon_0 > 0$, and $h\gamma_0 = h$. Thus $h(z)$ must be periodic with period 1, and we may assume that

$$h(z) = \sum_{n=1}^{\infty} a_n e^{-2inz}.$$

For every ε, $0 < \varepsilon < \varepsilon_0$, let G_ε denote the disc $|z| < \varepsilon$ in case 1, the half-plane $y > 1/\varepsilon$ in case 2, and let ∂G_ε be the boundary of G_ε, with the usual orientation. For $z \in G_\varepsilon \cup \partial G_\varepsilon$, set

$$\theta_\varepsilon(z) = \sum_{n=1}^{\infty} a_n \varepsilon^{-2n} \bar{z}^n \quad \text{(in case 1)},$$

$$\theta_\varepsilon(z) = \sum_{n=1}^{\infty} a_n e^{-2\pi n(i\bar{z} - 2/\varepsilon)} \quad \text{(in case 2)}.$$

One verifies that

$$\theta_\varepsilon | \partial G_\varepsilon = h | \partial G_\varepsilon$$

and that $\theta_\varepsilon \gamma_0 = \theta_\varepsilon$, which implies that

$$\left.\frac{\partial \theta_\varepsilon(t)}{\partial \bar{t}}\right|_{t=\gamma_0(z)} \gamma_0'(z)^{1-q}\, \overline{\gamma_0'(z)} = \frac{\partial \theta_\varepsilon(z)}{\partial \bar{z}}.$$

Also, $\lambda(z)^{q-2}\partial\theta_\varepsilon/\partial\bar{z}$ is bounded. This is trivial in case 1, and in case 2 it follows by noting that $\partial\theta_\varepsilon/\partial\bar{z}=O(e^{-2\pi y})$, $\lambda(z)=O(y^{-1})$ for $y\to+\infty$. (The second inequality is obtained by comparing the Poincaré metric $\lambda(z)|dz|$ in the component of Ω containing the half plane $y>0$, with the Poincaré metric $|dz|/y$ of that half plane.)

The observations just made imply that there is a Beltrami coefficient $\mu_\varepsilon(z)$, $z\in\Omega$, such that

$$\mu_\varepsilon(z)=\frac{\partial\theta(z)}{\partial\bar{z}}\quad\text{for } z\in G_\varepsilon,$$

$$\mu_\varepsilon(z)=0\quad\text{for } \gamma(z)\notin G_\varepsilon,\ \gamma\in\Gamma.$$

Let K_ε denote the union of all sets $\gamma(G_\varepsilon\cup\partial G_\varepsilon)$, $\gamma\in\Gamma$. The second condition on μ_ε can be rewritten as

$$\mu_\varepsilon|\Omega-K_\varepsilon=0.$$

Let F_ε be the potential of μ_ε, which vanishes at $A_1, ..., A_{2q-1}$, $2q-1$ given distinct points in Λ. Let l be the linear functional associated with $\mathcal{H}$, ψ the associated bounded automorphic form, F the potential of $\lambda^{2-2q}\bar{\psi}$ which vanishes at $A_1, ..., A_{2q-1}$, and set

$$E(z)=-l(f(z,\cdot))$$

where $f(z,\zeta)=f_{A_1,...,A_{2q-1}}(z,\zeta)$.

We claim that

$$F_\varepsilon(z)=E(z)\quad\text{for } z\notin K_\varepsilon.$$

Indeed, in case 1 there are m disjoint fundamental regions $w_1, ..., w_m$ such that $G_\varepsilon\subset(w_1\cup...\cup w_m)$ and we have, for $z\notin K_\varepsilon$,

$$\begin{aligned}F_\varepsilon(z)&=\langle f(z,\cdot),\mu_\varepsilon\rangle=\iint_{w_1}f(z,\zeta)\mu_\varepsilon(\zeta)\,d\zeta\wedge d\bar{\zeta}=\frac{1}{m}\iint_{w_1\cup...\cup w_m}f(z,\zeta)\mu_\varepsilon(\zeta)\,d\zeta\wedge d\bar{\zeta}\\&=\frac{1}{m}\iint_{G_\varepsilon}f(z,\zeta)\mu_\varepsilon(\zeta)\,d\zeta\wedge d\bar{\zeta}=\frac{1}{m}\iint_{G_\varepsilon}f(z,\zeta)\frac{\partial\theta_\varepsilon(\zeta)}{\partial\bar{\zeta}}\,d\zeta\wedge d\bar{\zeta}\\&=-\frac{1}{m}\int_{\partial G_\varepsilon}f(z,\zeta)h(\zeta)\,d\zeta=-l(f(z,\cdot))=E(z).\end{aligned}$$

In case 2 we note that $f(z,\zeta+1)=f(z,\zeta)$ so that for every fixed z, and $\eta=\operatorname{Im}\zeta>0$, $f(z,\zeta)$ can be represented by a Fourier series $\Sigma\, b_n e^{2\pi in\zeta}$. The cusp condition implies that $b_n=0$ for $n\leqslant 0$; hence $f(z,\zeta)=O(e^{-2\pi\eta})$, $\eta\to+\infty$. For $z\notin K_\varepsilon$, we have

$$
\begin{aligned}
F_\varepsilon(z) &= \langle f(z,\cdot),\mu_\varepsilon\rangle = \iint_{\substack{0<\xi<1\\ 1/\varepsilon<\eta}} f(z,\zeta)\,\mu_\varepsilon(\zeta)\,d\zeta\wedge d\bar\zeta \\
&= \lim_{R\to+\infty}\iint_{\substack{0<\xi<1\\ 1/\varepsilon<\eta<R}} \frac{\partial[f(z,\zeta)\,\theta_\varepsilon(\zeta)]}{\partial\bar\zeta}\,d\zeta\wedge d\bar\zeta \\
&= -\int_0^1 f(z,\xi+i/\varepsilon)\,\theta_\varepsilon(\xi+i/\varepsilon)\,d\xi + \lim_{R\to+\infty}\int_0^1 f(z,\xi+iR)\,\theta_\varepsilon(\xi+iR)\,d\xi \\
&= -\int_0^1 f(z,\xi+i/\varepsilon)\,h(\xi+i/\varepsilon)\,d\xi = -l(f(z,\cdot)) = E(z).
\end{aligned}
$$

Since we also have

$$
F_\varepsilon(z) = \frac{1}{2\pi i}\iint_{K_\varepsilon}\prod_{j=1}^{2q-1}\frac{z-A_j}{\zeta-A_j}\,\frac{\mu_\varepsilon(\zeta)}{\zeta-z}\,d\zeta\wedge d\bar\zeta,
$$

$F_\varepsilon(z)$ is holomorphic for $z\in\mathbf{C}-K_\varepsilon$, and since $\bigcap K_\varepsilon=\{\gamma(0);\,\gamma\in\Gamma\}$ in case 1 and $\bigcap K_\varepsilon=\emptyset$ in case 2, we conclude that $E(z)$ is holomorphic in $\Omega-\{\gamma(0);\,\gamma\in\Gamma\}$ in case 1, in Ω in case 2.

Also, $E_\gamma - E = F_\varepsilon\gamma - F_\varepsilon$ for $\gamma\in\Gamma$. Thus E is an Eichler integral, indeed a strongly parabolic one.

In case 1, E is regular at all cusps. Indeed, let C_1 be a cusp with vertex z_1. There exists a Möbius transformation α such that the $\alpha^{-1}(C_1)$ is a cusp $\alpha^{-1}(C)$ for the group $\hat\Gamma=\alpha^{-1}\Gamma\alpha$ is the half strip $0<\xi<1$, $\eta>0$, and the stabilizer of $\infty=\alpha^{-1}(z_1)$ in $\hat\Gamma$ is generated by $\hat\gamma_0(\zeta)=\zeta+1$. Now $\hat E=E\alpha$ coincides, for $\eta>0$, with the potential $\hat F_\varepsilon = F_\varepsilon\alpha$ of a Beltrami coefficient, provided ε is small enough. Hence, in this half plane, $\hat E(\zeta)=O(|\zeta|^{2q-2})$, $\zeta\to\infty$. Since $\hat\varphi=\partial^{2q-1}\hat E$ is holomorphic in a half plane $\eta>\eta_0>0$, it may be written there as $\hat\varphi(\zeta)=\sum l_n e^{2\pi i n\zeta}$. Then $\hat E(\zeta)=\sum b_n(2\pi i n)^{1-2q}e^{2\pi i n\zeta}+P(\zeta)$ where $P(\zeta)$ is a polynomial of degree at most $2q-1$. In view of the growth condition on $\hat E$, $\deg P\leqslant 2q-2$ and $b_n=0$ for $n<0$. Hence $\hat\varphi(\zeta)=O(e^{-2\pi\eta})$, $\eta\to+\infty$. Thus the cusp condition is satisfied and $\hat E$ is regular at $\alpha^{-1}(C)$. Hence E is regular at C_1.

In case 2 one sees in the same way that E is regular at every cusp C_1 not equivalent to C.

In case 1, let $\varepsilon>0$ be sufficiently small, and let $n>0$ be an integer. We shall show that

$$
\int_{|z|=\varepsilon} E(z)\,z^n\,dz = 2\pi i a_{n+1'}.
$$

This will imply that $E(z)-h(z)$ has a removable singularity at $z=0$.

Now, since $E(z)=F_\varepsilon(z)$ for $|z|=\varepsilon$, the integral considered equals

$$
\begin{aligned}
\int_{\partial G_\varepsilon} F_\varepsilon(z)\,z^n dz &= \iint_{G_\varepsilon} d[F_\varepsilon(z)\,z^n dz] = \iint_{G_\varepsilon}\frac{\partial F_\varepsilon(z)}{\partial\bar z}\,z^n d\bar z\wedge dz \\
&= \iint_{G_\varepsilon}\mu_\varepsilon(z)\,z^n d\bar z\wedge dz = \iint_{G_\varepsilon} d[\theta_\varepsilon(z)\,z^n dz] = \int_{\partial G_\varepsilon}\theta_\varepsilon(z)\,z^n dz = 2\pi i a_{n+1}.
\end{aligned}
$$

In case 2, $\partial^{2q-1}E$ is periodic with period 1. Hence, for $y>0$,

$$E(z)=\sum_{-\infty}^{+\infty} c_r e^{2\pi i r z}+P(z),$$

where $c_0=0$ and $P(z)$ is a polynomial of degree at most $2q-1$. For a fixed $\varepsilon>0$, we must have $E(x+i/\varepsilon)=F_\varepsilon(x+i/\varepsilon)=O(|x|^{2q-2})$ for $x\to\pm\infty$. Hence $\deg P\leqslant 2q-2$. Let $n>0$ be an integer. We shall show that $c_{-n}=a_n$. This will imply that $\partial^{2q-1}(E-h)$ approaches 0 as $z\to\infty$ in C.

Now the function $\theta_\varepsilon(z)$ is periodic with period 1 and bounded in the half plane $y>1/\varepsilon$. In this half plane $F_\varepsilon(z)-\theta_\varepsilon(z)$ is holomorphic, since $\bar\partial F_\varepsilon-\bar\partial\theta_\varepsilon=0$. Since $F_\varepsilon(z+1)-F_\varepsilon(z)$ is a polynomial of degree at most $2q-2$,

$$F_\varepsilon(z)-\theta_\varepsilon(z)=\sum_{j=1}^{\infty} d_j e^{2\pi i j z}+P_1(z)$$

where P_1 is a polynomial, $\deg P_1\leqslant 2q-1$. But since $F_\varepsilon(z)=O(|z|^{2q-2})$, $z\to\infty$, we have that $\deg P_1\leqslant 2q-2$. For $y=1/\varepsilon$ we have $E=F_\varepsilon$ and $h=\theta_\varepsilon$; thus

$$E(z)=h(z)+\sum_{j=1}^{\infty} d_j e^{2\pi i j z}+P_1(z),\quad (y=1/\varepsilon)$$

or

$$\sum_{-\infty}^{+\infty} c_r e^{2\pi i r z}=\sum_{n=1}^{\infty} a_n e^{-2\pi i n z}+\sum_{j=1}^{\infty} d_j e^{2\pi i j z}+P_1(z)-P(z),\qquad (y=1/\varepsilon)$$

which shows that $c_{-n}=a_n$.

We have shown that H is the principal part of E (at 0 or at C).

Next, for $z\in\Lambda$, $f(z,\zeta)$ is holomorphic in $\zeta\in\Omega$. Hence

$$F(z)=\langle f(z,\cdot),\lambda^{2-2q}\bar\psi\rangle=\langle f(z,\cdot),\mu_\varepsilon\rangle=F_\varepsilon(z).$$

Thus, for every $\gamma\in\Gamma$, $F\gamma-F=F_\varepsilon\gamma-F_\varepsilon$. Since $F_\varepsilon\gamma-F_\varepsilon=E\gamma-E$, E and F have the same period.

Finally, let Δ be a component of Ω such that $\gamma(0)\notin\Delta$ for all $\gamma\in\Gamma$ (in case 1) or $\gamma(C)\notin\Delta$ for all $\gamma\in\Gamma$ (in case 2). Let $D=\bigcup_{\gamma\in\Gamma}\gamma(\Delta)$. Then $\mu_\varepsilon|D=0$, hence, for every integrable automorphic form φ, $\langle\varphi,\mu_\varepsilon\rangle=0$ whenever $\varphi|\Omega-D=0$, hence $\psi|D=0$. Assume now that $z\in\Delta$. Then $f(z,\zeta)$ is holomorphic in $\zeta\in\Omega-D$. Set $\mathring{f}(z,\zeta)=f(z,\zeta)$ for $\zeta\in\Omega-D$, $\mathring{f}(z,\zeta)=0$ for $\zeta\in D$. Then $E(z)=F_\varepsilon(z)=\langle f(z,\cdot),\mu_\varepsilon\rangle=\langle\mathring{f}(z,\cdot),\mu_\varepsilon\rangle=(\mathring{f}(z,\cdot),\psi)=(f(z,\cdot),\psi)=F(z)$. Hence $E|\Delta=F|\Delta$, as asserted.

The theorem is proved.

Let us return for a moment to the case 1 considered above, and assume that $m=1$, so that 0 is an ordinary point. In this case one may assume that $\infty \in \Omega$ is also an ordinary point, since this can be achieved by conjugation.

Now if $\varphi(\zeta)$ is an automorphic form regular at 0, then

$$l(\varphi)=\sum_{n=1}^{\infty}\frac{a_n\varphi^{(n-1)}(0)}{n!}$$

and the Eichler integral $E(z) = -l(f(z,\cdot))$ is

$$E(z)=-\sum_{n=1}^{\infty}\frac{a_n}{n!}\frac{\partial^{n-1}f(z,\zeta)}{\partial\zeta^{n-1}}\bigg|_{\zeta=0}.$$

On the other hand, in view of the hypothesis on ∞, the Poincaré series considered by Ahlfors in [2],

$$\tilde{f}(z,\zeta)=\frac{1}{2\pi i}\sum_{\gamma\in\Gamma}\frac{\gamma'(\zeta)^q}{\gamma(\zeta)-z},$$

converges. Since $f(z,\zeta)=\tilde{f}(z,\zeta)+Q(z,\zeta)$ where

$$Q(z,\zeta)=\frac{1}{2\pi i}\sum_{\gamma\in\Gamma}\left\{\prod_{j=1}^{2q-1}\frac{z-A_j}{\gamma(\zeta)-A_j}-1\right\}\frac{\gamma'(\zeta)^q}{\gamma(\zeta)-z}$$

is a polynomial of degree $2q-2$ in z, with coefficients depending analytically on ζ, our E does not differ significantly from the integral used by Ahlfors.

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