RANDOM DIFFERENCE EQUATIONS AND RENEWAL THEORY FOR PRODUCTS OF RANDOM MATRICES

BY

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Introduction

In this paper we study the limit distribution of the solution Y_n of the difference equation

$$Y_n = M_n Y_{n-1} + Q_n, \quad n \geqslant 1, \tag{1.1}$$

where M_n and Q_n are random $d \times d$ matrices respectively d-vectors and Y_n also is a d-vector. Throughout we take the sequence of pairs (M_n, Q_n), $n \geqslant 1$, independently and identically distributed. The equation (1.1) arises in various contexts. We first met a special case in a paper by Solomon, [20] sect. 4, which studies random walks in random environments. Closely related is the fact that if $Y_n(i)$ is the expected number of particles of type i in the nth generation of a d-type branching process in a random environment with immigration, then $Y_n = (Y_n(1), \ldots, Y_n(d))$ satisfies (1.1) (Q_n represents the immigrants in the nth generation). (1.1) has been used for the amount of radioactive material in a compartment ([17]) and in control theory [9a]. Moreover, it is the principal feacture in a model for evolution and cultural inheritance by Cavalli-Sforza and Feldman [2]. Notice also that the dth order linear difference equation

$$y_n = \sigma_n^{(1)} y_{n-1} + \sigma_n^{(2)} y_{n-2} \ldots + \sigma_n^{(d)} y_{n-d} + q_n$$

can be brought into the form (1.1), if one takes

$$Y_n = (y_{n+d-1}, y_{n+d-2}, \ldots, y_n), \quad Q_n = (q_{n+d-1}, 0, \ldots, 0)$$

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and
$$M_n = \begin{pmatrix} \sigma_{n+d-1}^{(1)}, \dots, \sigma_{n+d-1}^{(d)} \\ 1 \quad 0 \quad 0 \\ \ddots \quad \ddots \\ 0 \quad 1 \quad 0 \end{pmatrix}. \tag{1.2}$$

Such random difference equations are mentioned in [0], section 4 and in [7], pp. 23, 125 and 181.

The solution of (1.1) is of course given by

$$Y_n = Q_n + M_n Q_{n-1} + M_n M_{n-1} Q_{n-2} + \dots + M_n M_{n-1} \dots M_2 Q_1 + M_n M_{n-1} \dots M_1 Y_0,$$

which for given Y_0 has the same distribution as

$$\sum_{k=1}^{n} M_1 \dots M_{k-1} Q_k + M_1 \dots M_n Y_0.$$

Put now for any d-(row) vector $x = (x(1) \dots, x(d))$ and for any $d \times d$ matrix $(m = (i, j))$

$$|x| = \left\{\sum_{i=1}^{d} x^2(i)\right\}^{\frac{1}{2}}, \quad \|m\| = \max_{|x|=1} xm.$$

It is known (¹) (see [5], Theorem 2) that if

$$E \log^+ \|M_1\| < \infty \tag{1.3}$$

then
$$\alpha \equiv \lim_{n\to\infty} \frac{1}{n} \log \|M_1 \dots M_n\| \quad \text{exists and is constant w.p.1.} \tag{1.4}$$

We shall assume that $\alpha < 0$ in which case $\|M_1 \dots M_n\| \to 0$ exponentially fast, and under very weak conditions on Q_n the series

$$R = \sum_{1}^{\infty} M_1 \dots M_{k-1} Q_k \tag{1.5}$$

will converge w.p.1. Then the distribution of Y_n converges to that of R, independently of Y_0. We note that conditions for the exponential convergence of $M_1 \dots M_n$ to 0 in the special case (1.2) have been given by Konstantinov and Nevelson [13]. Spitzer conjectured for the one dimensional case (i.e., $d = 1$; this is the situation of [20]) that R should be in the domain of attraction of a stable law. For the one dimensional case this is indeed not so hard to

(¹) In [5] $|x|$ denotes $\Sigma|x(i)|$, and the definition of $\|m\|$ is changed correspondingly. But it is easily seen that ratio between the present $\|M_1 \dots M_n\|$ and that of [5] is bounded away from 0 and ∞, so that (1.4) still holds.

prove (the long section 2 is only needed for $d>1$). A comparatively simple argument (see proof of Theorem 4) reduces the study of the tail of the distribution of R to that of

$$P\{\max_n |M_1 \dots M_n| > t\} = P\left\{\max_n \sum_1^n \log |M_i| > \log t\right\} \tag{1.6}$$

for large t. Since the M_i are independent and identically distributed the behavior of (1.6) can be found from renewal theory (see [4], section XI.6). Under reasonable conditions there exist $K>0$ and $\varkappa_1>0$ such that the probability in (1.6) behaves like $Kt^{-\varkappa_1}$ as $t\to\infty$, and also

$$0 < \lim_{t\to\infty} t^{\varkappa_1} P\{R>t\} < \infty.$$

In the d-dimensional case the proof in section 3 still goes through, but now the tail behavior of R reduces to that of

$$P\{\max_n |xM_1 \dots M_n| > t\} = P\left\{\max_n \sum_1^n \log \frac{|xM_1 \dots M_i|}{|xM_1 \dots M_{i-1}|} > \log t\right\} \tag{1.7}$$

for any given row vector x. This necessitates the development of revewal theory for the sequence of sums

$$\sum_1^n \log \frac{|xM_1 \dots M_i|}{|xM_1 \dots M_{i-1}|}.$$

We take this up first in section 2, where we show that (1.7) still behaves like $Kt^{-\varkappa_1}$ under suitable conditions. The simplest case is when M_n and Q_n have only positive entries, respectively components. We summarize our results for this situation, using the following notation:

$x=(x(1), \dots, x(d))$ stands for a generic row vector,

$$S_{d-1} = \{x \in \mathbf{R}^d: |x| = 1\},$$
$$S_+ = \{x \in \mathbf{R}^d: |x| = 1, x(i) \geq 0, 1 \leq i \leq d\} \subset S_{d-1}.$$

When $x \neq 0$ $$\tilde{x} = (x)^{\sim} = |x|^{-1} x \in S_{d-1}.$$

$\#\{\ \}$ denotes the number of elements in the set $\{\ \}$.

When m is a $d\times d$ matrix $m \geq 0$ ($m \gg 0$) means $m(i,j) \geq 0$ (>0) for $1 \leq i, j \leq d$. When $m\geq 0$, $\varrho(m)$ denotes its largest positive eigenvalue, the so called Frobenius eigenvalue ([6], vol. 2, p. 53).

$N_x(t) = \min\{n \geq 0: \log |xM_1 \dots M_n| > t\}$ ($=\infty$ when no such n exists).

On the event $\{N_x(t)<\infty\}$ we also define

$$W_x(t) = \log |xM_1 \dots M_{N_x(t)}| - t,$$

$$Z_x(t) = (xM_1 \dots M_{N_x(t)})^{\sim}.$$

THEOREM A. *Let $M_1, M_2, \dots$ be independent $d \times d$ matrices each with the same distribution μ such that*

$$P\{M_1 \geq 0\} = 1,$$

$$P\{M_1 \textit{ has a zero row}\} = 0, \tag{1.9}$$

and

$$E \log^+ \|M_1\| < \infty. \tag{1.10}$$

Assume also that the group generated by

$$\{\log \varrho(\pi) : \pi = m_1 \dots m_n \textit{ for some } n \textit{ and } m_i \in \operatorname{supp}(\mu) \textit{ and } \pi \gg 0\} \tag{1.11}$$

is dense in $\mathbf{R}$. *Then, there exists a constant $\alpha < \infty$ such that for each $x \in S_+$ one has w.p.1.*

$$\lim_{n\to\infty} \frac{1}{n} \|M_1 \dots M_n\| = \lim_{n\to\infty} \frac{1}{n} |xM_1 \dots M_n| = \alpha.$$

If $\alpha > 0$, then the limit distribution (as $t \to \infty$) of $Z_x(t)$, $W_x(t)$ exists and is independent of x. Also for $x \in S_+$ and $h \geq 1$

$$\lim_{n\to\infty} E\#\{n : t \leq |xM_1 \dots M_n| \leq th\} = \frac{\log h}{\alpha}. \tag{1.12}$$

If $\alpha < 0$, and if in addition to the above conditions there exists a $\varkappa_0 > 0$ for which

$$E\{\min_i (\sum_j M_1(i,j))\}^{\varkappa_0} \geq d^{\varkappa_0/2}, \tag{1.13}$$

and

$$E \|M_1\|^{\varkappa_0} \log^+ \|M_1\| < \infty, \tag{1.14}$$

then there exists a $\varkappa_1 \in (0, \varkappa_0]$ such that

$$\lim_{t\to\infty} t^{\varkappa_1} P\{\max_n |xM_1 \dots M_n| > t\} \tag{1.15}$$

exists and is strictly positive for $x \in S_+$.

THEOREM B. *Let $\{M_n, Q_n\}_{n \geq 1}$ be independent identically distributed, where the M_n are $d \times d$ matrices and the Q_n d-(column) vectors. Assume that the M_n satisfy all hypotheses of theorem A (including* (1.13) *and* (1.14)) *and that $\alpha < 0$. Assume also*

$$P\{Q_1 = 0\} < 1, \; P\{Q_1 \geq 0\} = 1, \quad E|Q_1|^{\varkappa_1} < \infty,$$

for the $\varkappa_1$ of (1.15). *Then for each $x \in S_{d-1}$*

$$\lim_{t\to\infty} t^{\varkappa_1} P\{xR \geq t\} \tag{1.16}$$

exists and is finite. For $x \in S_+$ *the limit in* (1.16) *is strictly positive.*

We want to point out that (1.12) is an analogue of Blackwell's renewal theorem (see [4], sect. XI.1) for products of random matrices and that the existence of the limit distribution of $W_x(t)$ is the analogue of the existence of the limit distribution of the residual waiting time in renewal theory ([4], p. 370).

Theorems A and B, together with some extensions are contained in Theorems 2–4. In section 4 we state without proof analogous results which do not presuppose $M_n \geq 0$ w.p.1. However, if we drop the condition $M_n \geq 0$ and if $d > 1$, then we have to add some absolute continuity requirements for the distribution μ of M_1. None of our results cover the following simple example: Let $d=2$, m_1, $m_2 \gg 0$ 2×2 matrices such that $\log \varrho(m_1)$ and $\log \varrho(m_2)$ generate a group which is dense in $\mathbf{R}$. Finally let m_3 be a rotation and take

$$P\{M_n = m_i\} = p_i,\ 1 \leq i \leq 3,$$

for some $p_i > 0$, $p_1+p_2+p_3=1$. We pose it as an open problem to prove appropriate forms of (1.12) and the existence of the limits (1.15) and (1.16) in this case. Perhaps it will be a little easier to solve similar problems for the special situation of [9], section 14.

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2. Renewal theory for products of positive matrices

Even though some of the renewal theory of this section is applicable to more general products of matrices (see Remark 1), the conditions are least cumbersome for products of positive matrics and the main results of this section are only formulated for such products. The basis for this section is a renewal theorem proved elsewhere [11] for (random) functions on a Markov chain. To be precise we consider a Markov chain $\{X_n\}_{n\geq 0}$ with stationary transition probabilities on a separable metric space S. Throughout $P^n(x, A) = P\{X_{k+n} \in A \mid X_k = x\}$ denotes the n-step transition probability for this chain, $P(x, A) = P^1(x, A)$ (1) and $\mathcal{S}$ the σ-field generated by the open sets of S, respectively $\mathcal{B}$ the collection of Borel sets of $\mathbf{R}$. We assume that another sequence $\{u_n\}_{n\geq 0}$ of random variables is defined on our probability space such that the distribution of u_i depends only on X_i and X_{i+1}, and not on the other

(1) Of course these have to satisfy the standard assumptions that $x \to P(x, A)$ is $\mathcal{S}$ measurable for fixed $A \in \mathcal{S}$ and $P(x, \cdot)$ is a probability measure on $\mathcal{S}$ for fixed $x \in S$. Moreover P^n is the n^{th} iterate of P, and the right hand side of (2.1) is assumed to be an $\mathcal{S}$ measurable function of x_0.

X_j or u_j when X_i and X_{i+1} are given. Formally, we assume that for each $x, y \in S$ there exists a distribution function $F(\cdot\,|x, y)$ such that for $A_i \in \mathcal{S}$, $B_i \in \mathcal{B}$

$$P\{X_i \in A_i,\, 0 \leqslant i \leqslant n,\ u_j \in B_j,\, 0 \leqslant j < n \,|\, X_0 = x_0\}$$
$$= I_{A_0}(x_0) \int_{A_1} P(x_0, dx_1) \int \dots \int_{A_n} P(x_{n-1}, dx_n) \prod_{i=0}^{n-1} F(B_i \,|\, x_i, x_{i+1}). \tag{2.1}$$

The expression $F(B\,|x, y)$ in (2.1) of course stands for

$$\int_B F(d\lambda\,|\,x, y).$$

Further notations and definitions to be used are as follows: $\mathcal{F}$ denotes the σ-field generated by the X_i and u_i, $i \geqslant 0$. P_x is the unique measure on $\mathcal{F}$ for which

$$P_x\{X_i \in A_i,\, 0 \leqslant i \leqslant n,\ u_j \in B_j,\, 0 \leqslant j < n\}$$

is given by the right hand side of (2.1). E_x denotes expectation w.r.t. P_x.

$d(\cdot,\cdot)$ is the distance function on S,

$$V_n = \sum_{i=0}^{n-1} u_i \quad (V_0 = 0), \tag{2.2}$$

$$N(t) = \min\{n \geqslant 0\colon V_n > t\} \ (= \infty \text{ if no such } n \text{ exists}). \tag{2.3}$$

On the event $\{N(t) < \infty\}$ we take

$$W(t) = V_{N(t)} - t, \tag{2.4}$$

$$Z(t) = X_{N(t)}. \tag{2.5}$$

$$C_k = \{x \in S\colon P_x\{V_m \geqslant mk^{-1} \text{ for all } m \geqslant k\} \geqslant \tfrac{1}{2}\}. \tag{2.6}$$

The following definition reduces to Feller's ([4], p. 362) when S is a one point set.

Definition. A function $g\colon S \times \mathbf{R} \to \mathbf{R}$ is called *directly Riemann* integrable if it is $\mathcal{S} \times \mathcal{B}$ measurable and satisfies (with $C_0 = \phi$)

$$\sum_{k=0}^{\infty} \sum_{l=-\infty}^{+\infty} (k+1) \sup\{|g(x, t)|\colon x \in C_{k+1} \setminus C_k,\, l \leqslant t \leqslant l+1\} < \infty, \tag{2.7}$$

and if for every fixed $x \in S$ and $0 < L < \infty$ the function $t \to g(x, t)$ is Riemann integrable on $[-L, +L]$.

Finally, if f is any function from $\Pi_{i=0}^{\infty}(S \times \mathbf{R})$ into $\mathbf{R}$ and $\varepsilon > 0$, we put

$$f^{\varepsilon}(x_0, v_0, x_1, v_1, \ldots)$$
$$= \lim_{n\to\infty} \sup \{f(y_0, w_0, y_1 w_1, \ldots) : d(x_i, y_i) + |v_i - w_i| < \varepsilon \text{ for } i \leqslant n\}.$$

Note that f^{ε} is automatically measurable w.r.t. $\Pi_{i=0}^{\infty}(S \times \mathcal{B})$.

We also need the following set of conditions:

I.1. There exists a probability measure φ on S such that $\varphi P = \varphi$ and

$$P_x\{X_n \in A \text{ for some } n\} = 1 \text{ for all } x \in S \text{ and}$$
$$\textit{open } A \text{ with } \varphi(A) > 0.$$

I.2.
$$\int \varphi(dx) \int P(x, dy) \int |\lambda| \, F(d\lambda \,|\, x, y) < \infty$$

and
$$\lim_{n\to\infty} \frac{V_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_0^{n-1} u_i$$
$$= \alpha \equiv \int \varphi(dx) \int P(x, dy) \int \lambda F(\lambda y \,|\, x, y) > 0 \quad \text{a.s.}(^1)$$

I.3. There exists a sequence $\{\zeta_\nu\}_{\nu\geqslant 1} \subset \mathbf{R}$ such that the group generated by $\{\zeta_\nu\}$ is dense in $\mathbf{R}$ and such that for each ζ_ν and $\delta > 0$ there exists a $y = y(\nu, \delta) \in S$ with the following property: For each $\varepsilon > 0$ there exist an $A \in \mathcal{S}$ with $\varphi(A) > 0$ and integers m_1, m_2 and a $\tau \in \mathbf{R}$ such that

$$P_x\{d(X_{m_1}, y) < \varepsilon, \; |V_{m_1} - \tau| \leqslant \delta\} > 0, \tag{2.8}$$

as well as
$$P_x\{d(X_{m_2}, y) < \varepsilon, \; |V_{m_2} - \tau - \zeta_\nu| \leqslant \delta\} > 0 \tag{2.9}$$

whenever $x \in A$.

I.4. For each fixed $x \in S$, $\varepsilon > 0$ there exists an $r_0 = r_0(x, \varepsilon) > 0$ such that for all functions $f: \Pi_{i\geqslant 0}(S \times \mathbf{R}) \to \mathbf{R}$ for which $f(X_0, V_0, X_1, V_1, \ldots)$ is an $\mathfrak{F}$-measurable function, and for all y with $d(x, y) < r_0$ one has

$$E_x f(X_0, V_0, X_1, V_1, \ldots) \leqslant E_y f^{\varepsilon}(X_0, V_0, X_1, V_1, \ldots) + \varepsilon \sup |f|$$

and
$$E_y f(X_0, V_0, X_1, V_1, \ldots) \leqslant E_x f^{\varepsilon}(X_0, V_0, X_1, V_1, \ldots) + \varepsilon \sup |f|.$$

In this setup the following theorem was proved in [11].

THEOREM 1. *If condition* I.1–I.4 *are satisfied, then there exists a finite measure* ψ *on* S *such that for every bounded, jointly continuous function* g: $S \times (0, \infty) \to \mathbf{R}$ *and every* $x \in S$

(1) We say that an event A occurs almost surely (a.s.) if $P_x(A) = 1$ for all $x \in S$.

$$\lim_{t\to\infty} E_x g(Z(t), W(t))$$
$$= \alpha^{-1}\int_S \varphi(dy)\int_{S\times(0,\infty)} P_y\{X_{N(0)}\in dz,\ V_{N(0)}\in d\lambda\}\int_{0<s\leqslant\lambda} g(z,s)\,ds. \tag{2.10}$$

Moreover, if $g: S\times\mathbf{R}\to\mathbf{R}$ *is jointly continuous and directly Riemann integrable, then for any* $x\in S$

$$\lim_{t\to\infty} E_x \sum_{n=0}^{\infty} g(X_n, t-V_n) = \alpha^{-1}\int_S \varphi(dy)\int_{-\infty}^{+\infty} g(y,s)\,ds. \tag{2.11}$$

Despite the forbidding appearance of conditions I Theorem 1 is useful when dealing with products of independent matrices (and probably for random walks on more general semigroups as well), as we proceed to explain. Let $\{M_n\}_{n\geqslant 1}$ be a sequence of random $d\times d$ matrices and put

$$\Pi_0 = \mathrm{I},\ \Pi_n = M_1 M_2 \dots M_n,\ n\geqslant 1.$$

Unless otherwise indicated all our vectors will be *row* vectors. If x is a d-vector, $|x|$ denotes its Euclidean norm, and if m is a $d\times d$ matrix, $\|m\|$ denotes its l_2 operator norm, i.e.,

$$\|m\| = \max_{|x|=1} |xm|.$$

For any $x\in\mathbf{R}^d$ with $|x|\neq 0$, we put $\tilde{x} = |x|^{-1}x$. When the expression for x is complicated we also use $(x)^{\sim}$ instead of $\tilde{x}$.

To apply Theorem 1 we take for our state space a closed subset S of the unit sphere

$$S_{d-1} = \{x = (x(1),\dots,x(d)) : |x|^2 = \sum_1^d x^2(i) = 1\}.$$

We define $\{X_n, u_n\}_{n\geqslant 0}$ as the following functions of X_0 and $\{M_n\}_{n\geqslant 1}$:

$$X_n = \begin{cases} (X_0\Pi_n)^{\sim} = (X_0 M_1\dots M_n)^{\sim} & \text{if } X_0\Pi_n \neq 0 \\ 0 & \text{if } X_0\Pi_n = 0, \end{cases} \tag{2.12}$$

$$u_n = \log\frac{|X_0\Pi_{n+1}|}{|X_0\Pi_n|}\ (= -\infty \text{ if } X_0\Pi_n = 0). \tag{2.13}$$

Note that for $X_0\in S_{d-1}$

$$V_n = \sum_{i=0}^{n-1} u_n = \log\frac{|X_0\Pi_n|}{|X_0|} = \log|X_0 M_1\dots M_n|. \tag{2.14}$$

Thus we are really looking at the action of the successive products of the matrices M_1, M_2, on d-space; V_n measures the size of the vector after n steps and X_n its direction. The probability measure is defined in terms of a measure μ on the set of $d\times d$ matrices, a $\varkappa\geqslant 0$, and a function $r(x)>0$ on S satisfying

$$r(x)=\int \mu(dM)\,|xM|^{\varkappa}\,r((xM)^{\sim}),\ x\in S. \tag{2.15}$$

For this to make sense, we shall of course require that for all $x\in S$, $\mu\{M: xM=0$ or $(xM)^{\sim}\notin S\}=0$. We now want

$$P\{M_{n+1}\in A\,|\,(X_i,u_i)_{i\leqslant n-1}, X_n=x_n\} = \frac{1}{r(x_n)}\int_{M\in A}\mu(dM)\,|x_nM|^{\varkappa}\,r((x_nM)^{\sim}). \tag{2.16}$$

By virtue of (2.15) this defines an honest probability distribution for M_{n+1}. The M_n are independent of each other and the preceding X_i in the case where μ is a probability measure, $\varkappa=0$ and $r(x)\equiv 1$; the more general case described here will be used later in this section. We shall slightly abuse notation and write P_x for the conditional measure given $X_0=x$ governing $\{X_n,u_n\}_{n\geqslant 0}$ *and* $\{M_n\}_{n\geqslant 1}$. I.e., if $\mathcal{M}$ is the Borel field in the space of $d\times d$ matrices,(1) then we think of P_x as a measure on $\Pi_{i\geqslant 0}(S\times\mathcal{B}\times\mathcal{M})$ rather than on $\Pi_{i\geqslant 0}(S\times\mathcal{B})$. E_x will still denote expectation w.r.t. P_x, and in accordance with (2.12), (2.13) and (2.16) we have for any positive measurable function f: $(S\times\mathbf{R})^{k+0}\to(0,\infty)$, $x\in S$ and $D_i\in\mathcal{M}$(2)

$$\begin{aligned} &E_x\{f(X_0,u_0,X_1,u_1,\ldots,X_k,u_k);\ M_i\in D_i,\ 1\leqslant i\leqslant k+1\}\\ &=\frac{1}{r(x)}\int_{D_1}\mu(dM_1)\ldots\int_{D_{k+1}}\mu(dM_{k+1})\,|x\Pi_{k+1}|^{\varkappa}\,r((x\Pi_{k+1})^{\sim})\\ &\qquad f\left(x,\log|x\Pi_1|,(x\Pi_1)^{\sim},\log\frac{|x\Pi_2|}{|x\Pi_1|},\ldots,(x\Pi_k)^{\sim},\log\frac{|x\Pi_{k+1}|}{|x\Pi_k|}\right). \end{aligned} \tag{2.17}$$

It is not hard to check that for $D_i=$ set of all $d\times d$ matrices, (2.17) agrees with (2.1) for suitable F and transition operator

$$P(X_{n+1}\in A\,|\,X_n=x)=P(x,A) \equiv\frac{1}{r(x)}\int_{(xM)^{\sim}\in A}\mu(dM)\,|xM|^{\varkappa}\,r((xM)^{\sim}).$$

Of course $P\{X_{n+k}\in A\,|\,X_n=x\}$ will again be given by $P^k(x,A)$, the kth iterate of P.

(1) The space is just $\mathbf{R}^{d^2}$ and $\mathcal{M}$ is generated by the open sets in this space.

(2) For any set of conditions $\mathcal{C}$, $E_x\{f;\mathcal{C}\}$ denotes the integral of f w.r.t P_x over the set where $\mathcal{C}$ is satisfied, i.e., $E_x\{f;\mathcal{C}\}=E_xfI_{\mathcal{C}}$.

We introduce one more concept to help us obtain the aperiodicity condition I.3. We call a $d \times d$ matrix π *feasible* if there exists on n and $m_1, \ldots, m_n \in \operatorname{supp}(\mu)$ such that $\pi = m_1 m_2 \ldots m_n$, and if in addition π has an (algebraically) simple eigenvalue $\varrho(\pi) > 0$ which exceeds all other eigenvalues of π in absolute value (i.e., if $\lambda \neq \varrho(\pi)$ is an eigenvalue of π, then $|\lambda| < \varrho(\pi)$). If this is the case we call $\varrho(\pi)$ a *feasible eigenvalue* and the corresponding right and left eigenvectors of unit length $\tilde{a}'(\pi)$ respectively $\tilde{b}(\pi)$ *feasible eigenvectors* (the prime in a' denotes transposition; a' is a column vector). Note that, by definition $\tilde{a}'$ and $\tilde{b}$ are the unique solutions of

$$\pi a' = \varrho(\pi)\, a', \ |a| = 1, b\pi = \varrho(\pi)\, b, \ |b| = 1.$$

PROPOSITION 1. *Let S be a closed subset of S_{d-1} such that*

$$\mu\{M: xM = 0 \quad or \quad (xM)^{\tilde{}} \notin S\} = 0, x \in S. \tag{2.18}$$

Assume further that r: $S \to (0, \infty)$ satisfies (2.15), *is continuous and bounded away from* 0 *and* ∞, *and that for all $x \in S$ (and P_x as in* (2.17))

$$P_x\{\exists C > 0 \quad with \quad |x\Pi_n| \geq C\|\Pi_n\| \quad for\ all \quad n\} = 1. \tag{2.19}$$

Finally, let the group generated by

$$\{\log \varrho(\pi): \pi \ \ feasible\ and \ \min_{y \in S} |y\tilde{a}'(\pi)| > 0\} \tag{2.20}$$

be dense in $\mathbf{R}$, *and*

$$\int \mu(dM)\, \|M\|^{\varkappa} \log^+ \|M\| < \infty. \tag{2.21}$$

Then, there exists a probability measure φ on S such that conditions I.1, I.3 *and* I.4 *are fulfilled and such that*

$$\int \varphi(dx) \int P(x, dy) \int \lambda^+ F(d\lambda \,|\, x, y) = \int \varphi(dx)\, E_x u_0^+$$
$$= \int \varphi(dx) \frac{1}{r(x)} \int \mu(dM)\, |xM|^{\varkappa} r((xM)^{\tilde{}}) \log^+ |xM| < \infty, \tag{2.22}$$

as well as

$$\lim_{n \to \infty} \frac{V_n}{n} = \alpha \equiv \int \varphi(dx)\, E_x u_0$$
$$= \int \varphi(dx) \frac{1}{r(x)} \int \mu(dM)\, |xM|^{\varkappa} r((xM)^{\tilde{}}) \log |xM| \quad a.s. \tag{2.23}$$

(However $-\infty \leq \alpha \leq 0$ is possible.)

Remark 1. The conditions (2.18)–(2.20) are somewhat awkward. However, as we shall see in Theorems 2 and 3, things become easy when μ is concentrated on the set of positive matrices. Of course, once we have an S for this situation, then we can also handle the case where μ is concentrated on matrices of the form $A^{-1}BA$, $B(i,j)\geqslant 0$, for some fixed non singular A. For then we only have to replace the former S by $\{(xA)^{\sim}: x\in S\}$. Another case in which (2.18) and (2.19) hold, and in which every $\pi=m_1 \dots m_n$, $m_i \in \operatorname{supp}(\mu)$ is feasible, occurs when all $m\in\operatorname{supp}(\mu)$ are of rank one, i.e. of the form $m(i,j)=a(i)\,b(j)$ for some d-vectors a and b, and when

$$\inf\left\{\sum_1^d \tilde b_1(l)\,\tilde a_2(l): m_1=(a_1(i)\,b_1(j)) \quad\text{and}\quad m_2=(a_2(i)\,b_2(j)) \quad \text{are in} \quad \operatorname{supp}(\mu) \quad \text{for some} \quad a_1, b_2\right\}>0.$$

Then we may take

$$S=\text{closure of}\quad \{\tilde b: \exists \text{ a such that} \quad m=(a(i)\,b(j))\in\operatorname{supp}(\mu)\}.$$

Proof of Proposition 1. We first prove condition I.4. For $x\in S$, $\delta>0$ let

$$\begin{aligned} E(x,\delta,k)&=\{(m_1,\dots,m_k): m_i\ d\times d \quad \text{matrices such that} \\ &\qquad |xm_1\dots m_l|\geqslant\delta\|m_1\dots m_l\|>0 \quad\text{for}\quad 1\leqslant l\leqslant k\}, \\ E(x,\delta)&=\{\{m_n\}_{n\geqslant1}: (m_1,\dots,m_k)\in E(x,\delta,k)\quad \text{for all}\quad k\}. \end{aligned}$$

Then for $|y-x|<\delta_1\delta$ and $(m_1,\dots,m_k)\in E(x,\delta,k)$ and $\pi_k=m_1\dots m_k$ we have

$$\begin{aligned}(1-\delta_1)\ |x\pi_k|&<|x\pi_k|-|y-x|\,\|\pi_k\| \\ &\leqslant|y\pi_k|<(1+\delta_1)|x\pi_k) \end{aligned}\tag{2.24}$$

as well as

$$\begin{aligned}&|(y\pi_k)^{\sim}-(x\pi_k)^{\sim}| \\ &=\frac{1}{|x\pi_k|\,|y\pi_k|}|(|x\pi_k|-|y\pi_k|)\,y\pi_k+|y\pi_k|\,(y\pi_k-x\pi_k)|\leqslant 2\,\delta_1.\end{aligned}$$

In other words, for any sample path with $X_0=y$, $|y-x|<\delta_1\delta$, and $(M_1,\dots,M_k)\in E(x,\delta)$ we have for sufficiently small δ_1 and $k\geqslant0$

$$|X_k-(x\Pi_k)^{\sim}|=|(y\Pi_k)^{\sim}-(x\Pi_k)^{\sim}|<2\,\delta_1,\ |V_k-\log|x\Pi_k\||=\left|\log\frac{|y\Pi_k|}{|x\Pi_k|}\right|<2\,\delta_1.\tag{2.25}$$

Consequently, for any bounded and $\mathfrak{F}$-measurable $f(X_0, V_0, X_1, V_1, \dots)$ and $|y-x|<\delta_1\delta(E^c(x,\delta)$ is the complement of $E(x,\delta))$

$$\begin{aligned}E_y f(X_0, V_0, X_1, V_1, \ldots) &\leqslant \sup |f|\, P_y\{E^c(x,\delta)\}\\ &+ E_y\{f(X_0, V_0, \ldots); E(x,\delta) \leqslant \sup |f|\, P_y\{E^c(x,\delta)\}\\ &+ E_y f^{2\delta_1}(x, 0, (x\Pi_1)^{\tilde{}}, \log|x\Pi_1|\,(x\Pi_2)^{\tilde{}}, \log|x\Pi_2|, \ldots); E(x,\delta)\}.\end{aligned} \tag{2.26}$$

But from (2.17), (2.24) and (2.25) we see that for $|x-y| \leqslant \delta_1 \delta$ and $D \in \mathfrak{M}^k$

$$\begin{aligned}P_y\{E(x,\delta,k) \cap D\} &= \frac{1}{r(y)} \int_{E(x,\delta,k)\cap D} \mu(dM_1) \ldots \mu(dM_k)\, |y\Pi_k|^\varkappa r((y\Pi_k)^{\tilde{}})\\ &\geqslant \left[\min_{|z_1-z_2|\leqslant 2\delta_1} \frac{r(z_1)}{r(z_2)}\right]^2 (1-\delta_1)^\varkappa \frac{1}{r(x)} \int_{E(x,\delta,k)\cap D} \mu(dM_1) \ldots \mu(dM_k)\, |x\Pi_k|^\varkappa r((x\Pi_k)^{\tilde{}})\\ &= \left[\min_{|z_1-z_2|\leqslant 2\delta_1} \frac{r(z_1)}{r(z_2)}\right]^2 (1-\delta_1)^\varkappa P_x\{E(x,\delta,k) \cap D\}.\end{aligned} \tag{2.27}$$

Since r is uniformly continuous on the compact set S, and since $P_x\{E^c(x,\delta)\} \to 0$ as $\delta \to 0$ (by (2.19)) it follows that for every $\varepsilon > 0$ and $x \in S$ we can choose $\delta > 0$, $\delta_1 > 0$ such that for all $y \in S$ with $|y-x| < \delta_1 \delta$

$$P_y\{E(x,\delta)\} = \lim_{k\to\infty} P_y\{E(x,\delta,k)\} \geqslant (1-\varepsilon)\, P_x\{E(x,\delta)\}$$

and

$$P_y\{E^c(x,\delta)\} \leqslant \varepsilon + P_x\{E^c(x,\delta)\} \leqslant 2\varepsilon. \tag{2.28}$$

Estimates similar to (2.27) and (2.28) show that for $|y-x| < \delta_1 \delta$ also

$$\begin{aligned}&E_y\{f^{2\delta_1}(x, 0, (x\Pi_1)^{\tilde{}}, \log|x\Pi_1|, \ldots); E(x,\delta)\}\\ &\quad\leqslant E_x\{f^{2\delta_1}(X_0, V_0, \ldots); E(x,\delta)\} + \varepsilon\, E_x\{|f^{2\delta_1}(X_0, V_0, \ldots)|; E(x,\delta)\}\\ &\quad\leqslant E_x f^{2\delta_1}(X_0, V_0, X_1, V_1, \ldots) + (\varepsilon + P_x\{E^c(x,\delta)\} \sup |f|.\end{aligned} \tag{2.29}$$

(2.26), (2.28) and (2.29) together imply the second inequality of I.4 and the first one is proved in the same way.

Next we prove I.1. For fixed m the map $x \to (xm)^{\tilde{}}$ is continuous from $S \to S$ at every x with $xm \neq 0$. It follows easily from this fact, (2.18) and (2.21) that

$$x \to E_x f(X_1) = \int P(x, dy)\, f(y)$$

is bounded and continuous whenever $f\colon S \to S$ is bounded and continuous. But then one easily proves (see [18], theorem IV.3.1) that for any $y_0 \in S$ any accumulation point of the sequence of measures

$$\frac{1}{n}\sum_{k=1}^{n} P^k(y_0, \cdot) \tag{2.30}$$

is an invariant measure for P. In order to satisfy the recurrence condition in I.1 we choose y_0 in a special way. Let π be a feasible matrix with feasible right and left eigenvectors $\tilde{a}' = \tilde{a}'(\pi)$, respectively $\tilde{b}(\pi)$. By bringing π into its Jordan canonical form it is easy to see (compare [12], pp. 1466–7) that we can then find a multiple a of $\tilde{a}$ such that $\tilde{b}a' = 1$ and that with this normalization

$$\lim_{k\to\infty}\left\{\frac{\pi}{\varrho(\pi)}\right\}^k = a'\tilde{b}.$$

Since π is feasible, $\pi = m_1 \dots m_n$ for some n and $m_i \in \operatorname{supp}(\mu)$. Thus also π^k is a product of $m_i's$, $m_i \in \operatorname{supp}(\mu)$ and for any x

$$\lim_{k\to\infty}\frac{x\pi^k}{(\varrho(\pi))^k} = (xa')\,\tilde{b}. \tag{2.31}$$

If now $\tilde{a}(\pi)$ is such that

$$\min_{y\in S}|y\tilde{a}'(\pi)| > 0, \tag{2.32}$$

then $xa' \neq 0$ for $x \in S$ and it follows that

$$\lim_{k\to\infty}(x\pi^k)^{\sim} = \tilde{b}. \tag{2.33}$$

Moreover, by (2.18) $(x\pi^k)^{\sim} \in$ closure of $S = S$, so that $\tilde{b}(\pi) \in S$ whenever (2.32) holds. One easily checks that in this case the convergence in (2.31) and (2.33) is uniform in $x \in S$, and consequently, for any neighborhood $U \subset S$ of $\tilde{b}$ there exists a $k = k(U)$, neighborhoods U_i of m_i and a $\delta = \delta(U) > 0$ (k, U_i and δ depend on π as well, but π is fixed for the moment) such that for all $x \in S$

$$\begin{aligned} &P_x\{X_{kn} \in U\} \\ &\geqslant P_x\{M_{rn+i} \in U_i,\ 1 \leqslant i \leqslant n,\ 0 \leqslant r < k\} \geqslant \delta(U) > 0. \end{aligned} \tag{2.34}$$

This, together with the extended Borel-Cantelli Lemma ([1], exercise 5.6.9), shows

$$P_x\{X_t \in U \quad \text{for some} \quad t\} = 1, \quad x \in S. \tag{2.35}$$

We now fix a specific feasible π such that $\tilde{a}(\pi_0)$ satisfies (2.32) (such π_0 exist because (2.20) generates $\mathbf{R}$) and take y_0 in (2.30) equal to $b_0 \equiv \tilde{b}(\pi_0)$. For φ we take now any weak accumulation point of (2.30) (with $y_0 = b_0$). As remarked we then have $\varphi P = \varphi$ and by definition of φ $P^l(b_0, A) > 0$ for some l whenever A is open and $\varphi(A) > 0$. It is not hard to see from I.4 that for such A there even exists a neighborhood $U_0 \subset S$ of b_0 such that

$$\inf_{y\in U_0} P^l(y, A) > 0.$$

(Apply I.4 to $f(X_0, V_0, \dots) = g_n(X_l)$ where $g_n: S \to [0, 1]$ is a sequence of continuous functions which increases to I_A and such that [1] $g_n^{1/n}(X_l) \leqslant I_A(X_l)$.) The strong Markov property then shows that for any x

$$P_x\{X_t \in A \text{ for some } t\} \geqslant P_x\{X_t \in U_0 \text{ for some } t\} \inf_{y \in U_0} P^l(y, A) = \inf_{y \in U_0} P^l(y, A) > 0$$

(by (2.35)). Again the extended Borel-Cantelli Lemma shows that $P_x\{X_t \in A \text{ for some } t\} = 1$, $x \in S$, thus proving I.1.

The same sort of arguments establish I.3. To see this, let π be a feasible matrix for which (2.32) holds, and let $\varepsilon > 0$ be given. Again using the uniformity of the convergence in (2.31) in (2.33) we can find an $l = l(\pi)$ such that

$$|(x\pi^l)^{\sim} - \tilde{b}(\pi)| \leqslant \frac{\varepsilon}{2}, \quad |(x\pi^{l+1})^{\sim} - \tilde{b}(\pi)| \leqslant \frac{\varepsilon}{2} \tag{2.36}$$

for all $x \in S$, and (because $\tilde{b}a' = 1$)

$$|\log|\tilde{b}\pi^l| - l \log \varrho(\pi)| \leqslant \frac{\varepsilon}{3}, \quad |\log|\tilde{b}\pi^{l+1}| - (l+1)\log \varrho(\pi)| \leqslant \frac{\varepsilon}{3}.$$

Then we can find a neighborhood $U \subset S$ of $\tilde{b}$ such that for $j = l$ or $(l+1)$ and $x \in U$

$$|\log|x\pi^j| - j \log \varrho(\pi)| \leqslant \frac{2\varepsilon}{3}. \tag{2.37}$$

If $\pi = m_1, \dots, m_n$ with $m_i \in \operatorname{supp}(\mu)$ then (2.34) again holds for this U and suitable k and $\delta(U)$. Thus, also for any $t \geqslant 0$

$$P^{t+kn}(b_0, U) = \int P^t(b_0, dy)\, P_y\{X_{kn} \in U\} \geqslant \delta(U) > 0$$

and consequently also

$$P(U) \geqslant \delta(U) > 0. \tag{2.38}$$

Moreover, as in (2.34), we deduce from (2.36) and (2.37) that there exist neighborhoods U_i of m_i, $1 \leqslant i \leqslant n$ such that for $x \in U$, $j = l$ or $(l+1)$ one has

$$\begin{aligned} &P_x\{|X_{jn} - \tilde{b}(\pi)| < \varepsilon \quad \text{and} \quad |V_{jn} - j \log \varrho(\pi)| < \varepsilon\} \\ &\qquad \geqslant P_x\{M_{rn+i} \in U_i,\ 1 \leqslant i \leqslant n,\ 0 \leqslant r \leqslant l\} > 0. \end{aligned} \tag{2.39}$$

(2.38) and (2.39) give us (2.8) and (2.9) if $\zeta_\nu = \log \varrho(\pi)$, $y = \tilde{b}(\pi)$, and $A = U$, $m_1 = ln$, $m_2 = (l+1)n$ and $\tau = l \log \varrho(\pi)$. The ζ_ν which can occur in this way run through the set (2.20) so that I.3 has been proved.

---

[1] In accordance with the definition of f^ε we mean by $g_n^{1/n}(X_l)$ sup $\{g_n(x): d(x, X_l) < n^{-1}\}$.

Lastly we prove (2.22) and (2.23). The measure ξ on $\mathcal{J}$ defined by

$$\begin{aligned}&\xi(X_i \in A_i,\, u_i \in B_i,\, 0 \leqslant i \leqslant k)\\ &= \int_S \varphi(dx)\, P_x\{X_i \in A_i,\, u_i \in B_i\, 0 \leqslant i \leqslant k\}\end{aligned}$$

is invariant under the shift. Moreover, by (2.21),

$$\begin{aligned}\int u_0^+ \, d\xi = \int \varphi(dx)\, E_x u_0^+ &= \int \varphi(dx) \frac{1}{r(x)} \int \mu(dM)\, |xM|^{\varkappa} r((xM)^{\sim}) \log^+ |xM| \\ &\leqslant \int \varphi(dx) \max_{z_1, z_2 \in S} \frac{r(z_1)}{r(z_2)} \int \mu(dM)\, \|M\|^{\varkappa} \log^+ \|M\| < \infty.\end{aligned}$$

This is (2.22); but more importantly it allows us to apply Birkhoff's ergodic theorem ([8], p. 18) (after a truncation argument as in [1], p. 116) to give us

$$\lim_{n\to\infty} \frac{1}{n} V_n = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} u_i \quad \text{exists a.e. } [\xi],$$

and hence,

$$P_x\left\{\lim_{n\to\infty} \frac{V_n}{n} = \lim_{n\to\infty} \frac{1}{n} \log |X_0 \Pi_n| \text{ exists}\right\} = 1 \tag{2.40}$$

for almost all x w.r.t. φ. We do not yet know that (2.40) holds with $x = b_0$, but assume

$$P_{b_0}\left\{\liminf_{n\to\infty} \frac{V_n}{n} < \limsup_{n\to\infty} \frac{V_n}{n}\right\} > 0.$$

Then there exist r and $\varepsilon > 0$ and for each l_1 an $l_2 > l_1$ with

$$P_{b_0}\left\{\frac{V_{n_1}}{n_1} \leqslant r - 2\varepsilon < r + 2\varepsilon \leqslant \frac{V_{n_2}}{n_2} \text{ for some } n_1, n_2 \in [l_1, l_2]\right\} \geqslant 2\varepsilon. \tag{2.41}$$

Let $f: \mathbf{R} \times \mathbf{R} \to [0, 1]$ be defined by

$$f(c, d) = \begin{cases} 1 & \text{if } c \leqslant r - 2\varepsilon \text{ and } d \geqslant r + 2\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Then by (2.41) and I.4 there exists an $r_0 = r_0(b_0, \varepsilon)$ such that

$$E_y f^{\varepsilon}\left(\min_{l_1 \leqslant n \leqslant l_2} \frac{V_n}{n}, \max_{l_1 \leqslant n \leqslant l_2} \frac{V_n}{n}\right) \geqslant -\varepsilon + E_{b_0} f\left(\min_{l_1 \leqslant n \leqslant l_2} \frac{V_n}{n}, \max_{l_1 \leqslant n \leqslant l_2} \frac{V_n}{n}\right) \geqslant \varepsilon \tag{2.42}$$

whenever $|y - b_0| < r_0$. Because

$$f^{\varepsilon}\left(\min_{l_1\leqslant n\leqslant l_2}\frac{V_n}{n},\ \max_{l_1\leqslant n\leqslant l_2}\frac{V_n}{n}\right)=0$$

when
$$\min_{l_1\leqslant n\leqslant l_2}\frac{V_n}{n}\geqslant r-\varepsilon \ \text{ or } \ \max_{l_1\leqslant n\leqslant l_2}\frac{V_n}{n}\leqslant r+\varepsilon,$$

(2.42) yields for each l_1 and $|y-b_0|<r_0$

$$P_y\left\{\inf_{n\geqslant l_1}\frac{V_n}{n}\leqslant r-\varepsilon\leqslant r+\varepsilon\leqslant\sup_{n\geqslant l_1}\frac{V_n}{n}\right\}\geqslant\lim_{l_2\to\infty}E_y f^{\varepsilon}\left(\min_{l_1\leqslant n\leqslant l_2}\frac{V_n}{n},\ \max_{l_1\leqslant n\leqslant l_2}\frac{V_n}{n}\right)\geqslant\varepsilon.$$

Thus,
$$P_y\left\{\liminf_{n\to\infty}\frac{V_n}{n}<\limsup_{n\to\infty}\frac{V_n}{n}\right\}\geqslant\varepsilon$$

whenever $|y-b_0|<r_0$. Since (see (2.38))

$$\varphi\{y:\ |y-b_0|<r_0\}>0, \tag{2.43}$$

this contradicts the validity of (2.40) for almost all x. Thus (2.40) holds with $x=b_0$. Now assume that $\lim n^{-1}V_n$ is not a constant a.e. $[P_{b_0}]$. As before we can then find r, $\varepsilon>0$ and l_1 such that for $l_2\geqslant l_1$,

$$P_{b_0}\left\{\frac{V_n}{n}\leqslant r-2\varepsilon\ \text{ for all }\ l_1\leqslant n\leqslant l_2\right\}\geqslant 2\varepsilon,$$

$$P_{b_0}\left\{\frac{V_n}{n}\geqslant r+2\varepsilon\ \text{ for all }\ l_1\leqslant n\leqslant l_2\right\}\geqslant 2\varepsilon.$$

As above this will imply for all y with $|y-b_0|<r_0$

$$P_y\left\{\frac{V_n}{n}\leqslant r-\varepsilon\ \text{ for all }\ n\geqslant l_1\right\}\geqslant\varepsilon,$$

$$P_y\left\{\frac{V_n}{n}\geqslant r+\varepsilon\ \text{ for all }\ n\geqslant l_1\right\}\geqslant\varepsilon. \tag{2.44}$$

This implies for any $x\in S$

$$P_x\left\{\liminf_{n\to\infty}\frac{V_n}{n}\leqslant r-\varepsilon\,|\,X_0,\ldots,X_k,u_0,\ldots,u_{k-1},M_1,\ldots,M_k\right\}\geqslant\varepsilon$$

a.e. $[P_x]$ on the set where $|X_k-b_0|<r_0$. From the martingale convergence theorem (see Cor. 5.22 in [1]) and the fact that (see I.1 and (2.43))

$$P_x\{|X_k-b_0|<r_0\ \ \text{infinitely often}\}=1,$$

it now follows that

$$P_x\left\{\liminf \frac{V_n}{n} \leqslant r-\varepsilon\right\}=1, \quad x \in S.$$

Similarly
$$P_x\left\{\limsup \frac{V_n}{n} \geqslant r-\varepsilon\right\}=1, \quad x \in S.$$

These equations again contradict (2.40). Thus for some α,

$$P_{b_0}\left\{\lim_{n\to\infty} \frac{V_n}{n}=\alpha\right\}=1.$$

Once again this shows for all $\varepsilon>0$ and $|y-b_0|<r_0(b_0, \varepsilon)$

$$P_y\left\{\alpha-\varepsilon \leqslant \liminf_{n\to\infty} \frac{V_n}{n} \leqslant \limsup_{n\to\infty} \frac{V_n}{n} \leqslant \alpha+\varepsilon\right\} \geqslant 1-\varepsilon$$

and a repetition of the argument following (2.44) gives for all $x \in S$

$$P_x\left\{\alpha-\varepsilon \leqslant \liminf_{n\to\infty} \frac{V_n}{n} \leqslant \limsup_{n\to\infty} \frac{V_n}{n} \leqslant \alpha+\varepsilon\right\}=1.$$

Since $\varepsilon>0$ is arbitrary this finally proves (2.23) except for the identification of α. However, we already know from the ergodic theorem (see [8], p. 18) that

$$\int \lim \frac{V_n}{n}\, d\xi=\int u_0\, d\xi=\int \varphi(dx)\, E_x u_0,$$

which completes the proof of Proposition 1. ∎

Proposition 1 now leads quickly to the next two theorems for positive matrices which constitute the main results of this section. We precede these theorems with some notation. Throughout all vectors are d-vectors and matrices of size $d \times d$. For a vector $x=(x(i), \ldots, x(d))$ we write $x \geqslant 0$ and $x \gg 0$ when $x(i) \geqslant 0$ respectively $x(i)>0$ for *all* $i \in [1, d]$. Similarly for a matrix $\pi, \pi \geqslant 0$ and $\pi \gg 0$ mean $\pi(i, j) \geqslant 0$ respectively $\pi(i, j)>0$ for *all* $1 \leqslant i, j \leqslant d$. If $\pi \gg 0$, then we know from the Perron-Frobenius theorem (see [6], vol. 2, p. 53) that there exists an (algebraically) simple eigenvalue $\varrho(\pi)$ of π such that $\varrho(\pi)>0$, and right and left eigenvectors a respectively b of π corresponding to $\varrho(\pi)$ can be chosen such that $a \gg 0$ and $b \gg 0$. $\varrho(\pi)$ exceeds all other eigenvalues of π in absolute value, and if a and b are normalized such that $ba'=\Sigma b(i)\, a(i)=1$ then (2.31) holds for all row vectors x. As before we consider the process $(X_n, u_n)_{n \geqslant 1}$ of (2.12), (2.13) and define V_n by (2.14). In the present situation it is somewhat neater to view the $N(t)$, $Z(t)$, $W(t)$ of (2.3)–(2.5) as functions of $M_1, \ldots, M_n$ with the initial point as a parameter. Thus we put [1]

[1] The definition (2.45) for $N_x(t)$ is the most natural one in the framework of Proposition 1. One should note, however, that it is the first time $|x\Pi_n|$ exceeds e^t, rather than t. This is why there is a factor $e^{\varkappa_1 t}$ in (2.62), rather than $t^{\varkappa_1}$ as in (2.63).

$$N_x(t) = \min\{n \geq 0\colon \log|xM_1 \dots M_n| > t\} = \min\{n \geq 0\colon |xM_1 \dots M_n| > e^t\}, \tag{2.45}$$

$$W_x(t) = \log|xM_1 \dots M_{N_x(t)}| - t, \tag{2.46}$$

$$Z_x(t) = (xM_1 \dots M_{N_x(t)})^{\sim}. \tag{2.47}$$

In the next theorem we use P *without subscript* for the measure governing the sequence $\{M_n\}_{n\geq 1}$, i.e., the product measure $\Pi_1^\infty \mu$; E is expectation w.r.t. P. These should not be confused with the P_x and E_x of (2.17).

THEOREM 2. *Let $M_1, M_2, \dots$ be a sequence of independent $d \times d$ matrices, each distributed according to the same probability measure μ. Assume that*

$$P\{M_1 \geq 0\} = 1, \tag{2.48}$$

$$P\{M_1 \quad \textit{has a zero row}\} = 0, \tag{2.49}$$

$$E \log^+ \|M_1\| < \infty, \tag{2.50}$$

and that the group generated by

$$\{\log \varrho(\pi)\colon \pi = m_1 \dots m_n \quad \textit{for some } n \textit{ and } m_i \in \operatorname{supp}(\mu), \textit{ and } \pi \gg 0\} \tag{2.51}$$

is dense in $\mathbf{R}$. *Then there exists a constant $\alpha < +\infty$ such that a.e.* $[P]$

$$\lim_{n\to\infty} \frac{1}{n} \log \|M_1 \dots M_n\| = \lim_{n\to\infty} \frac{1}{n} \log|xM_1 \dots M_n| = \alpha$$

for all $$x \in S_+ \equiv \{x = (x(1), \dots, x(d))\colon |x| = 1, x \geq 0\}.$$

If $\alpha > 0$ then for every bounded and jointly continuous function $g\colon S_+ \times (0, \infty) \to \mathbf{R}$

$$\lim_{n\to\infty} Eg(Z_x(t), W_x(t)) \quad \textit{exists for all} \quad x \in S_+ \quad \textit{and is independent of} \quad x. \tag{2.52}$$

Also if $\alpha > 0$ there exists a probability measure φ on S_+ such that for every jointly continuous function $g\colon S_+ \times \mathbf{R} \to \mathbf{R}$ which satisfies

$$\sum_{l=-\infty}^{+\infty} \sup\{|g(y, t)|\colon y \in S_+, l \leq t \leq l+1\} < \infty \tag{2.53}$$

one has $$\lim_{t\to\infty} E \sum_{n=0}^{\infty} \{g((x\Pi_n)^{\sim}, t - \log|x\Pi_n|)\}$$

$$= \alpha^{-1} \int_{S_+} \varphi(dy) \int_{-\infty}^{+\infty} g(y, s)\, ds, x \in S_+. \tag{2.54}$$

In particular for $x \in S_+, h \geqslant 1$,

$$\lim_{t\to\infty} E\#\{n: t \leqslant |xM_1 \dots M_n| \leqslant th\} = \frac{1}{\alpha}\log h. \tag{2.55}$$

Proof: We merely check that the conditions of Proposition 1 hold and that (2.53) implies direct Riemann integrability of g. The theorem is then immediate from Proposition 1 and Theorem 1. For S we take S_+ and in this theorem $\varkappa = 0$, $r(x) = 1$ for all $x \in S_+$. Clearly $xM(j) \geqslant 0$ when $x \in S_+$ and $M(l, j) \geqslant 0$ for $1 \leqslant l \leqslant d$, and even $xM(j) > 0$ for some j when no row of M is zero. Thus (2.18) is immediate from (2.48), (2.49). As we pointed out already, if $\pi \gg 0$, then we may take a right eigenvector a' corresponding to $\varrho(\pi)$ such that $a' \gg 0$. For any such a',

$$\min_{x\in S_+} xa' \geqslant \min_{x\in S_+} \min_i a(i) \sum_{j=1}^{d} x(j) > 0$$

so that the set (2.51) is contained in (2.20) and the group generated by (2.20) is indeed dense in $\mathbf{R}$. (2.21) reduces to (2.50) and finally (2.19) is proved as follows: Firstly, for any $x \in S_+$ and matrix $\pi \geqslant 0$

$$|x\pi| \geqslant d^{-\frac{1}{2}} \sum_{i=1}^{d} x\pi(i) \geqslant d^{-\frac{1}{2}} \min_l x(l) \sum_{j,i} \pi(j,i) \geqslant d^{-\frac{1}{2}} \min_l x(l)\, \|\pi\|.$$

Thus (2.19) holds for any $x \gg 0$ with $C = d^{-\frac{1}{2}} \min x(l)$. Now, since (2.51) generates a dense group in $\mathbf{R}$ there is an n_0 such that

$$P\{\Pi_{n_0} \gg 0\} > 0. \tag{2.56}$$

Thus
$$T \equiv \min\{n \geqslant n_0 : M_{n-n_0+1} M_{n-n_0+2} \dots M_n \gg 0\}$$

is finite with probability 1. Moreover, $x\Pi_T \gg 0$ for all $x \in S_+$ when $T < \infty$. Thus, for any $x \in S_+$ and $n \geqslant T$

$$\begin{aligned} |x\Pi_n| &= |(xM_1 \dots M_T)\, M_{T+1} \dots M_n| \\ &\geqslant d^{-\frac{1}{2}} \min_l (xM_1 \dots M_T)(l)\, \|M_{T+1} \dots M_n\| \\ &\geqslant d^{-\frac{1}{2}} (\|M_1 \dots M_T\|)^{-1} \min_l (xM_1 \dots M_T)(l)\, \|\Pi_n\|. \end{aligned}$$

Since also (by (2.18)) for any fixed $x \in S_+$

$$P\{\min_{n \leqslant T} |x\Pi_n| (\|\Pi_n\|)^{-1} > 0\} = 1,$$

(2.19) holds for any $x \in S_+$. Thus all hypotheses of Proposition 1 have been checked. (2.52) now follows from Theorem 1. Lastly we deduce the direct Riemann integrability of g from (2.53). We already know from Proposition 1 that for each $x \in S_+$

$$P\left\{\lim_{n\to\infty} \frac{1}{n} \log |x\Pi_n| = \alpha\right\} = 1. \tag{2.57}$$

Since we assumed $\alpha > 0$ we can find a k_0 such that for the coordinate vectors

$$e_i = (0, 0, \ldots, 1, 0, \ldots, 0) \in S_+, 1 \leqslant i \leqslant d,$$

(the non zero component of e_i is the i^{th} one)

$$P\{\log |e_i \Pi_m| \geqslant m k_0^{-1} + \tfrac{1}{2} \log d \;\text{ for all }\; m \geqslant k_0 \;\text{ and }\; 1 \leqslant i \leqslant d\} \geqslant \tfrac{1}{2}.$$

Also for $x \in S_+$

$$|x\Pi_m| \geqslant \max_l x(l) \min_i |e_i \Pi_m| \geqslant d^{-\frac{1}{2}} \min_i |e_i \Pi_m|. \tag{2.58}$$

We conclude from this that C_k is all of S_+ as soon as $k \geqslant k_0$ (see (2.6) for the definition of C_k). Thus (2.53) implies (2.7) and the ordinary Riemann integrability of $g(x, \cdot)$ on finite intervals for fixed x is implied by the continuity of g. Again (2.54) follows from Theorem 1. (2.55) is obtained if one takes $g(y, s) = 1$ for $-\log h \leqslant s \leqslant 0$ and 0 otherwise.(1)

We add one comment as to why

$$\lim_{n\to\infty} \frac{1}{n} \log \|M_1 \ldots M_n\| = \lim_{n\to\infty} \frac{1}{n} \log |xM_1 \ldots M_n|, \quad x \in S_+.$$

Clearly the left hand side is no less than the right hand side in this equation. On the other hand,

$$\|M_1 \ldots M_n\| \leqslant d^{\frac{1}{2}} \max_l |e_l M_1 \ldots M_n|$$

which together with (2.57) and (2.58) yields the desired equality. ■

THEOREM 3. *Let $M_1, M_2, \ldots$ be a sequence of independent $d \times d$ matrices, each distributed according to the same probability measure μ which is such that the group generated by* (2.51) *is dense in* $\mathbf{R}$. *Assume also that* (2.48)–(2.50) *are satisfied, but this time*

$$\lim_{n\to\infty} \frac{1}{n} \log \|M_1 \ldots M_n\| = \lim_{n\to\infty} \frac{1}{n} \log |xM_1 \ldots M_n| = \alpha < 0$$

a.e. $[P]$. *Assume in addition that there exists a $\varkappa_0 > 0$ for which*

(1) This g is not continuous so that strictly speaking this coice is not allowed for g in (2.54). However, following a common technique, we apply (2.54) to increasing and decreasing sequences of continuous functions which converge to the present g.

$$E\{\min_i (\sum_j M_1(i, j))\}^{\varkappa_0} \geq d^{\varkappa_0/2}, \tag{2.59}$$

and
$$E\|M_1\|^{\varkappa_0} \log^+ \|M_1\| < \infty. \tag{2.60}$$

Then there exists a $\varkappa_1 \in (0, \varkappa_0]$ and a continuous, strictly positive function r on S_+ such that

$$r(x) = \int \mu(dM)\, |xM|^{\varkappa_1} r((xM)^{\sim}) = E\,|xM_1|^{\varkappa_1} r((xM_1)^{\sim}), \quad x \in S_+. \tag{2.61}$$

In addition, for any bounded and jointly continuous function g: $S_+ \times (0, \infty) \to \mathbf{R}$ there is a finite constant $K = K(g)$ such that for $t \to \infty$(1) and $x \in S_+$ fixed

$$e^{\varkappa_1 t} E\{g(Z_x(t), W_x(t)); N_x(t) < \infty\} \to K(g) r(x). \tag{2.62}$$

When $g(z, s) > 0$ for all $(z, s) \in S_+ \times \mathbf{R}$ then $K(g) > 0$ and in particular there exists a $0 < K_1 < \infty$ such that for all $x \in S_+$ (2)

$$t^{\varkappa_1} P\{\max_n |xM_1 \dots M_n| > t\} \to K_1 r(x), \quad t \to \infty. \tag{2.63}$$

Remark 2. As the proof will show it suffices to take $e^{-\varkappa_1 t} g(x, t)$ bounded, instead of $g(x, t)$ itself.

Proof: Again this theorem will follow easily from Proposition 1 and Theorem 1 once we have found the desired $\varkappa_1$ and function r. This, however, is complicated and will be done in a number of separate steps.

Step 1. Define the linear operator $T_\varkappa$ on C, the space of continuous functions on S_+, by

$$T_\varkappa f(x) = E\,|xM_1|^\varkappa f((xM_1)^{\sim}), \quad f \in C, x \in S_+.$$

Its adjoint $T_\varkappa^*$ is a linear operator on the signed measures on S_+ and is determined by

$$\int T_\varkappa^* \nu(dx)\, f(x) = \int \nu(dx)\, T_\varkappa f(x) = \int \nu(dx)\, E\,|xM_1|^\varkappa f((xM_1)^{\sim}).$$

We show that for $0 \leq \varkappa \leq \varkappa_0$ there exists a probability measure $\nu_\varkappa$ on S_+ and a number $\varrho_\varkappa$, such that

$$0 < E[d^{-\frac{1}{2}} \min_i (\sum_j M_1(i, j))]^\varkappa \leq \varrho_\varkappa \leq E\|M_1\|^\varkappa \tag{2.64}$$

and
$$T_\varkappa^* \nu_\varkappa = \varrho_\varkappa \nu_\varkappa. \tag{2.65}$$

The existence of $\nu_\varkappa$ follows from Theorem 3.3 of [14] or Theorem 7 of [10]. We repeat the short proof, which will at the same time give us the estimate (2.64). We already showed in Theorem 2 that (2.48) and (2.49) imply (2.18). Moreover for any matrix m, the maps $x \to xm$

(1) See footnote p. 223.

(2) E. Arjas, Adv. Appl. Prob. *4* (1973) 258–270 can be used to obtain an expression for the Laplace transform of $\max_n |xM_1 \dots M_n|$ but it does not seem easy to obtain (2.63) from this.

and $x \to (xm)^{\sim}$ are continuous at each point where $xm \neq 0$. Thus $x \to (xM_1)^{\sim}$ is continuous with probability one, and from this one easily sees that $T_\varkappa$ takes C into C. As a matter of fact, if we put

$$\|f\| = \sup_{x \in S_+} |f(x)|, \quad f \in C,$$

then even

$$\|T_\varkappa\| = \sup_{\substack{\|f\|=1 \\ f \in C}} \|T_\varkappa f\| \leq \sup_{x \in S_+} E|xM_1|^\varkappa \leq E\|M_1\|^\varkappa$$
$$\leq E(1 + \|M_1\|^{\varkappa_0}), \quad 0 \leq \varkappa \leq \varkappa_0$$

so that $T_\varkappa$ is a continuous operator on C. It is also a positive operator, and if we put $e(x) \equiv 1$ on S_+, then by (2.18) $T_\varkappa e(x) > 0$ for all $x \in S_+$. It follows that

$$\tilde{T}_\varkappa \nu \equiv \left[\int T_\varkappa^* \nu(dx)\right]^{-1} T_\varkappa^* \nu = \left[\int \nu(dx)\, T_\varkappa e(x)\right]^{-1} T_\varkappa^* \nu$$

defines a continuous map from the set $\mathfrak{C}$ of probability measures on S_+ into itself, when $\mathfrak{C}$ is given the weak topology, i.e., ν_n converges to ν if and only if

$$\int_{S_+} f(x)\, \nu_n(dx) \to \int_{S_+} f(x)\, \nu(dx)$$

for every $f \in \mathfrak{C}$. $\mathfrak{C}$ is a compact convex set (see [14], [3], Sect. V.4 or [16], Sect. II.6) and by the Schauder-Tychonoff fixed point theorem ([3], Theorem V.10.5), $\tilde{T}_\varkappa$ has a fixed point $\nu_\varkappa$ in $\mathfrak{C}$. This proves (2.65) with

$$\varrho_\varkappa = \int \nu_\varkappa(dx)\, T_\varkappa e(x) = \int \nu_\varkappa(dx)\, E|xM_1|^\varkappa.$$

Clearly

$$\min_{x \in S_+} E|xM_1|^\varkappa \leq \varrho_\varkappa \leq E\|M_1\|^\varkappa.$$

This proves (2.64) since with probability 1 (see (2.49))

$$|xM_1| \geq d^{-\frac{1}{2}} \sum_j xM_1(j) \geq d^{-\frac{1}{2}} \sum_l x(l) \min_i \sum_j M_1(i,j) \geq d^{-\frac{1}{2}} \min_i \sum_j M_1(i,j) > 0.$$

We note that (2.64) and (2.59) imply

$$\varrho_0 = 1, \quad \varrho_{\varkappa_0} \geq 1. \tag{2.66}$$

Step 2. With $\varrho = \varrho_\varkappa$ as in Step 1, define

$$r(x,n) = r_\varkappa(x,n) = \varrho_\varkappa^{-n} E|x\Pi_n|^\varkappa = \varrho_\varkappa^{-n} T_\varkappa^n e(x), \quad n \geq 0, x \in S_+.$$

Let n_0 and $\tau \geq 0$ be such that

$$p \equiv P\{\Pi_{n_0}(i,j) \geqslant \tau \text{ for all } 1 \leqslant i,j \leqslant d\} > 0; \tag{2.67}$$

such an n_0 and τ exist because the set (2.51) is not empty. Then for all $x, y \in S_+, n \geqslant n_0$

$$\varrho_\varkappa^{-n_0} p\, d^{-\varkappa/2} \tau^\varkappa \leqslant r_\varkappa(x,n) \leqslant \varrho_\varkappa^{n_0} p^{-1} d^{\varkappa/2} \tau^{-\varkappa} \tag{2.68}$$

and for $x, y \in S_+, n \geqslant 0$,

$$|r_\varkappa(x,n) - r_\varkappa(y,n)| \leqslant (\varkappa+1)\, \varrho_\varkappa^{n_0} p^{-1} d^{\varkappa/2} \tau^{-\varkappa} |x-y|^{\min(1,\varkappa)}. \tag{2.69}$$

To prove these estimates observe first that

$$T_\varkappa^n f(x) = E\{|x\Pi_n|^\varkappa f((x\Pi_n)^\sim)\}, \quad x \in S_+, f \in C \tag{2.70}$$

(use induction on n) so that

$$\begin{aligned} \varrho_\varkappa^n = \int ((T_\varkappa^*)^n \nu_\varkappa)(dz) &= \int \nu_\varkappa(dz)\, E\,|z\Pi_n|^\varkappa \\ &\geqslant \min_{z \in S_+} E\{|z\Pi_n|^\varkappa ; \Pi_{n_0}(i,j) \geqslant \tau \text{ for all } 1 \leqslant \mathrm{i}, \mathrm{j} \leqslant d\}. \end{aligned} \tag{2.71}$$

But if $\qquad \Pi_{n_0}(i,j) \geqslant \tau$ for all $1 \leqslant i,j \leqslant d$,

then for $n \geqslant n_0$

$$\begin{aligned} |z\Pi_n| &\geqslant d^{-\frac{1}{2}} \sum_{i,j} z\Pi_{n_0}(i)\, M_{n_0+1} \ldots M_n(i,j) \\ &\geqslant d^{-\frac{1}{2}} \tau \sum_{i,j} M_{n_0+1} \ldots M_n(i,j) \geqslant d^{-\frac{1}{2}} \tau \|M_{n_0+1} \ldots M_n\|. \end{aligned} \tag{2.72}$$

By means of the independence of Π_{n_0} and $M_{n_0+1} \ldots M_n$ and (2.67) we conclude from (2.71) and (2.72) that

$$\varrho_\varkappa^n \geqslant p\, d^{-\varkappa/2} \tau^\varkappa E\,\|M_{n_0+1} \ldots M_n\|^\varkappa \tag{2.73}$$

or

$$\varrho_\varkappa^{n+n_0} \geqslant p\, d^{-\varkappa/2} \tau^\varkappa E\,\|\Pi_n\|^\varkappa, \quad n \geqslant 0. \tag{2.74}$$

This implies the right hand inequality in (2.68), because

$$E\,|x\Pi_n|^\varkappa \leqslant E\|\Pi_n\|^\varkappa.$$

If $0 \leqslant \varkappa \leqslant 1$ (2.69) follows from the inequality

$$\big||\alpha|^\varkappa - |\beta|^\varkappa\big| \leqslant |\alpha - \beta|^\varkappa, \quad \alpha, \beta \in \mathbf{R}, 0 \leqslant \varkappa \leqslant 1,$$

which implies

$$|E\,|x\Pi_n|^\varkappa - E\,|y\Pi_n|^\varkappa| \leqslant E\,|(x-y)\,\Pi_n|^\varkappa \leqslant |x-y|^\varkappa E\|\Pi_n\|^\varkappa.$$

Similarly, for $1 \leqslant \varkappa \leqslant \varkappa_0$ (2.69) follows from

$$||\alpha|^{\varkappa} - |\beta|^{\varkappa}| \leqslant \varkappa |\alpha - \beta| \max (|\alpha|^{\varkappa-1}, |\beta|^{\varkappa-1}).$$

As for the left hand inequality in (2.68), we have as in (2.71)–(2.73)

$$E|x\Pi_n|^{\varkappa} \geqslant p d^{-\varkappa/2} \tau^{\varkappa} E\|\Pi_{n-n_0}\|^{\varkappa}$$

and clearly (see (2.71)

$$\varrho_{\varkappa}^{n} = \varrho_{\varkappa}^{n_0} \int \nu(dz)\, E|z\Pi_{n-n_0}|^{\varkappa} \leqslant \varrho_{\varkappa}^{n_0} E\|\Pi_{n-n_0}\|^{\varkappa}. \tag{2.75}$$

Step 3. There exists a sequence $n_l \uparrow \infty$ such that

$$r_{\varkappa}(x) \equiv \lim_{l\to\infty} \frac{1}{n_l+1} \sum_{j=0}^{n_l} r_{\varkappa}(x, j) \text{ exists, } x \in S_+. \tag{2.76}$$

$r_{\varkappa}(\cdot)$ also satisfies (2.68) and (2.69) (with $r_{\varkappa}(x, n)$ replaced by $r_{\varkappa}(x)$) as well as

$$\varrho_{\varkappa} r_{\varkappa}(x) = T_{\varkappa} r_{\varkappa}(x) = E|xM_1|^{\varkappa} r_{\varkappa}((xM_1)^{\sim}). \tag{2.77}$$

The existence of the limit in (2.76) for suitable n_l follows from the Arzelà-Ascoli theorem ([3], Theorem IV.6.7) because the family of functions

$$x \to \frac{1}{n+1} \sum_{j=0}^{n} r_{\varkappa}(x, j), \quad n \geqslant 0,$$

on S_+ is equicontinuous (see (2.69)). In fact, we see that the convergence in (2.76) is uniform because S_+ is compact; consequently

$$T_{\varkappa} r = T_{\varkappa}\left(\lim_{l\to\infty} \frac{1}{n_l+1} \sum_{j=0}^{n_l} \varrho_{\varkappa}^{-j} T_{\varkappa}^{j} e\right)$$

$$\varrho_{\varkappa} \lim_{l\to 0} \frac{1}{n_l+1} \sum_{j=0}^{n_l} \varrho_{\varkappa}^{-j-1} T_{\lambda}^{j+1} e = \varrho_{\varkappa} r_{\varkappa},$$

which is just (2.77). Clearly $r_{\varkappa}(\cdot)$ also satisfies (2.68) and (2.69).

Step 4. Here we find our desired $r(\cdot)$ by showing that $\varrho_{\varkappa_1} = 1$ for some $\varkappa_1 \in (0, \varkappa_0]$. $r_{\varkappa_1}$ then has all the desired properties. More precisely, we show that $\log \varrho_{\varkappa}$ is a convex function of $\varkappa$ on $[0, \varkappa_0]$, which is continuous on $(0, \varkappa_0]$ and such that $\log \varrho_{\varkappa} < 0$ on $(0, \delta_1)$ for some $\delta_1 > 0$. In view of (2.66) this will be more than sufficient.

We start with the formula

$$\log \varrho_{\varkappa} = \lim_{n\to\infty} \frac{1}{n} \log E\|\Pi_n\|^{\varkappa}$$

which is immediate from (2.74) and (2.75). But one easily checks (by differentiating twice and appealing to Schwarz's inequality) that $n^{-1} \log E\|\Pi_n\|^{\varkappa}$ is a convex function of $\varkappa$ on $[0, \varkappa_0]$, and hence, so is its limit $\log \varrho_{\varkappa}$. This already shows that $\log \varrho_{\varkappa}$ is a continuous function of $\varkappa$ on $(0, \varkappa_0)$ and that there can only be a discontinuity at $\varkappa_0$ if $\lim_{\varkappa \uparrow \varkappa_0} \varrho_{\varkappa} < \varrho_{\varkappa_0}$. To show that this is impossible we use (2.73). For any $\varepsilon > 0$ we can first pick an $n \geqslant n_0$ such that

$$\{p d^{-\varkappa_0/2} \tau^{\varkappa_0} E\|\Pi_{n-n_0}\|^{\varkappa_0}\}^{1/n} \geqslant \varrho_{\varkappa_0}(1-\varepsilon)$$

and then $\varkappa$ so close to $\varkappa_0$ that for this n

$$p d^{-\varkappa/2} \tau^{\varkappa} E\|\Pi_{n-n_0}\|^{\varkappa} \geqslant (1-\varepsilon) p d^{-\varkappa_0/2} \tau^{\varkappa_0} E\|\Pi_{n-n_0}\|^{\varkappa_0}.$$

For such a $\varkappa$ one has $\varrho_{\varkappa} \geqslant \varrho_{\varkappa_0}(1-\varepsilon)^2$ so that $\varrho_{\varkappa}$ is indeed continuous at $\varkappa_0$.

Lastly, we observe that by Theorem 2

$$P\left\{\lim_{n\to\infty} \frac{1}{n} \log \|\Pi_n\| = \alpha\right\} = 1,$$

and that $n^{-1} \log^+ \|\Pi_n\|$ is uniformly integrable. Indeed,

$$0 \leqslant \frac{1}{n} \log^+ \|\Pi_n\| \leqslant \frac{1}{n} \sum_{i=1}^{n} \log^+ \|M_i\|$$

while by the L_1 form of the strong law of large numbers (see [8], p. 22)

$$E\left|\frac{1}{n} \sum_{i=1}^{n} \log^+ \|M_i\| - E \log^+ \|M_1\|\right| \to 0 \quad (n \to \infty).$$

Thus, by Fatou's lemma

$$\lim_{n\to\infty} E \frac{1}{n} \log \|\Pi_n\| \leqslant \alpha < 0.$$

Now fix n_1 such that $E \log \|\Pi_{n_1}\| < 0$. Another applictaion of Fatou's lemma shows that

$$\left[\frac{d}{d\varkappa} E\|\Pi_{n_1}\|^{\varkappa}\right]_{\varkappa=0} = \lim_{\varkappa \downarrow 0} \frac{E\|\Pi_{n_1}\|^{\varkappa} - 1}{\varkappa} \leqslant E \log \|\Pi_{n_1}\| < 0.$$

Since $E\|\Pi_{n_1}\|^0 = 1$ this shows that for some $\delta_1 > 0$ and $0 < \varkappa \leqslant \delta_1$,

$$E\|\Pi_{n_1}\|^{\varkappa} < 1 \quad \text{and} \quad \log \varrho_{\varkappa} = \lim_{l\to\infty} \frac{1}{l n_1} \log E\|\Pi_{l n_1}\|^{\varkappa} \leqslant \frac{1}{n_1} \log E\|\Pi_{n_1}\|^{\varkappa} < 0 \tag{2.79}$$

as desired.

Step 5. In this step we complete the proof of Theorem 3 by showing that Proposition 1 applies with P_x defined by (2.16) and (2.17) with $\varkappa_1$ for $\varkappa$ and $r_{\varkappa_1}$ for r. We already showed that (2.18) holds and that (2.20) generates a dense group in $\mathbf{R}$ in the proof of Theorem 2, and (2.21) is implied by (2.60). $r_{\varkappa_1}(\cdot)$ is continuous and bounded away from 0 and ∞ and satisfies (2.15) by step 3. Finally the proof of (2.19) given in Theorem 2 still goes through, provided we can show

$$P_x\{T<\infty\}=1. \tag{2.80}$$

(T is defined just below (2.56).) For this purpose we introduce the events

$$E_k=\{M_{k+1}\dots M_{k+n_0}(i,j)\geqslant\tau \quad \text{for all} \quad 1\leqslant i,j\leqslant d\},$$

for the τ of (2.67). Let $x\in S_+$ and $m_i\geqslant 0$ positive matrices, and use the abbreviation

$$r(x,m_i,M_i)=r_{\varkappa_1}((xm_1\dots m_k M_{k+1}\dots M_{k+n_0}\, m_{k+n_0+1}\dots m_n)^{\sim}).$$

We claim that for $k\leqslant n-n_0$

$$\begin{aligned}\int_{E_k}\dots\int &\mu(dM_{k+1})\dots\mu(dM_{k+n_0})\, r(x,m_i,M_i)\,|xm_1\dots m_k M_{k+1}\dots M_{k+n_0}m_{k+n_0+1}\dots m_n|^{\varkappa_1}\\ &\geqslant pd^{-\varkappa_1/2}\tau^{\varkappa_1}\{E\|\Pi_{n_0}\|^{\varkappa_1}\}^{-1}\min_{y\in S_+} r_{\varkappa_1}(y)\,\{\max_{z\in S_+} r_{\varkappa_1}(z)\}^{-1}\\ &\quad\times\int\dots\int\mu(dM_{k+1})\dots\mu(dM_{k+n_0})\, r(x,m_i,M_i)\,|xm_1\dots m_k M_{k+n_0}m_{k+n_0+1}\dots m_n|^{\varkappa_1}.\end{aligned} \tag{2.81}$$

To see this, observe that the integrand in the right hand side of (2.81) is at most

$$\max_{z\in S_+} r_{\varkappa_1}(z)\,|xm_1\dots m_k|^{\varkappa_1}\,\|M_{k+1}\dots M_{k+n_0}\|^{\varkappa_1}\,\|m_{k+n_0+1}\dots m_n\|^{\varkappa_1},$$

whereas the integrand on the left hand side is at least

$$\begin{aligned}&\min_{y\in S_+} r_{\varkappa_1}(y)\,\{d^{-\frac{1}{2}}\sum_i xm_1\dots m_k M_{k+1}\dots M_{k+n_0}m_{k+n_0+1}\dots m_n(i)\}^{\varkappa_1}\\ &\geqslant\min_{y\in S_+} r_{\varkappa_1}(y)\,d^{-\varkappa_1/2}\tau^{\varkappa_1}\{\sum_l xm_1\dots m_k(l)\sum_{j,i} m_{k+n_0+1}\dots m_n(j,i)\}^{\varkappa_1}\\ &\geqslant\min_{y\in S_+} r_{\varkappa_1}(y)\,d^{-\varkappa_1/2}\tau^{\varkappa_1}\,|xm_1\dots m_k|^{\varkappa_1}\,\|m_{k+n_0+1}\dots m_n\|^{\varkappa_1}.\end{aligned}$$

Thus (2.81) follows from $P\{E_k\}=p$ (see (2.67)). Now put

$$q=1-pd^{-\varkappa_1/2}\tau^{\varkappa_1}\{E\,\|\Pi_{n_0}\|^{\varkappa_1}\}\min_{y\in S_+} r_{\varkappa_1}(y)\,\{\max_{z\in S_+} r_{\varkappa_1}(z)\}^{-1}.$$

Then, one easily sees from (2.17) and (2.81) that

$P_x\{E_k$ does not occur $|M_i, i \leqslant k$ and $k+n_0<i\leqslant n\}\leqslant q$. Consequently

$$P_x\{T>ln_0\}\leqslant P_x\{E_{jn_0} \quad \text{does not occur for any} \quad 0\leqslant j<l\}<q^l,$$

which implies (2.80).

Thus Proposition 1 applies and we shall be able to use Theorem 1 once we show that the α in (2.23) is strictly positive, or that for some x $\lim n^{-1}V_n>0$ a.e. $[P_x]$. But by (2.79) there exist some $0<\delta<1$, $\gamma>0$ and constant $K_2<\infty$ for which

$$E\|\Pi_n\|^\delta\leqslant\{E\|\Pi_{n_1}\|^\delta\}^{[nn_1^{-1}]}E\|\Pi_{n-n_1[nn_1^{-1}]}\|^\delta\leqslant K_2e^{-\gamma(2+\delta)n}.$$

Consequently, for any $x\in S_+$

$$P\{|x\Pi_n|\geqslant e^{-\gamma n}\}\leqslant P\{\|\Pi_n\|^\delta\geqslant e^{-\gamma\delta n}\}\leqslant K_2e^{-2\gamma n} \tag{2.82}$$

and also

$$\begin{aligned}
&P_x\{|x\Pi_n|\leqslant e^{\gamma\varkappa_1^{-1}n}\}\\
&=\frac{1}{r_{\varkappa_1}(x)}E\{|x\Pi_n|^{\varkappa_1}r_{\varkappa_1}((x\Pi_n)^\sim);|x\Pi_n|\leqslant e^{\gamma\varkappa_1^{-1}n}\}\\
&\leqslant\frac{1}{r_{\varkappa_1}(x)}\max_y r_{\varkappa_1}(y)\,[e^{-\gamma\varkappa_1 n}+E\{|x\Pi_n|^{\varkappa_1};e^{-\gamma n}\leqslant|x\Pi_n|\leqslant e^{\gamma\varkappa_1^{-1}n}\}]\\
&\leqslant\frac{1}{r_{\varkappa_1}(x)}\max_y r_{\varkappa_1}(y)\,[e^{-\gamma\varkappa_1 n}+e^{\gamma n}K_2e^{-2\gamma n}].
\end{aligned}$$

It now follows from the Borel-Cantelli lemma that $V_n=\log|x\Pi_n|\geqslant n\gamma\varkappa_1^{-1}$ eventually, a.e. $[P_x]$, so that I.2 holds as well. We may now apply (2.10) to the function

$$g^*(x,t)\equiv[r_{\varkappa_1}(x)]^{-1}e^{-\varkappa_1 t}g(x,t),$$

which is bounded and continuous in (x,t). This yields the existence of

$$K(g)\equiv\lim_{t\to\infty}E_x\{[r(Z(t))]^{-1}e^{-\varkappa_1 W(t)}g(Z(t),\ W(t))\}$$

for some $K(g)$, independent of x. Because

$$\begin{aligned}
&E_x\{[r(Z(t))]^{-1}e^{-\varkappa_1 W(t)}g(Z(t),W(t))\}\\
&=\frac{1}{r(x)}\sum_{n=0}^{\infty}E\{|x\Pi_n|^{\varkappa_1}r((x\Pi_n)^\sim)\,(r((x\Pi_n)^\sim))^{-1}e^{-\varkappa_1[\log|x\Pi_n|-t]}\\
&\qquad\qquad\times g((x\Pi_n)^\sim,\log|x\Pi_n|-t);N_x(t)=n\}\\
&=\frac{e^{\varkappa_1 t}}{r(x)}E\{g(Z_x(t),W_x(t));N_x(t)<\infty\},
\end{aligned}$$

this proves (2.62). (2.63) is obtained by specializing g to $g(x, t) \equiv 1$ (and replacing t by $\log t$). The explicit formula (see (2.10))

$$K(g) = \alpha^{-1} \int \psi(dy) \int_{S \times (0,\infty)} P_y\{X_{N(0)} \in dz, V_{N(0)} \in d\lambda\} \int_{0<s\leqslant\lambda} g^*(z, s)\, ds$$

immediately shows $K(g) > 0$ whenever $g(z, s) > 0$ for all (z, s), so that the proof is complete. ∎

EXTENSION OF THEOREM 3. *Assume that the hypotheses of Theorem 3 are satisfied and that g: $S_{d-1} \times (0, \infty) \to \mathbf{R}$ is bounded and (jointly) continuous. Then for each $x \in S_{d-1}$*

$$\lim_{t\to\infty} e^{\varkappa_1 t} E\{g(Z_x(t), W_x(t)); N_x(t) < \infty\} \quad \textit{exists and is finite.} \tag{2.83}$$

(Note that the definitions (2.45)–(2.47) for N_x, W_x and Z_x need no change for $x \in S_{d-1}/S_+$. Of course (2.83) is already asserted by Theorem 3 if $x \in S_+$.)

We shall not prove this extension. It is proved by a reduction of (2.83) for general $x \in S_{d-1}$ to (2.62) for $x \in S_+$. This is done by means of a generalization of Lemma 3 in [5]. This lemma states that the directions of the rows of Π_n for large n differ very little (with high probability). If all rows of Π_n had exactly the same direction for some n_1, then Π_{n_1} would be of the form $\Pi_{n_1}(i, j) = a(i)\, b(j)$ for some d-vectors $a \geqslant 0$, $b \geqslant 0$. But then also, for any $x \in S_{d-1}$ and $n \geqslant n_1$,

$$x\Pi_n = (\sum_l x(l)\, a(l))\, |b|\, \tilde{b} M_{n_1+1} \ldots M_n.$$

Thus, after time n_1 the sequence $x\Pi_n$ is a constant multiple of $\tilde{b} M_{n_1+1} \ldots M_n$. If the factor

$$\sum_l x(l)\, a(l)\, |b| \tag{2.84}$$

is zero, then $|x\Pi_n|$ will not exceed large values of t at all. If, however, (2.84) is not zero, then given n_1, a, b, the conditional probability of $\max |x\Pi_n| > \exp t$ equals for large t

$$P\{\max |\tilde{b}\Pi_n| > |\sum x(l) a(l)|^{-1} |b|^{-1} e^t\},$$

whose asymptotic behavior we know from (2.63), because $\tilde{b} \in S_+$. Similarly the conditional expectation of $g(Z_x(t), W_x(t))$ over $N_x(t) < \infty$ would reduce to

$$E\{g(\pm Z_{\tilde{b}}(t^*), W_{\tilde{b}}(t^*); N_{\tilde{b}}(t^*) < \infty\}, \tag{2.85}$$

where $$t^* = t - \log(|\Sigma x(l)\, a(l)|\, |b|),$$

and the sign in front of $Z_{\tilde{b}}(t^*)$ in (2.85) is the sign of (2.84). The burden of the proof is to estimate the errors which arise because the rows of Π_{n_1} only have approximately the same direction for large n_1, rather than exactly the same direction.

3. Solutions of random difference equations; positive coefficients

In this section we study the limit distribution of the solution Y_n of the difference equation

$$Y_n = M_n Y_{n-1} + Q_n, \quad n \geqslant 1, \tag{3.1}$$

where the M_n are positive $d \times d$ matrices and the Y_n and Q_n are d dimensional column vectors. For given Y_0 the solution of (3.1) is of course

$$Y_n = Q_n + M_n Q_{n-1} + \dots + (M_n \dots M_2) Q_1 + (M_n \dots M_1) Y_0.$$

We assume throughout that the $\{M_n, Q_n\}_{n \geqslant 1}$ are independent, identically distributed. In this situation $Y_n - (M_n \dots M_1) Y_0$ has the same distribution as

$$R_n \equiv \sum_{k=1}^{n} M_1 \dots M_{k-1} Q_k$$

(the term corresponding to $k=1$ is Q_1). If

$$\frac{1}{n} \log \|M_1 \dots M_n\| \to \alpha < 0 \quad \text{w.p.1},$$

and

$$E|Q_1|^\gamma < \infty \quad \text{for some} \quad \gamma > 0,$$

then

$$P\{|Q_n| \leqslant e^{-\frac{1}{2}\alpha n} \text{ eventually}\} = 1$$

and R_n converges w.p.1 to

$$R = \sum_{k=1}^{\infty} M_1 \dots M_{k-1} Q_k.$$

Therefore, under the conditions of the theorem below the distribution of Y_n in (3.1) con verges to that of R for every fixed Y_0. The burden of Theorem 4 is that if the M_n are positive matrices, then this limit distribution is in the domain of attraction of a stable law.

THEOREM 4. *Let $\{M_n, Q_n\}_{n \geqslant 1}$ be independent identically distributed, and assume that the distribution μ of M_1 satisfies the conditions of Theorem 3 (including the condition $\alpha < 0$). If in addition*

$$P\{Q_1 = 0\} < 1, \tag{3.2}$$

$$E|Q_1|^{\varkappa_1} < \infty, \tag{3.3}$$

and[1]

$$P\{Q_1 \geqslant 0\} = 1, \tag{3.4}$$

then for each row vector $x \in S_{d-1}$

$$\lim_{t \to \infty} t^{\varkappa_1} P\{xR \geqslant t\} \text{ exists and is finite.} \tag{3.5}$$

---

[1] For column vectors q, the notation $q \geqslant 0$ again means that all components of q are positive.

There exists a $0<K_3<\infty$ *such that the limit in* (3.5) *equals* $K_3 r(x)$ *for all* $x\in S_+$. *In particular, the limit in* (3.5) *is strictly positive for* $x\in S_+$.

COROLLARY. *If the conditions of Theorem 4 hold with* $\varkappa_1 \neq 2$, *then* R *is in the domain of normal attraction of a* (*d-dimensional*) *stable law of index* $\min(\varkappa_1, 2)$. *If* $\varkappa_1=2$ *and* $R_1, R_2 \dots$ *are independent random vectors each with the distribution of* R, *then*

$$\frac{1}{\sqrt{n \log n}} \sum_1^n (R_i - ER)$$

converges in law to a normal distribution with zero mean on $\mathbf{R}^d$.

Remark 3. Even though we insist here on positive M_n (see (2.48)), it is not necessary to have Q_n positive. (3.5) and the corollary remain valid if (3.4) is replaced by the condition that there exist possible points (m, q') and (m, q'') for (M_1, Q_1), and $m_i \in \operatorname{supp} \mu$, $i\leqslant n_0$, such that $\pi = m_1 \dots m_{n_0} \gg 0$ and such that $b(\pi)(q'-q'')\neq 0$, where $b(\pi)$ is the left eigenvector of π corresponding to its Frobenius eigenvalue $\varrho(\pi)$. In this situation the limit in (3.5) will still be strictly positive for some $x\in S_{d-1}$, but not necessarily for all $x\in S_+$.

Proof: Again we shall break up the proof into a number of steps. The first 5 steps show in essence that $xR>t$ for large t occurs only if $|xM_1 \dots M_n| > \delta t$ for some n and suitable small $\delta>0$. This will allow us to apply Theorem 3 in step 6. We repeat some of the most frequent conventions from section 2.

$$\Pi_n = M_1 \dots M_n,$$
$$\tilde{x} = |x|^{-1} x \quad \text{for} \quad 0 \neq x \in \mathbf{R}^d,$$
$$e_0 = (d^{-\frac{1}{2}}, \dots, d^{-\frac{1}{2}}) \quad (\text{note } e_0 \in S_+).$$

Note that for any column vector q with components $q(i)\geqslant 0$

$$d^{\frac{1}{2}}|e_0 q| = \sum |q(i)| \geqslant |q| = \left\{\sum_{i+1}^{d} q^2(i)\right\}^{\frac{1}{2}} \geqslant |e_0 q|.$$

In addition we introduce

$$R^n = \sum_{k=n+1}^{\infty} M_{n+1} \dots M_{k-1} Q_k$$

(the term corresponding to $k=n+1$ is Q_{n+1}). We often use the relation

$$R = R_n + \Pi_n R^n; \tag{3.6}$$

and the fact that R^n is independent of (R_n, Π_n) and has the same distribution as R itself.

Step 1. There exists a $K_4 > 0$ such that

$$P\{|R|>t\} \geqslant K_4 t^{-\varkappa_1}, \quad t \geqslant 1. \tag{3.7}$$

This will be immediate from Theorem 3 because all the Q_n and M_n are positive (see (3.4) and (2.48)). Moreover, for suitable $\tau>0$ and n_0 (2.67) holds, whereas for some $\tau_1>0$ and i_0, by (3.2),

$$P\{Q_1(i_0) \geqslant \tau_1\} > 0.$$

If $|e_0 \Pi_n| > (\tau\tau_1)^{-1} t$ and $M_{n+1} \ldots M_{n+n_0}(i,j) \geqslant \tau$ for all i, j and $Q_{n+n_0+1}(i_0) \geqslant \tau_1$, then

$$|R| \geqslant |e_0 R| \geqslant |e_0 \Pi_n M_{n+1} \ldots M_{n+n_0} Q_{n+n_0+1}| \geqslant |e_0 \Pi_n| \tau\tau_1 > t$$

so that

$$\begin{aligned} P\{|R|>t\} \geqslant P\,&\{\text{there is an } n \text{ with } |e_0\Pi_n| > (\tau\tau_1)^{-1}t, \\ &M_{n+1} \ldots M_{n+n_0}(i,j) \geqslant \tau \quad \text{for all} \quad i,j,\, Q_{n+n_0+1}(i_0) \geqslant \tau_1\} \\ &\geqslant P\{\max_n |e_0 \Pi_n| > (\tau\tau_1)^{-1} t\}\, p\, P\{Q_1(i_0) \geqslant \tau_1\} \geqslant K_4 t^{-\varkappa_1} \end{aligned} \tag{3.8}$$

for some $K_4 > 0$ (see (2.63) and (2.67)).

Step 2. For some $K_5 > 0$ one has

$$P\{|R|>2t\} \geqslant K_5 P\{|R|>t\}. \tag{3.9}$$

This is almost immediate from (3.7) and the positivity of R_n, because, essentially as in (3.8)

$$\begin{aligned} P\{|R|>2t\} &\geqslant P\{e_0 \Pi_{n+n_0} R^{n+n_0} > 2t \quad \text{for some } n\} \\ &\geqslant P\,\{\text{for some } n\, |e_0\Pi_n| > 2\tau^{-1}, M_{n+1} \ldots M_{n+n_0}(i,j) \geqslant \tau \\ &\qquad \text{for all } i, j,\ |R^{n+n_0}| > t\} \geqslant K_5 P\{|R|>t\} \end{aligned}$$

with $K_5 > 0$.

Step 3. Define the ladder indices ξ_i by

$$\xi_0 = 0, \quad \xi_{i+1} = \min\{n > \xi_i \colon \|M_{\xi_i+1} \ldots M_n\| > 1\}.$$

We take $\xi_{i+1} = \infty$, when $\xi_i = \infty$ or when no n exist which satisfies the condition in the definition of ξ_{i+1}, and we put

$$\zeta = \max\{i \colon \xi_i < \infty\}.$$

Then, for every $\varepsilon > 0$ there exists a $k = k(\varepsilon)$ such that for all $x \in S_{d-1}$ and $t > 0$

$$P\{\zeta \geqslant k \quad \text{and} \quad |xR_n| > \varepsilon t \quad \text{for some} \quad n \leqslant \xi_{\zeta-k}\} \leqslant \varepsilon P\{|R|>t\}. \tag{3.10}$$

To prove (3.10), let

$$T(x, s) = \min\{n \geq 1\colon |xR_n| > s\} \quad (= \infty \text{ if no such } n \text{ exists}). \tag{3.11}$$

Then the left hand side of (3.10) is at most

$$\sum_{i,l=1}^{\infty} \int P\{\xi_{i-1} < T(x, \varepsilon t) \leq \xi_i = l\}\, P\{\xi_{i+k} < \infty \mid \xi_i = l\}. \tag{3.12}$$

But
$$\begin{aligned} P\{\xi_{i+k} < \infty \mid \xi_i = l\} = P\{&\text{there are at least } k \text{ finite ladder} \\ &\text{indices in the sequence } \|M_{l+1} \dots M_n\|, \quad n > l\} \\ \leq P\{&\|\Pi_n\| > 1 \quad \text{for some } n \geq k\}, \end{aligned}$$

and consequently the left hand side of (3.10) is at most

$$\begin{aligned} &P\{T(x, \varepsilon t) < \infty\} \cdot P\{\|\Pi_n\| > 1 \quad \text{for some } n \geq k\} \\ &\leq P\{|R_n| > \varepsilon t \quad \text{for some } n\} P\{\|\Pi_n\| > 1 \quad \text{for some } n \geq k\}. \end{aligned} \tag{3.13}$$

Since $|R| \geq |R_n|$ the first factor in the right hand side of (3.13) is at most

$$P\{|R| > \varepsilon t\} \leq (K_5)^{-1+\frac{\log \varepsilon}{\log 2}} P\{|R| > t\} \quad (\text{see } (3.9))$$

and the second factor can be made arbitrarily small by taking k large, because

$$\frac{1}{n} \log \|\Pi_n\| \to \alpha < 0 \quad \text{w.p.1} \tag{3.14}$$

(see Theorems 2 and 3). (3.10) is immediate from this.

Step 4. For every $\varepsilon > 0$ there exists a $\delta_0(\varepsilon) > 0$ and $t_0(\varepsilon) < \infty$ such that for all $x \in S_{d-1}$, $\delta \leq \delta_0(\varepsilon)$, $t \geq t_0(\varepsilon)$

$$P\{\text{there exists an } n \text{ with } |xR_n| > t \text{ but } \max_{l \leq n-1} |x\Pi_l| \leq \delta t\} \leq \varepsilon P\{|R| > t\}. \tag{3.15}$$

To prove (3.15) we note first that by (3.10)

$$P\left\{\zeta \geq k \text{ and } |xR_n| > \frac{t}{2} \text{ for some } n \leq \xi_{\zeta-k}\right\} \leq \frac{\varepsilon}{2} P\{|R| > t\}$$

for a suitable $k=k(\varepsilon)$. Therefore it suffices to prove that for any fixed k and $\delta_1=\delta(\varepsilon)>0$ sufficiently small and $t_0(\varepsilon)$ sufficiently large

$$P\{\zeta<k, T(x,t)<\infty\}\leqslant\frac{\varepsilon}{3}P\{|R|>t\},\quad t\geqslant t_0(\varepsilon), \tag{3.16}$$

as well as

$$P\left\{T(x,t)<\infty,\ \max_{l<T(x,t)}|x\Pi_l|\leqslant\delta_1 t,\quad \zeta\geqslant k,\quad \max_{n\leqslant\xi_{\zeta-k}}|xR_n|\leqslant\frac{t}{2}\right\}$$
$$\leqslant\frac{\varepsilon}{3}P\{|R|>t\},\ t\geqslant t_0(\varepsilon). \tag{3.17}$$

Let us fix x and k and abbreviate $\xi_{\zeta-k}$ to merely ξ for the remainder of this step. Then, when the event between braces in the left hand side of (3.17) occurs one has

$$T(x,t)>\xi\quad \text{(see (3.11))}$$

and hence necessarily

$$|x\Pi_\xi|\leqslant\delta_1 t,\ |xR_\xi|\leqslant\frac{t}{2},$$

$$\left|x\Pi_\xi\sum_{l=\xi+1}^{T(x,t)}M_{\xi+1}\dots M_{l-1}Q_l=|xR_{T(x,t)}-xR_\xi|>t-\frac{t}{2}=\frac{t}{2}.\right. \tag{3.18}$$

In turn, (3.18) is only possible if

$$|x\Pi_\xi|\,|R^\xi|=|x\Pi_\xi|\sup_{n>\xi}\left|\sum_{l=\xi+1}^{n}M_{\xi+1}\dots M_{l-1}Q_l\right|>\frac{t}{2}. \tag{3.19}$$

Thus, the left hand side of (3.17) is bounded by

$$P\{0<|x\Pi_\xi|\leqslant\delta_1 t\quad\text{and}\quad(3.19)\text{ occurs}\}, \tag{3.20}$$

and we shall show that (3.20) is at most $3^{-1}\varepsilon P\{|R|>t\}$. The same method which estimates (3.20) will handle (3.16) (actually (3.16) follows from (3.22)) so that we concentrate on (3.20). We break up (3.20) according to the value, i say, of $\zeta-k$. This shows that (3.20) equals

$$\sum_{i=0}^{\infty}\int_{0<s\leqslant\delta_1 t}P\Big\{\xi_i<\infty,\text{ there are exactly }k\text{ ladder indices in the sequence}$$
$$\|M_{\xi_i+1}\dots M_n\|,\ n>\xi_i,\ |x\Pi_{\xi_i}|\in ds\text{ and }s|R^{\xi_i}|>\frac{t}{2}\Big\}$$
$$=\sum_{i=0}^{\infty}\int_{0<s\leqslant\delta_1 t}P\{\xi_i<\infty,|x\Pi_{\xi_i}|\in ds\}\,P\left\{s|R|>\frac{t}{2},\ \zeta=k\right\}. \tag{3.21}$$

To estimate the sum of the measures appearing in the right hand side of (3.21) we pick n_2 such that

$$p_2 \equiv P\{\|\Pi_n\| < 1 \text{ for all } n \geqslant n_2\} > 0;$$

this is possible by virtue of (3.14). Clearly, if $\xi_i < \infty$ but

$$\|M_{\xi_i+1} \dots M_{\xi_i+n}\| < 1 \text{ for all } n \geqslant n_2$$

then $\xi_{i+n_2+1} = \infty$ or $\zeta \leqslant i + n_2$. Therefore

$$\begin{aligned}
&p_2 \sum_{i=0}^{\infty} P\{\xi_i < \infty, |x\Pi_{\xi_i}| \geqslant s\} \\
&= \sum_{i=0}^{\infty} P\{\xi_i < \infty, \|M_{\xi_i+1} \dots M_{\xi_i+n}\| < 1 \text{ for } n \geqslant n_2, |x\Pi_{\xi_i}| \geqslant s\} \\
&\leqslant \sum_{i=0}^{\infty} P\{\xi_i < \infty, \zeta \leqslant i + n_2, \max_n |x\Pi_n| \geqslant s\} \\
&\leqslant (n_2+1) \sum_{j=0}^{\infty} P\{\xi = j, \max_n |x\Pi_n| \geqslant s\} = (n_2+1) P\{\max_n |x\Pi_n| > s\} \\
&\leqslant (n_2+1) P\{\max_n |e_0 \Pi_n| \geqslant d^{-\frac{1}{2}} s\} \leqslant K_6 s^{-\varkappa_1}
\end{aligned}$$

for some $K_6 < \infty$, independent of x (see (2.63)). Therefore, if we replace

$$P\left\{s|R| > \frac{t}{2}, \zeta = k\right\}$$

by

$$\int_{2su>t} P\{|R| \in du, \zeta = k\}$$

and integrate over u first in the right hand side of (3.21) we get at most

$$\begin{aligned}
\int_{u \geqslant (2\delta_1)^{-1}} P\{|R| \in du, \zeta = k\} \sum_{i=0}^{\infty} P\{\xi_i < \infty, |x\Pi_{\xi_i}| \geqslant (2u)^{-1} t\} \\
\leqslant p_2^{-1} K_6 2^{\varkappa_1} t^{-\varkappa_1} E\{|R|^{\varkappa_1}; \zeta = k, |R| > (2\delta_1)^{-1}\}.
\end{aligned}$$

This bound for the left hand side of (3.17) can be made small w.r.t. $t^{-\varkappa_1}$ and hence w.r.t. $P\{|R| > t\}$ (see (3.7)) by choosing δ_1 small, provided

$$E\{|R|^{\varkappa_1}; \zeta = k\} < \infty. \tag{3.22}$$

The last task is therefore to prove (3.22) for any finite k. Now, on $\{\zeta = k\}$, $\xi_{k+1} = \infty$ and thus

$$|R| \leqslant \sum_{l \leqslant \xi_{k+1}} |\Pi_{l-1} Q_l| \leqslant \sum_{j=0}^{k} \|\Pi_{\xi_j}\| \sum_{\xi_j < l \leqslant \xi_{j+1}} \|M_{\xi_j+1} \dots M_{l-1}\| \, |Q_l|.$$

It follows that

$$E\{|R|^{\varkappa_1}; \zeta = k\} \leqslant (k+1)^{\varkappa_1} \sum_{j=0}^{k} E\|\Pi_{\xi_j}\|^{\varkappa_1} E\{\sum \|M_1 \ldots M_{l-1}\| \, |Q_l| \, I[l \leqslant \xi_1]\}^{\varkappa_1}. \tag{3.23}$$

Furthermore,

$$E\|\Pi_{\xi_j}\|^{\varkappa_1} \leqslant \{\min_{y \in S_{d-1}} r(y)\}^{-1} d^{\varkappa/2} E|e_0 \Pi_{\xi_j}|^{\varkappa_1} r((e_0 \Pi_{\xi_j})^{\sim}),$$

and by (2.61) the sequence

$$|e_0 \Pi_n|^{\varkappa_1} r((e_0 \Pi_n)^{\sim}), \quad n \geqslant 0,$$

is a positive martingale, so that for any stopping time N (and in particular for $N = \xi_j$)

$$E|e_0 \Pi_N|^{\varkappa_1} r((e_0 \Pi_N)^{\sim}) \leqslant \lim_{n \to \infty} E|e_0 \Pi_{\min(n,N)}|^{\varkappa_1} r((e_0 \Pi_{\min(n,N)})^{\sim}) = r(e_0)$$

(compare [15], V.T28 and its proof). Thus

$$E\|\Pi_{\xi_j}\|^{\varkappa_1} < \infty.$$

The expectation of the sum over l in the right hand side of (3.23) has to be treated differently for $\varkappa_1 > 1$ and $\varkappa_1 \leqslant 1$. We only do the case $\varkappa_1 \geqslant 1$ in detail; when $\varkappa_1 < 1$ the estimates are similar (but somewhat simpler) and based on the inequality

$$|a+b|^{\varkappa_1} \leqslant |a|^{\varkappa_1} + |b|^{\varkappa_1}, \quad \varkappa_1 \leqslant 1.$$

For $\varkappa_1 > 1$,

$$[E\{\sum \|\Pi_{l-1}\| \, |Q_l| \, I[l \leqslant \xi_1]\}^{\varkappa_1}]^{1/\varkappa_1} \leqslant \sum [E\{\|\Pi_{l-1}\| \, |Q_l| \, I[l \leqslant \xi_1]\}^{\varkappa_1}]^{1/\varkappa_1}$$
$$= \sum [E|Q_1|^{\varkappa_1}]^{1/\varkappa_1} [E\{\|\Pi_{l-1}\|^{\varkappa_1}; l \leqslant \xi_1\}]^{1/\varkappa_1}. \tag{3.24}$$

By definition of ξ_1, $\|\Pi_{l-1}\| \leqslant 1$ for $l \leqslant \xi_1$ so that for the γ and K_2 of (2.82)

$$E\{\|\Pi_{l-1}\|^{\varkappa_1}; l \leqslant \xi_1\} \leqslant e^{-\gamma\varkappa_1(l-1)} + 1\, P\{e^{-\gamma(l-1)} \leqslant \|\Pi_{l-1}\| \leqslant 1\}$$
$$\leqslant e^{-\gamma\varkappa_1(l-1)} + K_2 e^{-2\gamma(l-1)}. \tag{3.25}$$

Since $\gamma > 0$ in (2.82) and $E|Q|^{\varkappa_1} < \infty$ (see (3.3)) the expression in (3.24) is finite. This proves (3.22) and completes step 4.

Step 5. As in (2.45) let

$$N_x(s) = \min\{n \geqslant 0 : \log|x\Pi_n| > s\}.$$

Then for every $\varepsilon > 0$ there exists a $\delta_2 = \delta_2(\varepsilon) > 0$ and $t_1(\varepsilon) < \infty$ such that for all $x \in S_{d-1}$, $0 < \delta \leqslant \delta_2$, $t \geqslant t_1(\varepsilon)$

$$P\{N=N_x(\log(\delta t))<\infty \text{ and } x\Pi_N R^N>t(1+\varepsilon)\} - \varepsilon P\{|R|>t\} \leqslant P\{xR>t\}$$
$$\leqslant \varepsilon P\{|R|>t\} + P\{N=N_x(\log(\delta t))<\infty \text{ and } x\Pi_N R^N>t(1-\varepsilon)\}. \quad (3.26)$$

We shall only prove the second inequality in (3.26), leaving the first one to the reader. By step 4, if $\eta>0$, $\delta\varepsilon^{-1}\leqslant\delta_0(\eta)$, $\varepsilon t\geqslant t_0(\eta)$, then

$$\begin{aligned} P\{T(x,\varepsilon t)\leqslant N_x(\log(\delta t))\} \\ &\leqslant P\{\text{there exists an } n \text{ with } |xR_n|>\varepsilon t \text{ but } \max_{l\leqslant n-1}|x\Pi_l|\leqslant \delta\varepsilon^{-1}(\varepsilon t)\} \\ &\leqslant \eta P\{|R|>\varepsilon t\}. \end{aligned} \quad (3.27)$$

Now choose $\eta=\eta(\varepsilon)$ so small that

$$\eta P\{|R|>\varepsilon t\}\leqslant\varepsilon P\{|R|>t\}$$

(this can be done by (3.9)) and take $\delta_2=\varepsilon\delta_0(\eta)$. Then, since $xR>t$ implies $T(x,\varepsilon t)\leqslant T(x,t)<\infty$ for $\varepsilon<1$, we have for $\delta\leqslant\delta_2$, $t\geqslant\varepsilon^{-1}t_0(\eta)$

$$\begin{aligned} P\{xR>t\} &= P\{xR>t,\ T(x,\varepsilon t)<\infty\} \\ &\leqslant P\{N_x(\log(\delta t))<T(x,\varepsilon t)<\infty,\ xR>t\}+\varepsilon P\{|R|>t\}. \end{aligned} \quad (3.28)$$

But when $N=N_x(\log(\delta t))<T(x,\varepsilon t)$ and $xR>t$ then

$$|xR_N|<\varepsilon t \quad \text{and} \quad xR=xR_N+x\Pi_N R^N>t,$$

hence

$$x\Pi_N R^N>(1-\varepsilon)\,t.$$

Thus the first term in the right hand side of (3.28) is at most

$$P\{N=N_x(\log(\delta t))<\infty \quad \text{and} \quad x\Pi_N R^N>(1-\varepsilon)\,t\}$$

which, together with (3.28) proves the second inequality of (3.26).

Step 6. We now complete the proof of Theorem 4, by an application of Theorem 3. Firstly, by (3.26) and (3.9)

$$\begin{aligned} P\{|R|>d^{\frac12}t\}\leqslant P\{e_0R>t\}\leqslant \tfrac12 K_5^{\frac{\log d}{2\log 2}+1}P\{|R|>t\} \\ + P\{N_{e_0}(\log(\delta_3 t))<\infty\}\leqslant \tfrac12 P\{|R|>d^{\frac12}t\}+P\{\max_n|e_0\Pi_n|>\delta_3 t\} \end{aligned}$$

for some $\delta_3>0$. It now follows from (2.63) and (3.7) that for suitable $K_7<\infty$

$$K_4 t^{-\varkappa_1} \leqslant P\{|R| > t\} \leqslant K_7 t^{-\varkappa_1}. \tag{3.29}$$

Next fix $\varepsilon > 0$ and let h be a positive continuous function with support in $[1-\varepsilon, 1+\varepsilon]$ and such that

$$\int h(\lambda)\, d\lambda = 1.$$

Then clearly

$$\int h(\lambda)\, d\lambda P\{xR > \lambda(1-\varepsilon)^{-1} t\} \leqslant P\{xR > t\} \leqslant \int h(\lambda)\, d\lambda P\{xR > \lambda(1+\varepsilon)^{-1} t\} \tag{3.30}$$

and we now use (3.26) and (3.29) to estimate the extreme members of (3.30). E.g. to obtain an upper bound for sufficiently large t we apply (3.26) with t replaced by $\lambda(1+\varepsilon)^{-1} t$ and δ by

$$\lambda^{-1}(1+\varepsilon)\,\delta_4, \text{ where } (1-\varepsilon)^{-1}(1+\varepsilon)\,\delta_4 \leqslant \delta_2(\varepsilon).$$

This yields the bound

$$P\{xR > t\} \leqslant \varepsilon \int h(\lambda)\, d\lambda P\{|R| > \lambda(1+\varepsilon)^{-1} t\} + \int h(\lambda) d\lambda P\{N = N_x(\log \delta_4 t) < \infty, x\Pi_N R^N > \lambda(1-\varepsilon)(1+\varepsilon)^{-1} t\}. \tag{3.31}$$

The first term in the right hand side of (3.31) is at most

$$\varepsilon P\{|R| > (1-\varepsilon)(1+\varepsilon)^{-1} t\} \leqslant \varepsilon(1+\varepsilon)^{\varkappa_1}(1-\varepsilon)^{-\varkappa_1} K_7 t^{-\varkappa_1} \qquad \text{(see (3.29))}.$$

Now in the notation of section 2 (see (2.46), (2.47))

$$\log |x\Pi_N| - \log(\delta_4 t) = W_x(\log \delta_4 t)$$

and

$$(x\Pi_N)^{\sim} = Z_x(\log(\delta_4 t)).$$

The second term in the right hand side of (3.31) can therefore be written as

$$\begin{aligned}
&\int h(\lambda)\, d\lambda P\{N = N_x(\log \delta_4 t) < \infty, (x\Pi_N)^{\sim} R^N > \lambda\delta_4^{-1}(1-\varepsilon)(1+\varepsilon)^{-1}\delta_4 t |x\Pi_N|^{-1}\} \\
&= \int h(\lambda)\, d\lambda E\{P\{Z_x(\log \delta_4 t) R^* > \lambda\delta_4^{-1}(1-\varepsilon)(1+\varepsilon)^{-1} \exp - W_x(\log \delta_4 t)\}; N_x(\log \delta_4 t) < \infty\} \\
&= E\{g(Z_x(\log \delta_4 t), W_x(\log \delta_4 t); N_x(\log \delta_4 t) < \infty\},
\end{aligned} \tag{3.32}$$

where R^* has the distribution of R, but is independent of all $\{M_n, Q_n\}\, n \geqslant 1$, and

$$g(y, s) = \int h(\lambda)\, d\lambda P\{yR > \lambda\delta_4^{-1}(1-\varepsilon)(1+\varepsilon)^{-1} e^{-s}\}.$$

Since h is continuous with compact support it is not hard to see that $g(\cdot,\cdot)$ is bounded and continuous on $S_{d-1}\times\mathbf{R}$. Therefore by Theorem 3 and its extension

$$\lim_{s\to\infty} e^{\varkappa_1 s}E\{g(Z_x(s),\,W_x(s);\,N_x(s)<\infty\}=K(x,g) \tag{3.33}$$

for some finite $K(x,g)$, $x\in S_{d-1}$. It follows from the above estimates that

$$\limsup_{t\to\infty} t^{\varkappa_1}P\{xR>t\}\leqslant \varepsilon(1+\varepsilon)^{\varkappa_1}(1-\varepsilon)^{-\varkappa_1}K_7+\delta_4^{-\varkappa_1}K(x,g). \tag{3.35}$$

In exactly the same way, using the left hand rather than right hand inequalities in (3.30) and (3.26) (now replace t by $\lambda(1-\varepsilon)^{-1}t$ and δ by $\lambda^{-1}(1+\varepsilon)^2(1-\varepsilon)^{-1}\delta_4$) in (3.26)), we obtain

$$\liminf_{t\to\infty} t^{\varkappa_1}P\{xR>t\}\geqslant -\varepsilon K_7+\delta_4^{-\varkappa_1}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{2\varkappa_1}K(x,g). \tag{3.36}$$

Since the bounds in (3.35) and (3.36) can be brought arbitrarily close together, (3.5) follows. Note, however, that it is not meaningful to say the limit equals $\delta_4^{-\varkappa_1}K(x,g)$, because $h(\cdot)$ and $g(\cdot)$ depend on ε. Still, for fixed ε and $h(\cdot)$, we know from (2.62) that

$$K(x,g)=K(g)r(x)\quad\text{for}\quad x\in S_+$$

from which we easily see that the limit in (3.5) is $K_3r(x)$ for $x\in S_+$, and some $K_3<\infty$. $K_3>0$ because of (3.7) and

$$P\{e_0R>t\}\geqslant P\{|R|\geqslant d^{\frac{1}{2}}t\}. \qquad \blacksquare$$

The *Corollary* is immediate, because if $\varkappa_1>2$ then $E|R|^2<\infty$, and if $\varkappa_1<1$ and $R_1, R_2, \ldots$ are independent copies of R, then (see [4], Ch. XVII.5)

$$n^{-\varkappa_1^{-1}}x\sum_1^n R_j$$

converges in law to a (one-dimensional) stable distribution of index $\varkappa_1$ for each $x\in S_{d-1}$. For some x the limit law may be degenerate and concentrated on 0 only, but in any case, for any $y\in\mathbf{R}^d$

$$\lim_{n\to\infty} E\exp\left(in^{-\varkappa_1^{-1}}y\sum_1^n R_j\right)=\lim_{n\to\infty}E\exp\left(i|y|n^{-\varkappa_1^{-1}}\tilde{y}\sum_1^n R_j\right)$$

exists and is of the form

$$\exp(-|y|^{\varkappa_1}\Theta(\tilde{y})) \tag{3.37}$$

for a suitable function Θ on S_{d-1}. Thus the limit of the d-dimensional characteristic function of

$$n^{-\varkappa_1^{-1}}\sum_1^n R_j$$

exists and is clearly stable; (3.37) is even strictly stable in the terminology of [4], Def. VI.1.1. The same argument applies to $\sum_{j=1}^{n}(R_j - ER_1)$ for $1 < \varkappa_1 \leqslant 2$, but for $\varkappa_1 = 1$ we need a special trick. Let $R_1^s, R_2^s, \ldots$ be independent, and each with the distribution

$$P\{R_j^s \in A\} = \tfrac{1}{2}[P\{R \in A\} + P\{-R \in A\}].$$

Then

$$E e^{iy \cdot R_j^s} = \operatorname{Re} E e^{iy \cdot R},$$

and again by [4], Ch. XVII.5 we get from (3.5) with $\varkappa_1 = 1$

$$\lim_{n\to\infty} E \exp\left(in^{-1} y \sum_1^n R_j^s\right) = \exp(-|y|\,\Theta(\tilde{y}))$$

for suitable Θ. Thus R_1^s is in the domain of normal attraction of a stable law of index 1. By Theorem 4.2 of [19] or the multidimensional analogue of [4], Ch. XVII.5 this implies

$$0 < \lim_{t\to\infty} t\, P\{|R_1^s| > t\} < \infty, \tag{3.38}$$

and

$$\lim_{t\to\infty} P\{|R_1^s| > t, \tilde{R}_1^s \in A\}\,[P\{|R_1^s| > t\}]^{-1} = H(A) \tag{3.39}$$

in the weak sense for a suitable finite measure H on S_{d-1}. However, $P\{|R_1^s| > t\} = P\{|R| > t\}$ and, since $P\{R_1 \geqslant 0\} = 1$ under the conditions of Theorem 4, we also have

$$P\{|R_1^s| > t, \tilde{R}_1^s \in A\} = \tfrac{1}{2} P\{|R| > t, \tilde{R} \in A\}$$

for all A in the positive orthant of S_{d-1}. It follows that (3.38) and (3.39) hold with R_1^s replaced by R and $H(A)$ by $2\,H(A \cap$ positive orthant). Now we can use the sufficiency part of Theorem 4.2 in [19] to conclude that the distribution of $n^{-1}\sum_1^n R_j - K_n$ converges to a stable law of index 1 for suitable vectors K_n.

Remark 4. The proof of the corollary in the case $\varkappa_1 = 1$ used the fact that $R \geqslant 0$ w.p. 1. It would still work if R were concentrated in an open half space, i.e., if $P\{x_0 R > 0\} = 1$ for some $x_0 \in S_{d-1}$. However, in general one cannot conclude that $n^{-1}\sum_1^n R_j - K_n$ converges in distribution to a stable law of index 1 from the existence of $\lim_{t\to\infty} tP\{xR > t\}$, $x \in S_{d-1}$ only.

4. Solutions of random difference equations; general coefficients

As in section 3 we want to find the asymptotic distribution of Y_n defined by (3.1), but now without the restriction $M_n \geqslant 0$. Again we are interested in proving (3.5). In the one dimensional case ($d = 1$) there is no difficulty in generalizing Theorem 4 because the basic results (2.62), (2.63) and (2.83) can be obtained directly from Theorem 1 or other renewal theorems. (S_0 consists only of the two points $+1$ and -1 and we therefore only have to do renewal theory for functions on a *finite* Markov chain.) However, for $d>1$ the proof of (2.62) and (2.63) breaks down in several places when we drop the condition $M_n \geqslant 0$. (Most notably in the verification of condition (2.19) in the proof of Proposition 1.) Nevertheless it is possible to derive appropriate forms of (2.62), and (2.63) and (2.83) if we make assumptions on the absolute continuity of the distribution μ of M_1. This does, however, require an alternate form of the renewal Theorem 1. Such an alternate theorem was stated in [11]. We state here without proof the results which can be obtained for $d=1$ by means of Theorem 1 and for $d>1$ by means of the alternate renewal theorem.

THEOREM 5. ($d=1$). *Let M_n and Q_n, $n \geqslant 1$, be (real valued) random variables such that the pairs (M_n, Q_n), $n \geqslant 1$, are independent and identically distributed. Assume that*

$$E \log |M_1| < 0, \tag{4.1}$$

but that for some $\varkappa_1 > 0$

$$E|M_1|^{\varkappa_1} = 1, \tag{4.2}$$

$$E|M_1|^{\varkappa_1} \log^+ |M_1| < \infty, \tag{4.3}$$

$$0 < E|Q_1|^{\varkappa_1} < \infty. \tag{4.4}$$

If in addition $\log |M_1|$ does not have a lattice distribution(1) *and Q_1 is not a constant times $(1-M_1)$, i.e.,*

$$P\{Q_1 = (1-M_1)r\} < 1 \tag{4.5}$$

for each fixed r, then the series

$$R = \sum_{k=1}^{\infty} M_1 \dots M_{k-1} Q_k$$

*converges w.p.*1. *and the distribution of the solution Y_n of* (3.1) *converges to that of R, independently of Y_0, Moreover*

$$\lim_{t\to\infty} t^{\varkappa_1} P\{R > t\} \text{ and } \lim_{t\to\infty} t^{\varkappa_1} P\{R < -t\} \tag{4.6}$$

exist and are finite.

At least one of the limits in (4.6) *is strictly positive.*

We need one further piece of notation for the general d-dimensional case. If M is any $d \times d$ matrix and M' its transpose, then MM' is symmetric and positive definite. We put

$$\lambda(M) = (\text{smallest eigenvalue of } MM')^{\frac{1}{2}}.$$

We also remind the reader that we defined the term "feasible" just before Proposition 1 in section 2.

(1) In the present situation $-\infty$ may be a possible value of $\log|M_1|$. If this is so, the condition here means that the possible values of $\log|M_1|$ which are not $-\infty$ generate a dense group in $\mathbf{R}$.

THEOREM 6. *Let M_n and Q_n be $d\times d$ matrices, respectively d column vectors such that the pairs (M_n, Q_n), $n\geqslant 1$, are independent identically distributed and assume*

$$E \log^+ \|M_1\| < \infty. \tag{4.7}$$

Then $$\alpha \equiv \lim_{n\to\infty} \frac{1}{n} \log \|\Pi_n\| \text{ exists and is } < +\infty \text{ w.p.1.}$$

If $\alpha < 0$ and (4.9) *below holds, then the series*

$$R = \sum_{n=1}^{\infty} M_1 \dots M_{n-1} Q_n$$

converges w.p.1 and the distribution of the solution Y_n of (3.1) *converges to that of R, independently of Y_0. If in addition the conditions* (i)–(vi) *below hold, then for some $\varkappa_1 \in (0, \varkappa_0]$*

$$\lim_{t\to\infty} t^{\varkappa_1} P\{xR > t\} \tag{4.8}$$

exists and is strictly positive for all $x \in S_{d-1}$.

Here are the conditions (i)–(vi):

(i) $P\{M_1$ is singular$\} = 0$.

(ii) For every open $U \subset S_{d-1}$ and $x \in S_{d-1}$ there exists an n with $P\{(x\Pi_n)^{\sim} \in U\} > 0$.

(iii) There exists an n, a cube $C \subset \mathbf{R}^{d^2}$ and a $\gamma_0 > 0$ such that the distribution of Π_n has a nonsingular (w.r.t. Lebesgue measure on $\mathbf{R}^{d^2}$) component with a density at least γ_0 on C.

(iv) The group generated by $\{\log \varrho(\pi): \pi$ feasible$\}$ is dense in R.

(v) For every fixed column vector r $P\{Q_1 = (I - M_1)\, r\} < 1$.

(vi) There exists a $\varkappa_0 > 0$ such that

$$E[\lambda(M_1)]^{\varkappa_0} \geqslant 1,$$

$$E\|M_1\|^{\varkappa_0} \log^+ \|M_1\| < \infty,$$

and $$0 < E|Q_1|^{\varkappa_0} < \infty. \tag{4.9}$$

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