THE FOURIER TRANSFORM ON SEMISIMPLE LIE GROUPS OF REAL RANK ONE

BY

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1. Introduction

Let G be a connected semisimple Lie group with finite center and let K be a maximal compact subgroup of G. We assume that $\operatorname{rank}(G) = \operatorname{rank}(K)$ and that $\operatorname{rank}(G/K) = 1$. Let T be a Cartan subgroup of G contained in K. We write $\mathfrak{G}$ for the Lie algebra of G and $\mathfrak{G}_{\mathbf{C}}$ for the complexification of $\mathfrak{G}$. If $G_{\mathbf{C}}$ is the simply connected, complex analytic group corresponding to $\mathfrak{G}_{\mathbf{C}}$, we assume that G is the real analytic subgroup of $G_{\mathbf{C}}$ corresponding to $\mathfrak{G}$.

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Let y be a semisimple element in G and let G_y denote the centralizer of y in G. Then G_y is unimodular, and we denote by $d_{G/G_y}(\dot{x})$ a G-invariant measure on G/G_y. If we write ${}^x y = xyx^{-1}$, $x \in G$, then the map

$$f \mapsto \int_{G/G_y} f({}^{\dot{x}}y)\, d_{G/G_y}(\dot{x}), \quad f \in C_c^\infty(G),$$

defines an invariant distribution Λ_y on G which is actually a tempered distribution.

In this paper, we give explicit formulas for the Fourier transform of Λ_y, that is, we determine a linear functional $\hat{\Lambda}_y$ such that

$$\Lambda_y(f) = \hat{\Lambda}_y(\hat{f}), \quad f \in C_c^\infty(G).$$

Here, we regard $\hat{f}$ as being defined on the space of tempered invariant eigendistributions on G. This space contains the characters of the principal series and the discrete series for G along with some "singular" invariant eigendistributions whose character theoretic nature has not yet been completely determined (see 2. C).

Apart from the intrinsic interest of our results relative to harmonic analysis on G, the Fourier transforms of the invariant distributions Λ_y arise naturally in the context of Selberg's trace formula. Thus, let Γ be a discrete subgroup of G such that G/Γ is compact. Let λ be the (left) regular representation of G on $L^2(G/\Gamma)$. Then λ can be decomposed as the direct sum of irreducible unitary representations of G, and each irreducible unitary representation π of G occurs in λ with finite multiplicity m_π.

We write

$$\lambda = \sum_{\pi \in \hat{G}} \oplus\, m_\pi \pi,$$

where $\hat{G}$ denotes the set of equivalence classes of irreducible unitary representations of G. The basic problem here is the determination of those $\pi \in \hat{G}$ for which $m_\pi > 0$ and, moreover, the determination of an explicit formula for m_π.

Let $d_G(x)$ denote a Haar measure on G. For f in a suitable class of complex valued functions on G, the operator $\lambda(f) = \int_G f(x)\, \lambda(x)\, d_{\dot{G}}(x)$ is of the trace class and

$$\operatorname{tr} \lambda(f) = \sum_{\pi \in \hat{G}} m_\pi \hat{f}(\pi) \quad (\hat{f}(\pi) = \operatorname{tr} \pi(f)). \tag{1.1}$$

On the other hand ([2], Ch. 1), we can write

$$\operatorname{tr} \lambda(f) = \sum_{\{y\}} \mu(G_y/\Gamma_y) \int_{G/G_y} f({}^{\dot{x}}y)\, d_{G/G_y}(\dot{x}), \tag{1.2}$$

where $\{y\}$ runs through the conjugacy classes in Γ and $\mu(G_y/\Gamma_y)$ is the volume of G_y/Γ_y.

Now, the idea is to get information about the multiplicities m_π by equating (1.1) and (1.2). The first step in this program is the computation of the Fourier transform of the terms which occur in (1.2), that is, the computation of $\hat{\Lambda}_y$ for $y \in \Gamma$. Since G/Γ is compact, every element of Γ is semisimple so that the formulas in the present paper provide the necessary information. Some aspects of the above program have been carried out for $G = \mathbf{SL}(2, \mathbf{R})$ in [2], Ch. 1, and, in somewhat more detail, by R. Langlands in a course given at Princeton in 1966. In particular, Langlands shows that the multiplicity of those members of the discrete series of $\mathbf{SL}(2, \mathbf{R})$ which do not have L^1 matrix coefficients is not given by an analogue of the multiplicity formula for those discrete series which have L^1 matrix coefficients. The multiplicity formula for the non-L^1 discrete series contains an additional term of -1. It is our intention to use the formulas in this paper and methods similar to those of Langlands to obtain multiplicity information for real rank one groups.

We now outline the contents of the paper. In § 2, we summarize some results of Harish-Chandra. The entire paper relies heavily on the work of Harish-Chandra, an account of which may be found in [11]. In general, we adopt the notation of [11]. In § 3, we consider the case when y is a regular element in G. The basic case is when $y \in T$. All the remaining results in the paper stem from this case. The Plancherel formula for G, first given by Harish-Chandra [4 f)] and Okamoto [6], is derived in § 4 by a simple application of Harish-Chandra's limit formula [4 a)], [4 c)]. Our method differs from that of the authors cited above. In § 5, we take y to be a semisimple, non-regular element in G. The formula for $\hat{\Lambda}_y(\hat{f})$ can again be computed from the results of § 3 by applying a theorem of Harish-Chandra ([4 g)], p. 33). We mention, in passing, that the case when y is a unipotent element may also be treated by our methods, that is, by applying an appropriate differential operator to a regular orbit and then taking a limit. Unfortunately, the explicit form of the differential operator is unknown to us at this writing. (1)

Some of the results of this paper were announced in [8 a)], and some examples are discussed in [8 b)]. For $\mathbf{SL}(2, \mathbf{R})$, our formulas may be found in [1], [2], [4 b)]. Similar results for $\mathbf{SL}(2, k)$, k a non-archimedean local field appear in [7]. We would like to express our appreciation to J. Arthur, C. Rader and N. Wallach for their helpful comments.

2. Some results of Harish-Chandra

2. A. The structure of $\mathfrak{G}$ and G

We retain the notation of the Introduction. Let $\mathfrak{t}$ be the Lie algebra of T and $\mathfrak{t}_{\mathbf{C}}$ the complexification of $\mathfrak{t}$. Then $\mathfrak{t}$ (resp. $\mathfrak{t}_{\mathbf{C}}$) is a Cartan subalgebra of $\mathfrak{G}$ (resp. $\mathfrak{G}_{\mathbf{C}}$). Proceeding

(1) Results in this direction have been obtained recently by Ranga Rao.

as in [4 f)], § 24, we fix a singular imaginary root α_t of the pair $(\mathfrak{G}_C, \mathfrak{t}_C)$ and a point Γ in $\mathfrak{t}$ such that $\pm\alpha_t$ are the only roots of the pair $(\mathfrak{G}_C, \mathfrak{t}_C)$ which vanish at Γ. Denote by $\mathfrak{G}_\Gamma$ the centralizer of Γ in $\mathfrak{G}$, and let $\mathfrak{c}_\Gamma$ and $\mathfrak{l}_\Gamma$ be the center of $\mathfrak{G}_\Gamma$ and the derived algebra of $\mathfrak{G}_\Gamma$ respectively.

The subalgebra $\mathfrak{l}_\Gamma$ is isomorphic over $\mathbf{R}$ to $\mathfrak{sl}(2, \mathbf{R})$, and we may select a basis H^*, X^*, Y^* for $\mathfrak{l}_\Gamma$ over $\mathbf{R}$ such that $[H^*, X^*] = 2X^*$, $[H^*, Y^*] = -2Y^*$, $[X^*, Y^*] = H^*$. Then $\mathfrak{t} = \mathbf{R}(X^* - Y^*) + \mathfrak{c}_\Gamma$ and $\mathfrak{a} = \mathbf{R}H^* + \mathfrak{c}_\Gamma$ form a complete set of non-conjugate Cartan subalgebras of $\mathfrak{G}$. Put $\mu = \exp[\sqrt{-1}\,(\pi/4)\,(X^* + Y^*)] \in G_C$. Then $(\mathfrak{t}_C)^\mu = \mathfrak{a}_C$, the complexification of $\mathfrak{a}$ and, if $\alpha_{\mathfrak{a}} = (\alpha_t)^\mu$ is the μ-transform of α_t, we have $\alpha_{\mathfrak{a}}(H^*) = 2$ and $\alpha_{\mathfrak{a}}$ vanishes identically on $\mathfrak{c}_\Gamma$. We order the space of real linear functions λ on $\mathbf{R}H^* + \sqrt{-1}\,\mathfrak{c}_\Gamma$ by stipulating that $\lambda > 0$ whenever $\lambda(H^*) > 0$. We then obtain a set of positive roots for the pair $(\mathfrak{G}_C, \mathfrak{t}_C)$ by demanding that the μ-transform of such a root be positive when considered as a root of $(\mathfrak{G}_C, \mathfrak{a}_C)$.

Let A be the Cartan subgroup of G associated with $\mathfrak{a}$, and let A^0 be the identity component of A. Then, setting $A_K = A \cap K$, $A_K^0 = A^0 \cap K$ and $A_{\mathfrak{p}} = \{\exp(tH^*) : t \in \mathbf{R}\}$, we have

$$A = A_K A_{\mathfrak{p}} \quad \text{and} \quad A^0 = A_K^0 A_{\mathfrak{p}}.$$

Put $Z(A_{\mathfrak{p}}) = K \cap \exp\{\sqrt{-1}\,\mathbf{R}H^*\}$. Then $Z(A_{\mathfrak{p}}) = \{1, \gamma\}$ is a group of order two with $\gamma = \exp[\pi(X^* - Y^*)] = \exp(\sqrt{-1}\,\pi H^*) \neq 1$. We have $A_K = Z(A_{\mathfrak{p}})\,A_K^0$.

Set $\mathfrak{t}_1 = \mathfrak{c}_\Gamma$, $\mathfrak{t}_2 = \mathbf{R}(X^* - Y^*)$ and let T_1 and T_2 be the analytic subgroups of T corresponding to $\mathfrak{t}_1$ and $\mathfrak{t}_2$ respectively. T_1 and T_2 are compact and $T_1 \cap T_2 \subset Z(A_{\mathfrak{p}})$. Since $A_K = T_1 \cup \gamma T_1 (T_1 = A_K^0)$, it follows that A_K has one or two connected components according to whether γ lies in $T_1 \cap T_2$ or not. Now, if M is the centralizer of $A_{\mathfrak{p}}$ in K and M^0 is the identity component of M, then $M = M^0 \cup \gamma M^0$.

If no simple factor of G is isomorphic to $\mathbf{SL}(2, \mathbf{R})$, it follows from the classification of real rank one groups [9] that M is connected or, equivalently $T_1 \cap T_2 = \{1, \gamma\}$. In this case $T_1 = A_K$, a maximal torus in M.

For the group $\mathbf{SL}(2, \mathbf{R})$, our results are well-known and may be found in [8 b)]. *Throughout the remainder of this paper, we assume that M is connected.*

Write $G = KA_{\mathfrak{p}}N^+$, the Iwasawa decomposition of G, and set $P = MA_{\mathfrak{p}}N^+$. Then P is a minimal (and maximal) parabolic subgroup of G.

2. B. The invariant integral on G

We first establish a normalization of certain invariant measures. Let $x \mapsto \dot{x}$, $x \in G$, denote the canonical projection of G on G/T (or G/A). We take a G-invariant measure

$d_{G/T}(\dot{x})$ on G/T which is normalized as in [11], v. II, Ch. 8. If we choose a Haar measure $d_T(t)$ on T normalized so that the volume of T is one, then a Haar measure $d_G(x)$ on G is fixed by the formula

$$\int_G f(x)\, d_G(x) = \int_{G/T} \int_T f(xt)\, d_T(t)\, d_{G/T}(\dot{x}),$$

for $f \in C_c(G)$.

Let $d_{A_\mathfrak{p}}(h_\mathfrak{p})$ be the Haar measure on $A_\mathfrak{p}$ which is the transport via the exponential map of the canonical Haar measure on the Lie algebra of $A_\mathfrak{p}$ associated with the Euclidean structure derived from the Killing form of $\mathfrak{G}$. Since $A_\mathfrak{p} = \{\exp tH^* : t \in \mathbf{R}\}$, we have

$$d_{A_\mathfrak{p}}(h_\mathfrak{p}) = c_A\, dt, \tag{2.1}$$

where c_A is a positive constant and dt is normalized Lebesgue measure on $\mathbf{R}$. We normalize Haar measure $d_{A_K}(h_K)$ on A_K so that the volume of A_K is one. Now a Haar measure $d_A(h)$ on A is fixed by the formula $d_A(h) = d_{A_K}(h_K)\, d_{A_\mathfrak{p}}(h_\mathfrak{p})$ where $h = h_K h_\mathfrak{p}$. A G-invariant measure $d_{G/A}(\dot{x})$ on G/A is then determined by the formula

$$\int_G f(x)\, d_G(x) = \int_{G/A} \int_A f(xh)\, d_A(h)\, d_{G/A}(\dot{x}),$$

for $f \in C_c(G)$.

Let G' be the set of regular elements in G and set $T' = T \cap G'$, $A' = A \cap G'$. Put $G^e = \bigcup_{x \in G} xT'x^{-1}$, the *elliptic set* in G, and $G^h = \bigcup_{x \in G} xA'x^{-1}$, the *hyperbolic set* in G. Then $G' = G^e \cup G^h$ (disjoint union) and

$$\int_G f(x)\, d_G(x) = \int_{G^e} f(x)\, d_G(x) + \int_{G^h} f(x)\, d_G(x), \tag{2.2}$$

for $f \in C_c(G)$. Let Δ_T, Δ_A, ε_R^T, ε_R^A, $W(G, T)$ and $W(G, A)$ be defined as in [4 d)] (in particular, $W(G, T)$ is the Weyl group of K). For $x \in G$, write ${}^{x}t = xtx^{-1}$, $t \in T$, and ${}^{x}h = xhx^{-1}$, $h \in A$. If $f \in C_c^\infty(G)$ and $t \in T'$ the *invariant integral of f* (relative to T) is defined by

$$\Phi_f^T(t) = \Delta_T(t) \int_{G/T} f({}^{\dot{x}}t)\, d_{G/T}(\dot{x}). \tag{2.3}$$

Similarly, if $h \in A'$, the *invariant integral of f* (relative to A) is

$$\Phi_f^A(h) = \varepsilon_R^A(h)\, \Delta_A(h) \int_{G/A} f({}^{\dot{x}}h)\, d_{G/A}(\dot{x}). \tag{2.4}$$

From Weyl's formula ([4 g)], p. 110), and (2.2), it follows that

$$\int_{G^e} f(x)\,d_G(x) = [W(G,T)]^{-1}\int_T \overline{\Delta_T(t)}\,\Phi_f^T(t)\,d_T(t),$$
$$\int_{G^h} f(x)\,d_G(x) = [W(G,A)]^{-1}\int_A \overline{\Delta_A(h)}\,\varepsilon_R^A(h)\,\Phi_f^A(h)\,d_A(h). \tag{2.5}$$

It is known [4 d)] that $\Phi_f^T \in C^\infty(T')$ (in general, Φ_f^T does not extend to a C^∞ function on all of T). Relative to the operation of $W(G,T)$ on T, we have

$$\Phi_f^T(wt) = \det(w)\,\Phi_f^T(t) \tag{2.6}$$

for $w \in W(G,T)$, $t \in T'$. The function Φ_f^A is in $C^\infty(A')$ and extends to a compactly supported C^∞ function on all of A since the pair $(\mathfrak{G}_{\mathbf{C}}, \mathfrak{a}_{\mathbf{C}})$ has no singular imaginary roots [see [4 d)], § 22). The general formula for the transformation of Φ_f^A relative to the action of $W(G,A)$ is given in [4 f)], p. 103. We are interested in two special cases.

$$\Phi_f^A(h_K h_\mathfrak{p}) = \Phi_f^A(h_K h_\mathfrak{p}^{-1}),\ h_K \in A_K,\ h_\mathfrak{p} \in A_\mathfrak{p}. \tag{2.7}$$

If $w \in W(M, A_K)$, the Weyl group of M, then w may be considered as an element of $W(G,A)$, and we have

$$\Phi_f^A(wh) = \det(w)\,\Phi_f^A(h),\ h \in A,\ w \in W(M, A_K). \tag{2.8}$$

2. C. The characters of the discrete series

The unitary character group $\hat{T}$ of T may be identified with a lattice L_T in the dual space of $\sqrt{-1}\,\mathfrak{t}$, and, for $\tau \in L_T$, the corresponding character $\xi_\tau \in \hat{T}$ is given by

$$\xi_\tau(\exp H) = e^{\tau(H)}, \quad H \in \mathfrak{t}. \tag{2.9}$$

The Weyl group $W(\mathfrak{G}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$ acts on L_T and hence on $\hat{T}$ by the prescription

$$w\tau(H) = \tau(w^{-1}H),\ \xi_{w\tau}(\exp H) = e^{w\tau(H)}, \quad H \in \mathfrak{t},\ \tau \in L_T.\ (^1) \tag{2.10}$$

We say that $\tau \in L_T$ is *regular* if $w\tau \neq \tau$ for all $w \neq 1$ in $W(\mathfrak{G}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$; otherwise τ is said to be *singular*. The set of regular τ will be denoted by L'_T and the set of singular τ by L_T^s. The character ξ_τ is called regular or singular accordingly.

To each $\tau \in L_T$, there is associated a central eigendistribution Θ_τ on G characterized uniquely by certain properties ([4 e)], p. 281, [4 f)], p. 90). Θ_τ is locally summable on G and analytic on G'. We have

(1) For convenience, we write $w\tau$ for $w \cdot \tau$.

$$\Theta_\tau(t) = \Delta(t)^{-1} \sum_{w \in W(G,T)} \det(w)\, \xi_{w\tau}(t),\ t \in T'. \tag{2.11}$$

Note that, if $\tau \in L_T^s$ and, moreover, τ is fixed by a non-trivial element of $W(G,T)$, then Θ_τ is identically zero on T'.

On A', the behavior of Θ_τ is slightly more complicated. Set

$$A_\mathfrak{p}^+ = \{h_\mathfrak{p} \in A_\mathfrak{p} : \alpha_\mathfrak{a}(\log h_\mathfrak{p}) > 0\} = \{\exp(tH^*) : t > 0\},$$

$$A_\mathfrak{p}^- = \{h_\mathfrak{p} \in A_\mathfrak{p} : \alpha_\mathfrak{a}(\log h_\mathfrak{p}) < 0\} = \{\exp(tH^*) : t < 0\},$$

and define

$$A^+ = A_K A_\mathfrak{p}^+ \cap A', \quad A^- = A_K A_\mathfrak{p}^- \cap A'.$$

For $\tau \in L_T$, put

$$c(\tau : A^+) = \begin{cases} -1 & \text{if} \quad \tau(\sqrt{-1}(X^* - Y^*)) > 0 \\ +1 & \text{if} \quad \tau(\sqrt{-1}(X^* - Y^*)) < 0 \\ 0 & \text{if} \quad \tau(\sqrt{-1}(X^* - Y^*)) = 0. \end{cases}$$

$$c(\tau : A^-) = -c(\tau : A^+).$$

Then, we have

$$\Theta_\tau(h) = \Delta_A(h)^{-1} \sum_{w \in W(G,T)} \det(w)\, \xi_{w\tau}(h_K)\, c(w\tau : A^\pm) \exp(-|(w\tau)^\mu(\log h_\mathfrak{p})|), \tag{2.13}$$

where $h = h_K h_\mathfrak{p}$ and the sign in $c(w\tau : A^\pm)$ is chosen to correspond to $h \in A^+$ or $h \in A^-$. Again, it is easy to see that $\Theta_\tau \equiv 0$ on A' if τ is fixed by a non-trivial element of $W(G,T)$.

For $\tau \in L_T'$, put $s = (\frac{1}{2}) \dim(G/K)$ and $\varepsilon(\tau) = \operatorname{sgn}\{\prod_{\alpha \in P_T}(\tau, \alpha)\}$ where P_T denotes the set of positive roots of $(\mathfrak{G}_\mathrm{C}, \mathfrak{t}_\mathrm{C})$. Then ([4 g)], p. 96)

$$T_\tau = (-1)^s \varepsilon(\tau) \Theta_\tau \tag{2.14}$$

is the character of a representation in the discrete series for G and all discrete series characters are obtained in this way. Moreover, $T_{\tau_1} = T_{\tau_2}$ if and only if τ_1 and τ_2 are conjugate under $W(G,T)$.

Even though the invariant eigendistributions Θ_τ, $\tau \in L_T^s$, do not correspond to characters of the discrete series for G, these eigendistributions do appear discretely in the Fourier transform of the invariant integral. The need for these Θ_τ arises from the fact that Fourier analysis on T requires the use of the full character group of T. The character theoretic nature of Θ_τ, $\tau \in L_T^s$, has been settled in only a few special cases.

2. D. The characters of the principal series

For $\chi \in \hat{A}_K$, the unitary character group of A_K, denote by $\log \chi$ the linear function on $\mathfrak{t}_1$ defined by

$$\chi(\exp H) = e^{\langle H, \log \chi \rangle},\ H \in \mathfrak{t}_1. \tag{2.15}$$

Let P_I^+ be the set of positive imaginary roots of the pair $(\mathfrak{G}_{\mathbf{C}}, \mathfrak{a}_{\mathbf{C}})$, and let W_I be the subgroup of $W(G, A)$ which is generated by the Weyl reflections associated with the elements of P_I^+. W_I may be identified with the Weyl group $W(M, A_K)$ in a natural way. An element $\chi \in \hat{A}_K$ is called *regular* if $w\chi \neq \chi$ for all $w \neq 1$ in W_I. Otherwise, χ is called *singular*. If χ is a regular character in $\hat{A}_K$, we set

$$\varepsilon(\chi) = \operatorname{sgn}\{\prod_{\alpha \in P_I^+} (\log \chi, \alpha)\}. \tag{2.16}$$

The unitary character group $\hat{A}_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$ is isomorphic to $\mathbf{R}$ and, for $\nu \in \mathbf{R}$, we define the corresponding unitary character on $A_{\mathfrak{p}}$ by

$$h_{\mathfrak{p}}^{\sqrt{-1}\nu} = e^{\sqrt{-1}\nu(\log h_{\mathfrak{p}})},\ h_{\mathfrak{p}} \in A_{\mathfrak{p}}.$$

Let $f \in C_c^\infty(G)$. The Fourier transform $\hat{\Phi}_f^A$ of the invariant integral Φ_f^A is defined on $\hat{A}_K \times \hat{A}_{\mathfrak{p}}$ by

$$\hat{\Phi}_f^A(\chi, \nu) = (2\pi)^{-\frac{1}{2}} \int_{A_K} \int_{A_{\mathfrak{p}}} \chi(h_K)\, h_{\mathfrak{p}}^{\sqrt{-1}\nu}\, \Phi_f^A(h_K h_{\mathfrak{p}})\, d_{A_K}(h_K)\, d_{A_{\mathfrak{p}}}(h_{\mathfrak{p}}), \quad \chi \in \hat{A}_K, \nu \in \mathbf{R}. \tag{2.17}$$

If χ is singular, it follows immediately from (2.8) that $\hat{\Phi}_f^A(\chi, \nu) = 0$ for all ν in $\mathbf{R}$.

Now suppose that χ is a regular element in $\hat{A}_K$ and ν is an arbitrary element of $\hat{A}_{\mathfrak{p}}$. Then, if $r_I = [P_I^+]$, the distribution

$$T^{(\chi,\nu)}(f) = (2\pi)^{\frac{1}{2}}(-1)^{r_I}\varepsilon(\chi)\,\hat{\Phi}_f^A(\chi, \nu), \quad f \in C_c^\infty(G), \tag{2.18}$$

is the character of a representation of the principal series for G and, moreover all principal series characters have this form for suitable (regular) $\chi \in \hat{A}_K$, $\nu \in \hat{A}_{\mathfrak{p}}$ (see [11], v. II, Epilogue). If χ is singular, we set $\varepsilon(\chi) = 1$ and define $T^{(\chi,\nu)}$ by (2.18). Of course, $T^{(\chi,\nu)} \equiv 0$ for singular χ, but, as is the case for Θ_τ, $\tau \in L_T^s$, we need the formal expression for $T^{(\chi,\nu)}$ for all $(\chi, \nu) \in \hat{A}_K \times \hat{A}_{\mathfrak{p}}$ when we work with the Fourier transform on $\hat{A}_K \times \hat{A}_{\mathfrak{p}}$.

Finally, it follows from (2.7) that

$$T^{(\chi,\nu)} = T^{(\chi,-\nu)}, \quad \chi \in \hat{A}_K, \nu \in \mathbf{R}. \tag{2.19}$$

3. The Fourier transform of a regular orbit

3. A. The Fourier transform of a regular elliptic orbit

Fix $f \in C_c^\infty(G)$. Then $\Phi_f^T \in L^1(T)$ and, as pointed out above, $\Phi_f^T \in C^\infty(T')$. For $\tau \in L^T$, we denote by $\hat{\Phi}_f^T(\tau)$ the Fourier coefficient of Φ_f^T at τ.

LEMMA 3.1. *Let* $\tau \in L_T$. *Then*

$$\hat{\Phi}_f^T(\tau) = (-1)^r \left(\Theta_\tau(f) - \int_{G^h} f(x)\, \Theta_\tau(x)\, d_G(x) \right)$$

where $r = 2^{-1}(\dim(G) - \operatorname{rank}(G))$.

Proof. We have

$$\Theta_\tau(f) = \int_{G^e} f(x)\, \Theta_\tau(x)\, d_G(x) + \int_{G^h} f(x)\, \Theta_\tau(x)\, d_G(x).$$

From (2.3), (2.5) and (2.11), we obtain

$$\int_{G^e} f(x)\, \Theta_\tau(x)\, d_G(x) = (-1)^r [W(G,T)]^{-1} \sum_{w \in W(G,T)} \det(w) \int_T \Phi_f^T(t)\, \xi_{w\tau}(t)\, dt = (-1)^r \hat{\Phi}_f^T(\tau). \|$$

Remark. The next step in our development is the consideration of the Fourier series of Φ_f^T. For this, we must give explicit form to the type of convergence we use relative to the lattice L_T. Let $\{\alpha_1, \dots, \alpha_l\}$ be the set of simple roots for the pair $(\mathfrak{G}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$ relative to the given ordering. Adopting the customary notation, we let

$$H_i = \frac{2}{\alpha_i(H_{\alpha_i})} H_{\alpha_i},\ i = 1, \dots, l.$$

If $\{\Lambda_1, \dots, \Lambda_l\}$ is the dual basis to $\{H_1, \dots, H_l\}$, then $L_T = \{\sum_{i=1}^l m_i \Lambda_i : m_i \in \mathbf{Z}\}$. For any positive integer m, define $L_T^m = \{\sum_{i=1}^l m_i \Lambda_i : -m \leqslant m_i \leqslant m\}$. Summability relative to L_T is then defined by

$$\sum_{\tau \in L_T} = \lim_{m \to \infty} \sum_{\tau \in L_T^m}. \tag{3.2}$$

For the remainder of this section, we fix an element $t_0 \in T'$.

LEMMA 3.3.

$$\Phi_f^T(t_0) = (-1)^r \sum_{\tau \in L_T} \Theta_\tau(f)\, \overline{\xi_\tau(t_0)} + I_f(t_0),$$

where

$$I_f(t_0) = (-1)^{r+1} \sum_{\tau \in L_T} \overline{\xi_\tau(t_0)} \int_{G^h} f(x)\, \Theta_\tau(x)\, d_G(x).$$

Proof. From the properties of Φ_f^T, it follows that the Fourier series of Φ_f^T converges to $\Phi_f^T(t_0)$ at t_0 (see [5]). Thus, $\Phi_f^T(t_0) = \sum_{\tau \in L_T} \hat{\Phi}_f^T(\tau)\, \overline{\xi_\tau(t_0)}$, and, from [4 e)], p. 316, we conclude that the series $\sum_{\tau \in L_T} \Theta_\tau(f)\, \overline{\xi_\tau(t_0)}$ converges absolutely. The assertion of the lemma is now clear. $\|$

Remarks. (i) Since Φ_f^T is, in general, only piecewise smooth, we cannot assert that the series for $\Phi_f^T(t_0)$ converges absolutely.

(ii) The results of Lemma 3.1 and Lemma 3.3 obviously are valid for groups G having split rank greater than one if we interpret G^h to be the complement of G^e in G'. For groups of split rank one, there is exactly one non-compact Cartan subgroup (up to conjugacy). Thus, to complete the inversion formula, we must express $I_f(t_0)$ in terms of the principal series associated to this non-compact Cartan subgroup or, more precisely, the invariant distributions $T^{(\chi,\nu)}$ introduced in 2. D.

From [4 e)], p. 309, we have $\bar{\Delta}_A = (-1)^{r+1}\Delta_A$ (r as in Lemma 3.1). Since $\varepsilon_R^A = 1$ on A^+ and $\varepsilon_R^A = -1$ on A^-, it follows from (2.5) and (2.13) that

$$\int_{G^h} f(x)\,\Theta_\tau(x)\,d_G(x) = (-1)^{r+1}[W(G,A)]^{-1} \sum_{w\in W(G,T)} \det(w)$$
$$\times \left\{ \int_{A^+} c(w\tau : A^+)\,\xi_{w\tau}(h_K)\exp\left(-\left|(w\tau)^\mu(\log h_{\mathfrak{p}})\right|\right)\Phi_f^A(h)\,d_A(h) \right.$$
$$\left. - \int_{A^-} c(w\tau : A^-)\,\xi_{w\tau}(h_K)\exp\left(-\left|(w\tau)^\mu(\log h_{\mathfrak{p}})\right|\right)\Phi_f^A(h)\,d_A(h) \right\}.$$

Now, using (2.7) and (2.12), we obtain

$$\begin{aligned}\int_{G^h} f(x)\,\Theta_\tau(x)\,d_G(x) &= (-1)^{r+1}[W(G,A)]^{-1} \sum_{w\in W(G,T)} \det(w)\\ &\times 2\int_{A^+} c(w\tau : A^+)\,\xi_{w\tau}(h_K)\exp\left(-\left|(w\tau)^\mu(\log h_{\mathfrak{p}})\right|\right)\Phi_f^A(h)\,d_A(h).\end{aligned} \tag{3.4}$$

Denote the integral over A^+ in (3.4) by $I_f^+(\tau : w)$. Then

$$I_f(t_0) = 2[W(G,A)]^{-1} \sum_{\tau\in L_T} \overline{\xi_\tau(t_0)} \sum_{w\in W(G,T)} \det(w)\,I_f^+(\tau : w). \tag{3.5}$$

For m a fixed positive integer, we consider the partial sum

$$\sum_{\tau\in L_T^m} \overline{\xi_\tau(t_0)} \sum_{w\in W(G,T)} \det(w)\,I_f^+(\tau : w) = \sum_{w\in W(G,T)} \det(w) \sum_{\tau\in L_T^m} \overline{\xi_\tau(t_0)}\,I_f^+(\tau : w).$$

The lattice L_T is $W(G,T)$ stable, and, if $w\in W(G,T)$, we define

$$wL_T^m = \{w\tau : \tau\in L_T^m\}.$$

Setting $I_f^+(\tau : 1) = I_f^+(\tau)$, we can then write the last sum as

$$\sum_{w\in W(G,T)} \det(w) \sum_{\tau\in w^{-1}L_T^m} \overline{\xi_{w\tau}(t_0)}\,I_f^+(\tau). \tag{3.6}$$

For further analysis, it is necessary to decompose the lattice L_T as in [4 f)], § 24. Let $L_T^* = \{\tau \in L_T : \tau(\sqrt{-1}\,(X^* - Y^*)) = 0\}$, a sublattice of L_T, and let L_0 be the lattice generated by L_T^* and α_t, that is $L_0 = \mathbf{Z}\alpha_t + L_T^*$. Then L_T/L_0 is a group of order two, and there exists an element τ_0 in $L_T \setminus L_0$(1) such that $\tau_0(\sqrt{-1}\,(X^* - Y^*)) = 1$. Observe that L_T^* may be identified with the (unitary) character group of $T_1/Z(A_\mathfrak{p})$. In particular, $\xi_\tau|_{T_2} = 1$ for $\tau \in L_T^*$. We also note that $\xi_{n\alpha}|_{T_1} \equiv 1$.

Fix $w \in W(G, T)$. The inner sum in (3.6) may be written

$$\sum_{\tau \in L_0 \cap w^{-1} L_T^m} \overline{\xi_{w\tau}(t_0)}\, I_f^+(\tau) + \sum_{\substack{\tau \in L_0 \\ \tau + \tau_0 \in w^{-1} L_T^m}} \overline{\xi_{w(\tau+\tau_0)}(t_0)}\, I_f^+(\tau + \tau_0). \tag{3.7}$$

We shall treat each of the last two sums separately.

If $\tau \in L_0$, we write $\tau = n\alpha_t + \tau^*$, $\tau^* \in L_T^*$. Then, with the understanding that the sums are over $L_0 \cap w^{-1} L_T^m$, we have

$$\sum_\tau \overline{\xi_{w\tau}(t_0)}\, I_f^+(\tau) = \sum_{\tau = n\alpha_t + \tau^*} \overline{\xi_{wn\alpha_t}(t_0)}\ \overline{\xi_{w\tau^*}(t_0)}$$
$$\times \int_{A_K} \int_{A_\mathfrak{p}} c(n\alpha : A^+)\, \overline{\xi_{n\alpha_t + \tau^*}(h_K)} \exp\left(-\left|(n\alpha_t)^\mu (\log h_\mathfrak{p})\right|\right) \Phi_f^A(h_K h_\mathfrak{p})\, d_{A_K}(h_K)\, d_{A_\mathfrak{p}}(h_\mathfrak{p}).$$

At this point, we write $\log h_\mathfrak{p} = tH^*$ and use the measure given by (2.1). Since

$$c(n\alpha_t : A^+) = \begin{cases} -1 & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ 1 & \text{if } n < 0, \end{cases}$$

and $\xi_{n\alpha_t}(h_K) = 1$, $h_K \in A_K = T_1$, we see that

$$\sum_\tau \overline{\xi_{w\tau}(t_0)}\, I_f^+(\tau) = c_A \left\{ \sum_{n<0} \overline{\xi_{n\alpha_t}(t_2(w))} \int_0^\infty e^{-|2nt|} dt \right.$$
$$\times \sum_{\tau^*} \overline{\xi_{\tau^*}(t_1(w))} \int_{A_K} \xi_{\tau^*}(h_K)\, \Phi_f^A(h_K \exp(tH^*))\, d_{A_K}(h_K) - \sum_{n>0} \overline{\xi_{n\alpha_t}(t_2(w))} \int_0^\infty e^{-2nt} dt$$
$$\left. \times \sum_{\tau^*} \overline{\xi_{\tau^*}(t_1(w))} \int_{A_K} \xi_{\tau^*}(h_K)\, \Phi_f^A(h_K \exp(tH^*))\, d_{A_K}(h_K) \right\}, \tag{3.8}$$

where we have written

$$w^{-1} t_0 = t_1(w)\, t_2(w),\ t_1(w) \in T_1,\ t_2(w) \in T_2. \tag{3.9}$$

This last decomposition is unique up to $Z(A_\mathfrak{p}) = T_1 \cap T_2 = \{1, \gamma\}$.

(1) We denote by $A \setminus B$ the set theoretic difference of sets A and B.

Define

$$F(h_K; h_{\mathfrak{p}}; \tau_1) = \xi_{\tau_1}(h_K)\,\Phi_f^A(h_K h_{\mathfrak{p}}) + \xi_{\tau_1}(\gamma h_K)\Phi_f^A(\gamma h_K h_{\mathfrak{p}}), \quad h_K \in A_K, h_{\mathfrak{p}} \in A_{\mathfrak{p}}^+, \tau_1 \in L_T. \tag{3.10}$$

LEMMA 3.11. *For $h_{\mathfrak{p}}$ and τ_1 fixed, the function $h_K \mapsto F(h_K; h_{\mathfrak{p}}; \tau_1)$ may be regarded as a function on $T_1/Z(A_{\mathfrak{p}})$, and, for any element h'_K in A_K,*

$$\sum_{\tau \in L_T^*} \overline{\xi_\tau(h'_K)} \int_{A_K} \xi_\tau(h_K)\,\xi_{\tau_1}(h_K)\Phi_f^A(h_K h_{\mathfrak{p}})\,d_{A_K}(h_K) = \tfrac{1}{2}\,F(h'_K; h_{\mathfrak{p}}; \tau_1).$$

Moreover, the series converges absolutely and uniformly in h'_K.

Proof. Since $\Phi_f^A \in C_c^\infty(A)$, the lemma follows from elementary Fourier analysis on $T_1/Z(A_{\mathfrak{p}})$. ||

Note that the sum $\sum_{\tau \in L_T^*}$ in the lemma may be taken as the limit of any sequence of partial sums due to the absolute convergence.

LEMMA 3.12.

$$\lim_{m\to\infty} \sum_{\tau \in L_0 \cap w^{-1}L_T^m} \overline{\xi_{w\tau}(t_0)}\, I_f^+(\tau) = (c_A/2) \sum_{a \in Z(A_{\mathfrak{p}})} \int_0^\infty \Phi_f^A(a t_1(w)\,h_t) \left[\frac{e^{-2t}(e^{-2\sqrt{-1}\,\theta_w} - e^{2\sqrt{-1}\,\theta_w})}{1 - 2e^{-2t}\cos 2\theta_w + e^{-4t}}\right] dt,$$

where $h_t = \exp(tH^*)$ and θ_w is determined by the equation $t_2(w) = \exp(\theta_w(X^* - Y^*))$.

(As indicated after (3.9), the value of θ_w is unique only up to $\{1, \gamma\}$. However, the expression above is independent of the choice of θ_w.)

Proof. We have $\overline{\xi_{n\alpha_t}(t_2(w))} = e^{2\sqrt{-1}\,n\theta_w}$, and $\theta_w \not\equiv 0 \pmod{\pi}$ since $t_0 \in T'$. From (3.8), we consider the partial sums

$$c_A \sum_{n<0} e^{2\sqrt{-1}\,n\theta_w} \int_0^\infty e^{-|2nt|}\,dt \sum_{\tau^*} \overline{\xi_{\tau^*}(t_1(w))} \int_{A_K} \xi_{\tau^*}(h_K)\,\Phi_f^A(h_K h_t)\,d_{A_K}(h_K)$$

and

$$c_A \sum_{n>0} e^{2\sqrt{-1}\,n\theta_w} \int_0^\infty e^{-2nt}\,dt \sum_{\tau^*} \overline{\xi_{\tau^*}(t_1(w))} \int_{A} \xi_{\tau^*}(h_K)\,\Phi_f^A(h_K h_t)\,d_{A_K}(h_K),$$

where $n\alpha_t + \tau^* \in L_0 \cap w^{-1}L_T^m$. From lemma 3.11, we see that the partial sums $\sum_{\tau^*}$ are uniformly bounded in t and are supported in a fixed compact set relative to $A_{\mathfrak{p}}$. Moreover, for any positive integer $\varkappa$,

$$\left|\sum_{n=1}^{\varkappa} e^{\pm 2\sqrt{-1}\,n\theta_w}\, e^{-2nt}\right| \leq \frac{2}{1 - \cos 2\theta_w}.$$

Thus, it follows from the bounded convergence theorem ([10], p. 345) that we may compute the limit as $m \to \infty$ for each of the above sums by taking

$$c_A \int_0^\infty \left[\lim_{m\to\infty} \sum_{n<0} e^{2\sqrt{-1}n\theta_w} e^{-|2nt|} \sum_{\tau^*} \overline{\xi_{\tau^*}(t_1(w))} \int_{A_K} \cdots \right] dt,$$

and

$$c_A \int_0^\infty \left[\lim_{m\to\infty} \sum_{n>0} e^{2\sqrt{-1}n\theta_w} e^{-|2nt|} \sum_{\tau^*} \overline{\xi_{\tau^*}(t_1(w))} \int_{A_K} \cdots \right] dt.$$

With the help of Lemma 3.11, this leads to

$$\begin{aligned}\lim_{m\to\infty} \sum_{\tau\in L_0 \cap w^{-1}L_T^m} & \overline{\xi_{w\tau}(t_0)}\, I_f^+(\tau) \\ &= (c_A/2) \int_0^\infty \left[\sum_{a\in Z(A_{\mathfrak{p}})} \Phi_f^A(at_1(w)\, h_t) \right] \left(\frac{e^{-2\sqrt{-1}\theta_w} e^{-2t}}{1-e^{-2\sqrt{-1}\theta_w} e^{-2t}} \right) dt \\ &\quad - (c_A/2) \int_0^\infty \left[\sum_{a\in Z(A_{\mathfrak{p}})} \Phi_f^A(at_1(w)\, h_t) \right] \left(\frac{e^{2\sqrt{-1}\theta_w} e^{-2t}}{1-e^{2\sqrt{-1}\theta_w} e^{-2t}} \right) dt.\end{aligned}$$

The conclusion of the lemma follows by addition. ‖

We next consider the second sum in (3.7). Observe that, for $\tau^* \in L_T^*$,

$$c(n\alpha_{\mathfrak{t}} + \tau^* + \tau_0 : A^+) = c(n\alpha_{\mathfrak{t}} + \tau_0 : A^+) = \begin{cases} -1 & \text{if } n \geqslant 0 \\ 1 & \text{if } n < 0, \end{cases}$$

and that $\xi_{\tau_0}(\gamma) = e^{-\sqrt{-1}\pi} = -1$.

LEMMA 3.13.

$$\begin{aligned}\lim_{m\to\infty} \sum_{\substack{\tau\in L_0 \\ \tau+\tau_0 \in w^{-1}L_T^m}} & \overline{\xi_{w(\tau+\tau_0)}(t_0)}\, I_f^+(\tau+\tau_0) \\ &= (c_A/2) \int_0^\infty \left[\Phi_f^A(t_1(w)\, h_t) - \Phi_f^A(\gamma t_1(w)\, h_t) \right] \left[\frac{e^{-2t}(e^t + e^{-t})\,(e^{-\sqrt{-1}\theta_w} - e^{\sqrt{-1}\theta_w})}{1 - 2e^{-2t}\cos 2\theta_w + e^{-4t}} \right] dt.\end{aligned}$$

Proof. Proceeding as above, we obtain

$$\begin{aligned}\sum_{\substack{\tau\in L_0 \\ \tau+\tau_0 \in w^{-1}L_T^m}} & \overline{\xi_{w(\tau+\tau_0)}(t_0)}\, I_f^+(\tau+\tau_0) \\ &= c_A\, \overline{\xi_{w\tau_0}(t_0)} \left\{ \sum_{n<0} \overline{\xi_{n\alpha_{\mathfrak{t}}}(t_2(w))} \int_0^\infty e^{-|2n+1|t}\, dt \sum_{\tau^*} \overline{\xi_{\tau^*}(t_1(w))} \int_{A_K} \xi_{\tau^*+\tau_0}(h_K)\, \Phi_f^A(h_K h_t)\, d_{A_K}(h_K) \right. \\ &\quad \left. - \sum_{n\geqslant 0} \xi_{n\alpha_{\mathfrak{t}}} \overline{(t_2(w))} \int_0^\infty e^{-|2n+1|t} dt \sum_{\tau^*} \overline{\xi_{\tau^*}(t_1(w))} \int_{A_K} \xi_{\tau^*+\tau_0}(h_K)\, \Phi_f^A(h_K h_t)\, d_{A_K}(h_K) \right\},\end{aligned}$$

where $\tau = n\alpha_t + \tau^* \in L_0$ and $\tau + \tau_0 \in w^{-1} L_T^m$. Now the result is obtained with the use of Lemma 3.11 and the same techniques that were employed in the proof of Lemma 3.12. ‖

The results of Lemma 3.12 and Lemma 3.13 may be combined to yield the following proposition.

PROPOSITION 3.14.

$$I_f(t_0) = (-1)^{r+1} \sum_{\tau \in L_T} \overline{\xi_\tau(t_0)} \int_{G^h} f(x)\, \Theta_\tau(x)\, d_G(x)$$

$$= [W(G,A)]^{-1} c_A \sum_{w \in W(G,T)} \det(w) \left\{ \int_0^\infty \Phi_f^A(t_1(w)\, h_t) \left[\frac{e^t(e^{-\sqrt{-1}\theta_w} - e^{\sqrt{-1}\theta_w})}{1 - 2e^t \cos\theta_w + e^{2t}}\right] dt \right.$$

$$\left. + \int_0^\infty \Phi_f^A(\gamma t_1(w)\, h_t) \left[\frac{e^t(e^{-\sqrt{-1}(\theta_w+\pi)} - e^{\sqrt{-1}(\theta_w+\pi)})}{1 - 2e^t \cos(\theta_w+\pi) + e^{2t}}\right] dt \right\}.$$

We emphasize once more that, for each $w \in W(G,T)$, θ_w is determined only modulo π, or, in other terms, we may choose θ_w so that $-\pi < \theta_w < 0$ or $0 < \theta_w < \tau$. The formula for $I_f(t_0)$ is, of course, independent of the choice of θ_w since $t_1(w)$ must be replaced by $\gamma t_1(w)$ if θ_w is replaced by $\theta_w \pm \pi$.

The Fourier transform of Φ_f^A is given by (2.17). Since $\Phi_f^A \in C_c^\infty(A)$, we have

$$\Phi_f^A(h_K h_t) = c_A^{-1} (2\pi)^{-\frac{1}{2}} \sum_{\chi \in \hat{A}_K} \overline{\chi(h_K)} \int_{-\infty}^\infty e^{-\sqrt{-1}\nu t} \hat{\Phi}_f^A(\chi,\nu)\, d\nu, \tag{3.15}$$

where $d\nu$ is normalized Lebesgue measure on $\mathbf{R}$.

PROPOSITION 3.16.

$$I_f(t_0) = [W(G,A)]^{-1} \sum_{w \in W(G,T)} \det(w)$$

$$\times \left\{ (e^{-\sqrt{-1}\theta_w} - e^{\sqrt{-1}\theta_w}) (2\pi)^{-\frac{1}{2}} \sum_{\chi \in \hat{A}_K} \overline{\chi(t_1(w))} \int_{-\infty}^\infty \hat{\Phi}_f^A(\chi,\nu) \int_0^\infty \frac{e^{\sqrt{-1}\nu t} e^t}{1 - 2e^t \cos\theta_w + e^{2t}}\, dt\, d\nu \right.$$

$$\left. + (e^{-\sqrt{-1}(\theta_w+\pi)} - e^{\sqrt{-1}(\theta_w+\pi)}) (2\pi)^{-\frac{1}{2}} \sum_{\chi \in \hat{A}_K} \overline{\chi(\gamma t_1(w))} \int_{-\infty}^\infty \hat{\Phi}_f^A(\chi,\nu) \int_0^\infty \frac{e^{-\sqrt{-1}\nu t} e^t}{1 + 2e^t \cos\theta_w + e^{2t}}\, dt\, d\nu \right\}.$$

Proof. Since all the series and integrals involved converge absolutely, the proof follows from Proposition 3.14, (3.15) and the fact that $\gamma \in A_K$. ‖

Now, using the fact that $\hat{\Phi}_f^A(\chi,\nu) = \hat{\Phi}_f^A(\chi,-\nu)$ (see (2.7) and (2.18)), we can write

$$\int_{-\infty}^\infty \hat{\Phi}_f^A(\chi,\nu) \int_0^\infty \frac{e^{-\sqrt{-1}\nu t} e^t}{1 \mp 2e^t \cos\theta_w + e^{2t}}\, dt\, d\nu = \frac{1}{2} \int_{-\infty}^\infty \hat{\Phi}_f^A(\chi,\nu) \int_0^\infty \frac{\lambda^{\sqrt{-1}\nu}}{1 \mp 2\lambda \cos\theta_w + \lambda^2}\, d\lambda, \tag{3.17}$$

where $d\lambda$ is normalized Lebesgue measure on $\mathbf{R}$.

The inner integrals in (3.17) can be evaluated using the formulas in [3], p. 297. This yields the following proposition.

PROPOSITION 3.18. *Suppose* $0 < \theta_w < \pi$, $w \in W(G, T)$. *Then*

$$I_f(t_0) = [W(G, A)]^{-1} \sum_{w \in W(G, T)} \det(w)$$
$$\times \left\{ \sqrt{-1}\,(\pi/2)^{\frac{1}{2}} \sum_{\chi \in \hat{A}_K} \overline{\chi(t_1(w))} \int_{-\infty}^{\infty} \hat{\Phi}_f^A(\chi, \nu) \left[\frac{\sinh(\nu(\theta_w - \pi))}{\sinh(\nu\pi)}\right] d\nu \right.$$
$$\left. + \sqrt{-1}\,(\pi/2)^{\frac{1}{2}} \sum_{\chi \in \hat{A}_K} \overline{\chi(\gamma t_1(w))} \int_{-\infty}^{\infty} \hat{\Phi}_f^A(\chi, \nu) \left[\frac{\sinh(\nu\theta_w)}{\sinh(\nu\pi)}\right] d\nu \right\}.$$

If $-\pi < \theta_w < 0$, *then* $\sinh(\nu(\theta_w - \pi))$ *must be replaced by* $\sinh(\nu(\theta_w + \pi))$ *in the first integral.*

We are now in a position to state the final inversion formula for $\Phi_f^T(t_0)$.

THEOREM 3.19. *Suppose that* $t_0 \in T'$. *For* $w \in W(G, T)$, *we write* $w^{-1} t_0 = t_1(w)\, t_2(w)$ *where* $t_1(w) \in T_1$ *and* $t_2(w) = \exp(\theta_w(X^* - Y^*)) \in T_2$. *Then, if* $f \in C_c^\infty(G)$ *and* $0 < \theta_w < \pi$ *for all* $w \in W(G, T)$, *we have*

$$\Phi_f^T(t_0) = (-1)^r \sum_{\tau \in L_T} \Theta_\tau(f)\, \overline{\xi_\tau(t_0)}$$
$$+ (\sqrt{-1}/2)\,(-1)^{r_I} [W(G, A)]^{-1} \sum_{w \in W(G, T)} \det(w) \sum_{\chi \in \hat{A}_K} \varepsilon(\chi)$$
$$\times \left\{ \overline{\chi(t_1(w))} \int_{-\infty}^{\infty} T^{(\chi, \nu)}(f) \left[\frac{\sinh(\nu(\theta_w - \pi))}{\sinh(\nu\pi)}\right] d\nu \right.$$
$$\left. + \overline{\chi(\gamma t_1(w))} \int_{-\infty}^{\infty} T^{(\chi, \nu)}(f) \left[\frac{\sinh(\nu\theta_w)}{\sinh(\nu\pi)}\right] d\nu \right\}.$$

If $-\pi < \theta_w < 0$, *then* $\sinh(\nu(\theta_w - \pi))$ *must be replaced by* $\sinh(\nu(\theta_w + \pi))$ *in the first integral.*

Proof. This follows from (2.18), Lemma 3.3 and Proposition 3.18. ||

The first sum can be formulated in a more representation theoretic way. Denote by $\hat{G}_d$ the set of equivalence classes of representations in the discrete series for G. If $\omega \in \hat{G}_d$, we write $\hat{f}(\omega)$ for $T_\tau(f) = T_\omega(f)$ where $\tau \in L_T'$ corresponds to ω and $T_\tau = T_\omega$ is given by (2.14). Then

$$\sum_{\tau \in L_T} \Theta_\tau(f)\, \overline{\xi_\tau(t_0)} = \sum_{\tau \in L_T^S} \Theta_\tau(f)\, \overline{\xi_\tau(t_0)} + \overline{\Delta_T(t_0)} \sum_{\omega \in \hat{G}_d} \hat{f}(\omega)\, \overline{T_\omega(t_0)}. \tag{3.20}$$

Remark. Suitably interpreted, Theorem 3.19 can be applied to $\mathbf{SL}(2, \mathbf{R})$ (see [8 b)]).

3. B. The Fourier transform of a regular hyperbolic orbit

The analysis involved in this section is completely elementary and has already been indicated in 3. A. We isolate the result here for future reference.

Let $h=h_K h_t \in A'$. Then, from (2.17), (2.18) and (3.15), we have

$$\Phi_f^A(h)=c_A^{-1}(2\pi)^{-1}(-1)^{r_I}\sum_{\chi\in\hat{A}_K}\varepsilon(\chi)\overline{\chi(h_K)}\int_{-\infty}^{\infty}e^{-\sqrt{-1}\nu t}T^{(\chi,\nu)}(f)\,d\nu,\quad f\in C_c^\infty(G). \tag{3.21}$$

4. The Plancherel formula for G

In this section, we derive the Plancherel formula for G from the inversion formula for Φ_f^T (Theorem 3.19). The derivation is quite simple. As in 2. C, let P_T denote the set of positive roots of the pair $(\mathfrak{G}_{\mathbf{C}},\mathfrak{t}_{\mathbf{C}})$ and set $\Pi^T=\prod_{\alpha\in P_T}H_\alpha$. We regard Π^T as a differential operator on T. Then, if $f\in C_c^\infty(G)$, the function $\Pi^T\Phi_f^T$ extends to a continuous function on T. With the measures normalized as in 2. B, we have

$$f(1)=M_G^{-1}\Phi_f^T(1;\Pi^T), \tag{4.1}$$

where $M_G=(2\pi)^r(-1)^s$ (see [4 a)], [4 c)]; the constant M_G is determined in [11], Ch. VIII).

Thus, we apply Π^T to Φ_f^T at a point $t_0\in T'$ and compute the limit of $\Pi^T\Phi_f^T(t_0)$ as t_0 approaches 1 through the regular elements in T.

THEOREM 4.2. (*The Plancherel formula*). *Let $f\in C_c^\infty(G)$ and denote by P_A the set positive roots of the pair $(\mathfrak{G}_{\mathbf{C}},\mathfrak{a}_{\mathbf{C}})$. Set $\hat{A}_K^{\pm}=\{\chi\in\hat{A}_K:\chi(\gamma)=\pm 1\}$. Then*

$$\begin{aligned}
f(1)=M_G^{-1}\sum_{\tau\in L_T'}\Big[\prod_{\alpha\in P_T}(\tau,\alpha)\Big]\Theta_\tau(f)+M_G^{-1}(\sqrt{-1}/2)\,([W(G,T)]/[W(G,A)])\\
\times\Bigg\{\sum_{\chi\in\hat{A}_K^+}\varepsilon(\chi)\int_{-\infty}^{\infty}T^{(\chi,\nu)}(f)\coth\Big(\frac{\pi\nu}{2}\Big)\Big[\prod_{\alpha\in P_A}\Big(\log\chi+\frac{\sqrt{-1}\nu}{2}\alpha_{\mathfrak{a}},\alpha\Big)\Big]d\nu\\
+\sum_{\chi\in\hat{A}_K^-}\varepsilon(\chi)\int_{-\infty}^{\infty}T^{(\chi,\nu)}(f)\tanh\Big(\frac{\pi\nu}{2}\Big)\Big[\prod_{\alpha\in P_A}\Big(\log\chi+\frac{\sqrt{-1}\nu}{2}\alpha_{\mathfrak{a}},\alpha\Big)\Big]d\nu\Bigg\}.
\end{aligned}$$

Our proof of the Plancherel formula differs from that of Harish-Chandra [4 f)], and avoids the use of the principal value integral ([4 e)], p. 308). The remainder of this section is devoted to proving Theorem 4.2.

Fix an element $t_0\in T'$ and consider the series $M_G^{-1}(-1)^r\sum_{\tau\in L_T}\Pi^T\overline{\xi_\tau(t_0)}\,\Theta_\tau(f)$. From [4 e)], it follows that this series converges absolutely and uniformly in a neighborhood of t_0 in T'. We conclude that

$$\lim_{t_0 \to 1} M_G^{-1}(-1)^r \Pi^T\Big(\sum_{\tau \in L_T} \overline{\xi_\tau(t_0)}\, \Theta_\tau(f)\Big) = M_G^{-1}(-1)^r \sum_{\tau \in L'_T} \Big[\prod_{\alpha \in P_T} (-\tau, \alpha)\Big] \Theta_\tau(f)$$

$$= M_G^{-1} \sum_{\tau \in L'_T} \Big[\prod_{\alpha \in P_T} (\tau, \alpha)\Big] \Theta_\tau(f). \tag{4.3}$$

This is the contribution of the discrete series to the Plancherel formula.

Next, assume that $\chi \in \hat{A}_K^+$ and extend χ trivially to all of T. We must consider the application of Π^T to

$$\overline{\chi(t_1(w))} \int_{-\infty}^{\infty} T^{(\chi,\nu)}(f) \left[\frac{\sinh(\nu(\theta_w \mp \pi)) + \sinh(\nu\theta_w)}{\sinh(\nu\pi)}\right] d\nu$$

$$= \overline{\chi(t_1(w))} \int_{-\infty}^{\infty} T^{(\chi,\nu)}(f) \left[\frac{\sinh\left(\nu\left(\theta_w \mp \dfrac{\pi}{2}\right)\right)}{\sinh\left(\dfrac{\nu\pi}{2}\right)}\right] d\nu, \tag{4.4}$$

where, as in Theorem 3.19, $w \in W(G, T)$, $t_0 \in T'$, $w^{-1}t_0 = t_1(w)\, t_2(w)$, $t_1(w) \in T_1$, $t_2(w) \in T_2$; and $t_2(w) = \exp(\theta_w(X^* - Y^*))$. In (4.4), we take $\sinh(\nu(\theta_w - (\pi/2)))$ if $0 < \theta_w < \pi$ and $\sinh(\nu(\theta_w + (\pi/2)))$ if $-\pi < \theta_w < 0$. Since we are interested in t_0 only in a neighborhood of 1, we may assume that $-\pi/2 < \theta_w < \pi/2$ for all $w \in W(G, T)$.

Since $\chi | T_2 \equiv 1$ and $\alpha_t | \mathfrak{t}_1 \equiv 0$, we can write

$$\overline{\chi(t_1(w))} = \overline{w\chi(t_0)}$$

and

$$\sinh\left(\nu\left(\theta_w \mp \frac{\pi}{2}\right)\right) = (\tfrac{1}{2})\, [e^{\mp \nu\pi/2}(w\xi_{\alpha_t}(t_0))^{\sqrt{-1}\nu/2} - e^{\pm \nu\pi/2}(w\xi_{\alpha_t}(t_0))^{-\sqrt{-1}\nu/2}].$$

In the last expression, we work with principal branch of the argument, and our restriction on θ_w eliminates any ambiguity

Now set

$$F_\chi^\pm(w : \nu : t_0) = \overline{w\chi(t_0)}\, [e^{\mp \nu\pi/2}(w\xi_{\alpha_t}(t_0))^{\sqrt{-1}\nu/2} - e^{\pm \nu\pi/2}(w\xi_{\alpha_t}(t_0))^{-\sqrt{-1}\nu/2}]. \tag{4.5}$$

From the properties of $T^{(\chi,\nu)}$ ([11], v. I, Ch. 5), it is clear that Π^T applied to (4.4) is equal to

$$(\tfrac{1}{2}) \int_{-\infty}^{\infty} T^{(\chi,\nu)}(f)\, [\Pi^T F_\chi^\pm(w : \nu : t_0)/\sinh(\nu\pi/2)]\, d\nu, \tag{4.6}$$

and if we consider the sum of the terms (4.6) over $\hat{A}_K^+$, it is also clear that the resulting series converges absolutely and uniformly in a neighborhood of t_0 in T'. We conclude, from Theorem 3.19 and (4.1), that

$$M_G^{-1}(\sqrt{-1}/2)(-1)^{r_I}[W(G,A)]^{-1}\sum_{w\in W(G,T)}\det(w)\sum_{\chi\in\hat{A}_K^+}\varepsilon(\chi)$$
$$\times(\tfrac{1}{2})\int_{-\infty}^{\infty}T^{(\chi,\nu)}(f)\,[\lim_{t_0\to 1}\Pi^T F_\chi^{\pm}(w:\nu:t_0)/\sinh(\nu\pi/2)]\,d\nu \tag{4.7}$$

represents the contribution to the Plancherel formula of the principal series indexed by $\chi\in\hat{A}_K^+$. Note that t_0 runs through a sequence of elements in T', and, for each t_0 and $w\in W(G,T)$, we take F_χ^+ or F_χ^- according to whether $0<\theta_w<\pi/2$ or $-\pi/2<\theta_w<0$ respectively.

If we work in a sufficiently small neighborhood of 0 in $\mathfrak{G}$, we can write

$$(\xi_{\alpha_t}(t_0\exp sH_\alpha))^{\pm\sqrt{-1}\nu/2}=(\xi_{\alpha_t}(t_0))^{\pm\sqrt{-1}\nu/2}(\xi_{\alpha_t}(\exp sH_\alpha))^{\pm\sqrt{-1}\nu/2},$$

$\alpha\in P_T$, $s\in\mathbf{R}$, and then

$$F_\chi^{\pm}(w:\nu:t_0:\Pi^T)=\Pi^T F_\chi^{\pm}(w:\nu:t_0)$$
$$=\det(w)\,\overline{w\chi(t_0)}\left\{e^{\mp\pi\nu/2}(w\xi_{\alpha_t}(t_0))^{\sqrt{-1}\nu/2}\left[\prod_{\alpha\in P_T}\left(\log\bar{\chi}+\frac{\sqrt{-1}\,\nu}{2}\alpha_t,\alpha\right)\right]\right.$$
$$\left.-e^{\pm\nu\pi/2}(w\xi_{\alpha_t}(t_0))^{-\sqrt{-1}\nu/2}\left[\prod_{\alpha\in P_T}\left(\log\bar{\chi}-\frac{\sqrt{-1}\,\nu}{2}\alpha_t,\alpha\right)\right]\right\}. \tag{4.8}$$

Note that the factor $\det(w)$ in (4.8) cancels with $\det(w)$ in (4.7).

Evidently

$$\prod_{\alpha\in P_T}\left(\log\bar{\chi}\pm\frac{\sqrt{-1}\,\nu}{2}\alpha_t,\alpha\right)=\prod_{\alpha\in P_A}\left(\log\bar{\chi}\pm\frac{\sqrt{-1}\,\nu}{2}\alpha_{\mathfrak{a}},\alpha\right),$$

and we claim that

$$\prod_{\alpha\in P_A}\left(\log\bar{\chi}+\frac{\sqrt{-1}\,\nu}{2}\alpha_{\mathfrak{a}},\alpha\right)=-\prod_{\alpha\in P_A}\left(\log\bar{\chi}-\frac{\sqrt{-1}\,\nu}{2}\alpha_{\mathfrak{a}},\alpha\right). \tag{4.9}$$

This claim is substantiated by the following observations.

(i) If α is compact, then $(\alpha_{\mathfrak{a}},\alpha)=0$.

(ii) If $\alpha=\alpha_{\mathfrak{a}}$, the unique positive real root of the pair $(\mathfrak{G}_{\mathbf{C}},\mathfrak{a}_{\mathbf{C}})$, then $(\log\bar{\chi},\alpha_{\mathfrak{a}})=0$.

(iii) If α is a positive complex root, then

$$\left(\log\bar{\chi}+\frac{\sqrt{-1}\,\nu}{2}\alpha_{\mathfrak{a}},\alpha\right)\left(\log\bar{\chi}+\frac{\sqrt{-1}\,\nu}{2}\alpha_{\mathfrak{a}},\bar{\alpha}\right)$$
$$=\left(\log\bar{\chi}-\frac{\sqrt{-1}\,\nu}{2}\alpha_{\mathfrak{a}},\alpha\right)\left(\log\bar{\chi}-\frac{\sqrt{-1}\,\nu}{2}\alpha_{\mathfrak{a}},\bar{\alpha}\right),$$

$\bar{\alpha}$ the conjugate of α. This last equality follows from the fact that $(\alpha_{\mathfrak{a}}, \alpha) = (\alpha_{\mathfrak{a}}, \bar{\alpha})$ since $\alpha_{\mathfrak{a}}$ is real, and the fact that $(\log \bar{\chi}, \alpha) + (\log \bar{\chi}, \bar{\alpha}) = 0$.

From (4.8) and (4.9), we have

$$F_{\chi}^{\pm}(w:\nu:1;\Pi^T) = 2\det(w)\cosh(\nu\pi/2)\left[\prod_{\alpha\in P_A}\left(\log\bar{\chi} + \frac{\sqrt{-1}\,\nu}{2}\alpha_{\mathfrak{a}}, \alpha\right)\right]. \tag{4.10}$$

Since $\log\bar{\chi} = -\log\chi$, we have

$$\prod_{\alpha\in P_A}\left(\log\bar{\chi} + \frac{\sqrt{-1}\,\nu}{2}\alpha_{\mathfrak{a}}, \alpha\right) = \prod_{\alpha\in P_A}\left(-\log\chi + \frac{\sqrt{-1}\,\nu}{2}\alpha_{\mathfrak{a}}, \alpha\right)$$
$$= (-1)^r\prod_{\alpha\in P_A}\left(\log\chi - \frac{\sqrt{-1}\,\nu}{2}\alpha_{\mathfrak{a}}, \alpha\right) = (-1)^{r+1}\prod_{\alpha\in P_A}\left(\log\chi + \frac{\sqrt{-1}\,\nu}{2}\alpha_{\mathfrak{a}}, \alpha\right)$$
$$= (-1)^{r_I}\prod_{\alpha\in P_A}\left(\log\chi + \frac{\sqrt{-1}\,\nu}{2}\alpha_{\mathfrak{a}}, \alpha\right).$$

the last equality following from the fact that the number of positive complex roots of the pair $(\mathfrak{G}_C, \mathfrak{a}_C)$ is even. Thus

$$F_{\chi}^{\pm}(w:\nu:1;\Pi^T) = 2\det(w)\cos(\nu\pi/2)(-1)^{r_I}\left[\prod_{\alpha\in P_A}\left(\log\chi + \frac{\sqrt{-1}\,\nu}{2}\alpha_{\mathfrak{a}}, \alpha\right)\right]. \tag{4.11}$$

An entirely analogous procedure can be followed for $\chi\in\hat{A}_{\bar{K}}$ (see (5.15) ff.) to complete the derivation of the Plancherel formula.

5. The Fourier transform of a semisimple orbit

Let y be a semisimple element in G, and let G_y be the centralizer of y in G. Then G_y is unimodular, and we denote by $d_{G/G_y}(\dot{x})$ a G-invariant measure on G/G_y. In this section, we compute the Fourier transform of the invariant distribution

$$f \mapsto \int_{G/G_y} f(^{\dot{x}}y)\, d_{G/G_y}(\dot{x}), \quad f\in C_c^\infty(G). \tag{5.1}$$

Since the distribution (5.1) is invariant, we may assume that $y\in A\cup T$. The cases when y is a regular element were treated in section 3, so we also assume that $y\notin A'\cup T'$.

Let $\mathfrak{G}_y$ be the centralizer of y in $\mathfrak{G}$, and let $\mathfrak{j}_y$ be a Cartan subalgebra of $\mathfrak{G}_y$ which is fundamental in $\mathfrak{G}_y$. Then, $\mathfrak{j}_y$ is a Cartan subalgebra of $\mathfrak{G}$ since rank $(\mathfrak{G}) = $ rank $(\mathfrak{G}_y)$. (Of course, $\mathfrak{j}_y$ need not be fundamental in $\mathfrak{G}$.) If J_y is the Cartan subgroup of G corresponding to $\mathfrak{j}_y$, then, by conjugating (if necessary), we may assume that $J_y = A$ or $J_y = T$.

Now denote by P_y^+ the set of positive roots of the pair $(\mathfrak{G}_y, \mathfrak{j}_y)$ and set $\Pi_y = \prod_{\alpha\in P_y^+} H_\alpha$. If Φ_f^y, $f \in C_c^\infty(G)$, is the invariant integral of f relative to J_y, then, according to a result of Harish-Chandra ([4 g)], p. 33), there exists a constant $M_y \neq 0$ such that

$$\int_{G/G_y} f(\dot{x}y)\, d_{G/G_y}(\dot{x}) = M_y \Phi_f^y(y; \Pi_y). \tag{5.2}$$

It is possible to combute M_y for a certain normalization of the relevant invariant measures.

In the remainder of this section, we compute $\Phi_f^y(y; \Pi_y)$ for the cases $J_y = A$ and $J_y = T$. In either case, we set

$$r_y = [P_y^+]. \tag{5.3}$$

The idea is the same as that used in the derivation of the Plancherel formula, that is, we compute $\Phi_f^y(x; \Pi_y)$ at a regular element x by using the formulas of section 3 and then let x approach y through a sequence of regular elements.

5. A. $J_y = A$

We consider the differential operator Π_y applied to Φ_f^A, where Φ_f^A is given by (3.21). If we write

$$y = y_K y_{\mathfrak{p}},\ y_K \in A_K,\ y_{\mathfrak{p}} \in A_{\mathfrak{p}},$$

then we have

$$\begin{aligned}\Phi_f^y(y; \Pi_y) &= \Phi_f^A(y; \Pi_y^A)\\ &= c_A^{-1}(2\pi)^{-1}(-1)^{r_I} \sum_{\chi\in\hat{A}_K} \varepsilon(\chi)\,\overline{\chi(y_K)} \int_{-\infty}^{\infty} e^{-\sqrt{-1}\nu t_y}\left[\prod_{\alpha\in P_y^+}\left(-\log\chi - \frac{\sqrt{-1}\,\nu}{2}\langle\alpha_{\mathfrak{a}}, \alpha\rangle\right)\right] T^{(\chi,\nu)}(f)\, d\nu,\end{aligned}$$

where $y_{\mathfrak{p}} = \exp(t_y H^*)$. The necessary facts relating to convergence have already been indicated in section 4.

Thus,

$$\begin{aligned}\Phi_f^A(y; \Pi_y^A) &= c_A^{-1}(2\pi)^{-1}(-1)^{r_I + r_y} \sum_{\chi\in\hat{A}_K} \varepsilon(\chi)\,\overline{\chi(y_K)}\\ &\quad\times \int_{-\infty}^{\infty} e^{-\sqrt{-1}\nu t_y}\left[\prod_{\alpha\in P_y^+}\left(\log\chi + \frac{\sqrt{-1}\,\nu}{2}\langle\alpha_{\mathfrak{a}}, \alpha\rangle\right)\right] T^{(\chi,\nu)}(f)\, d\nu.\end{aligned} \tag{5.4}$$

5. B. $J_y = T$

Here, we have $\Phi_f^y(y; \Pi_y) = \Phi_f^T(y; \Pi_y^T)$. If y is central, then $\Phi_f^T(y; \Pi_y^T)$ can be computed by a simple variant of the derivation of the Plancherel formula as presented in section 4. So, we assume that y is not central and proceed in a fashion similar to that of section 4. All the convergence arguments necessary in this section were used in the proof of the Plancherel formula so we shall not mention them explicitly here.

We first observe that the contribution of the invariant distributions $\Theta_\tau(f); \tau \in L_T$, to the formula for $\Phi_f^T(y; \Pi_y^T)$ is given by

$$(-1)^{r+r_y} \sum_{\tau \in L_T} \left[\prod_{\alpha \in P_y^+} (\tau, \alpha) \right] \Theta_\tau(f) \, \overline{\xi_\tau(y)}. \tag{5.5}$$

Now consider the contribution of the principal series. Let $W_y(G, T)$ be the subgroup of $W(G, T)$ generated by the compact roots in P_y^+. If G_y^0 is the identity component of G_y, then $W_y(G, T)$ is the quotient of the normalizer of T in G_y^0 by T. Choose elements $w_1 = 1$, $w_2, \dots, w_N$ in $W(G, T)$ such that

$$W(G, T) = \bigcup_{i=1}^{N} W_y(G, T)\, w_i \ \text{(disjoint union)}. \tag{5.6}$$

If $w \in W(G, T)$, we can write $w = w_y w_i$, $w_y \in W_y(G, T)$, for some i, $1 \leqslant i \leqslant N$. Moreover, we have

$$w_i^{-1} y = y_1(w_i)\, y_2(w_i),\ y_1(w_i) \in T_1,\ y_2(w_i) \in T_2. \tag{5.7}$$

Since the decomposition (5.7) is unique only up to $\{1, \gamma\}$, we may assume that

$$y_2(w_i) = \exp(\theta_{w_i}(X^* - Y^*)), \quad -\pi/2 \leqslant \theta_{w_i} < \pi/2. \tag{5.8}$$

Let t_0 be a regular element in T satisfying the conditions of section 4, that is, for $w \in W(G, T)$,

$$w^{-1} t_0 = t_1(w)\, t_2(w);\ t_2(w) = \exp(\theta_w(X^* - Y^*))$$

with $0 < |\theta_w| < \pi/2$. If we take $\chi \in \hat{A}_K^+$, apply the differential operator Π_y^T to $F_\chi^\pm(w : \nu : t_0)$ (see (4.5)), and then take the limit as t_0 approaches y through a sequence of regular elements which satisfy the conditions above, we obtain

$$\begin{aligned} F_\chi^\pm(w : \nu : y; \Pi_y^T) = \overline{w\chi(y)} \Bigg\{ & e^{\mp \nu\pi/2} (w\xi_{\alpha_t}(y))^{\sqrt{-1}\nu/2} \left[\prod_{\alpha \in P_y^+} \left(w\left(\log \overline{\chi} + \frac{\sqrt{-1}\,\nu}{2} \alpha_t \right), \alpha \right) \right] \\ & - e^{\pm \nu\pi/2} (w\xi_{\alpha_t}(y))^{-\sqrt{-1}\nu/2} \left[\prod_{\alpha \in P_y^+} \left(w\left(\log \overline{\chi} - \frac{\sqrt{-1}\,\nu}{2} \alpha_t \right), \alpha \right) \right] \Bigg\}. \end{aligned} \tag{5.9}$$

(Here, as before, we have extended χ trivially to all of T.)

If $w = w_y w_i$, $w_y \in W_y(G, T)$, then $\overline{w\chi(y)} = \overline{w_i \chi(y)}$, $w\xi_{\alpha_t}(y) = w_i \xi_{\alpha_t}(y)$ and

$$\prod_{\alpha\in P_y^+}\left(w\left(\log\bar{\chi}\pm\frac{\sqrt{-1}\,\nu}{2}\alpha_t\right),\alpha\right)=\det(w_y)\prod_{\alpha\in P_y^+}\left(w_i\left(\log\bar{\chi}\pm\frac{\sqrt{-1}\,\nu}{2}\right),\alpha\right).$$

Thus, for any fixed i, $1\leqslant i\leqslant N$,

$$\begin{aligned}(\tfrac{1}{2})&\sum_{w\in W_y(G,T)w_i}\det(w)\,F_\chi^\pm(w:\nu:y;\Pi_y^T)\\ &=(\tfrac{1}{2})\,[W_y(G,T)]\det(w_i)\,\overline{w_i\chi(y)}\left\{e^{\mp\nu\pi/2}e^{\nu\theta_{w_i}}\left[\prod_{\alpha\in P_y^+}\left(w_i\left(\log\bar{\chi}+\frac{\sqrt{-1}\,\nu}{2}\alpha_t\right),\alpha\right)\right]\right.\\ &\quad\left.-e^{\pm\nu\pi/2}e^{-\nu\theta_{w_i}}\left[\prod_{\alpha\in P_y^+}\left(w_i\left(\log\bar{\chi}-\frac{\sqrt{-1}\,\nu}{2}\alpha\right),\alpha\right)\right]\right\},\end{aligned}\tag{5.10}$$

where θ_{w_i} is defined by (5.8).

In the formula for $\Phi_j^T(y;\Pi_y^T)$, we use F_χ^+ in the case $0\leqslant\theta_{w_i}<\pi/2$ and F_χ^- if $-\pi/2\leqslant\theta_{w_i}\leqslant 0$. Although this appears to present a difficulty when $\theta_{w_i}=0$, we shall see below that this difficulty is easily resolved. Of course, we have a similar formula if $\chi\in\hat{A}_K^-$.

At this point, we analyze the product

$$\prod_{\alpha\in P_y^+}\left(w_i\left(\log\bar{\chi}+\frac{\sqrt{-1}\,\nu}{2}\alpha_t\right),\alpha\right)$$

in some special cases.

LEMMA 5.11. *Suppose that $\chi\in\hat{A}_K$ and $w_j^{-1}y\in T_1=A_K$ for some j, $1\leqslant j\leqslant N$. Then*

$$\prod_{\alpha\in P_y^+}\left(w_j\left(\log\bar{\chi}+\frac{\sqrt{-1}\,\nu}{2}\alpha_t\right),\alpha\right)=-\prod_{\alpha\in P_y^+}\left(w_j\left(\log\bar{\chi}-\frac{\sqrt{-1}\,\nu}{2}\alpha_t\right),\alpha\right).$$

Proof. Suppose first that $j=1$, that is, $w_j=w_1=1$. Then $y\in A_K$ and G_y contains both T and A. It follows that G_y is a split rank one group and the conclusion of the lemma may be obtained in the same fashion as (4.9). If $w_j\neq 1$, then $G_{w_j^{-1}y}=w_j^{-1}G_yw_j$ and $\alpha\in P_y^+$ if and only if $w_j^{-1}\alpha\in P^+_{w_j^{-1}y}$. Thus

$$\begin{aligned}\prod_{\alpha\in P_y^+}\left(w_j\left(\log\bar{\chi}+\frac{\sqrt{-1}\,\nu}{2}\alpha_t\right),\alpha\right)&=\prod_{\alpha\in P_{w_j^{-1}y}}\left(\log\bar{\chi}+\frac{\sqrt{-1}\,\nu}{2}\alpha_t,\alpha\right)\\ &=-\prod_{\alpha\in P_{w_j^{-1}y}}\left(\log\bar{\chi}-\frac{\sqrt{-1}\,\nu}{2}\alpha\,,\alpha\right)=-\prod_{\alpha\in P_y^+}\left(w_j\left(\log\bar{\chi}-\frac{\sqrt{-1}\,\nu}{2}\alpha_t\right),\alpha\right).\ \|\end{aligned}$$

COROLLARY 5.12. *Suppose that $\chi\in\hat{A}_K^+$* and *$w_j^{-1}y\in A_K$ for some j, $1\leqslant j\leqslant N$. Then $\theta_{w_j}=0$ and*

$$(\tfrac{1}{2})\sum_{w\in W_y(G,T)w_j}\det(w)\,F_\chi^\pm(w:\nu:y;\Pi_y^T)$$
$$=[W_y(G,T)]\det(w_j)\,\overline{w_j\chi(y)}\cosh(\nu\pi/2)\sum_{\alpha\in P_y^+}\left(w_j\left(\log\overline{\chi}+\frac{\sqrt{-1}\,\nu}{2}\alpha_t\right),\alpha\right).$$

Remark. The ambiguity in (5.10) is resolved by Corollary 5.12. If $\theta_{w_i}=0$, then the choice of either F_χ^+ or F_χ^- leads to the same result just as in the proof of the Plancherel formula.

LEMMA 5.13. *Suppose that* $w_j^{-1}y\in T_2\setminus T_1$ *and that* $w_j^{-1}y=y_2(w_j)$ *(see* (5.7) *and* (5.8)) *for some* j, $1\leqslant j\leqslant N$. *Then,*

$$\prod_{\alpha\in P_y^+}\left(w_j\left(\log\overline{\chi}+\frac{\sqrt{-1}\,\nu}{2}\alpha_t\right),\alpha\right)=\prod_{\alpha\in P_y^+}\left(w_j\left(\log\overline{\chi}-\frac{\sqrt{-1}\,\nu}{2}\alpha_t\right),\alpha\right).$$

Proof. Assume first that $j=1$, that is, $w_j=w_1\equiv 1$. Write $y=\exp(\theta_1(X^*-Y^*))$, where $-\pi/2\leqslant\theta_1<0$ or $0<\theta_1<\pi/2$. Now, G_y^0 is compact and P_y^+ is made up of compact roots. If $\alpha\in P_y^+$, we claim that $\alpha|\mathfrak{t}_2\equiv 0$. In fact, $\xi_\alpha(y)=e^{\theta_1\alpha(X^*-Y^*)}=1$ so that $\theta_1\alpha(X^*-Y^*)=2\pi\sqrt{-1}\,n$ for some integer n.

From [4f)], p. 121, we have $\sqrt{-1}(X^*-Y^*)=2H_{\alpha_t}/(\alpha_t,\alpha_t)$ which implies

$$\theta_1\alpha(X^*-Y^*)=\frac{\theta_1}{\sqrt{-1}}\frac{2(\alpha,\alpha_t)}{(\alpha_t,\alpha_t)}=2\pi\sqrt{-1}\,n.$$

From the theory of root systems, we know that

$$\left|\frac{2(\alpha,\alpha_t)}{(\alpha_t,\alpha_t)}\right|\in\{0,1,2,3\}.$$

The restrictions on θ_1 imply that $n=0$ and $\alpha(X^*-Y^*)=0$.

Now, for any $\alpha\in P_y^+$, we have

$$\left(\log\overline{\chi}+\frac{\sqrt{-1}\,\nu}{2}\alpha_t,\alpha\right)=(\log\overline{\chi},\alpha)=\left(\log\overline{\chi}-\frac{\sqrt{-1}\,\nu}{2}\alpha_t,\alpha\right)$$

so that

$$\prod_{\alpha\in P_y^+}\left(\log\overline{\chi}+\frac{\sqrt{-1}\,\nu}{2}\alpha_t,\alpha\right)=\prod_{\alpha\in P_y^+}\left(\log\overline{\chi}-\frac{\sqrt{-1}\,\nu}{2}\alpha_t,\alpha\right).$$

The remainder of the proof for $w_j\neq 1$ is similar to the proof of Lemma 5.11. ||

COROLLARY 5.14. *Suppose that* $\chi\in\hat{A}_K^+$, $w_j^{-1}y\in T_2\setminus T_1$ *and that* $w_j^{-1}y=y_2(w_j)$ *for some* j, $1\leqslant j\leqslant N$. *Then*

$$(\tfrac{1}{2}) \sum_{w \in W(G,T) w_j} \det(w)\, F_\chi^\pm(w:\nu:y;\Pi_y^T)$$
$$= [W_y(G,T)] \det(w_j) \sinh\left(\nu\left(\theta_{w_i} \mp \frac{\pi}{2}\right)\right) \prod_{\alpha \in P_y^+} (w_j(\log \bar{\chi}), \alpha),$$

where we take F_χ^+ and $\sinh(\nu(\theta_{w_i} - (\pi/2)))$ if $0 < \theta_{w_i} < \pi/2$, and we take F_χ^- and $\sinh(\nu(\theta_{w_i} + (\pi/2)))$ if $-\pi/2 \leqslant \theta_{w_i} < 0$.

In the case when $\chi \in \hat{A}_K^-$, we must consider the application of Π_y^T to

$$\overline{\chi(t_1(w))} \int_{-\infty}^{\infty} T^{(\chi,\nu)}(f) \left[\frac{\sinh(\nu(\theta_w \mp \pi)) - \sinh(\nu\theta_w)}{\sinh(\nu\pi)}\right] d\nu$$
$$= \overline{\chi(t_1(w))} \int_{-\infty}^{\infty} T^{(\chi,\nu)}(f) \left[\frac{+\cosh\left(\nu\left(\theta_w \mp \frac{\pi}{2}\right)\right)}{\cosh(\nu\pi/2)}\right] d\nu.$$

We extend χ to a function on T as follows. For $t \in T$, we write

$$t = t_1 t_2, \quad t_1 \in T_1, \quad t_2 \in T_2,$$
$$t_2 = \exp(\theta_2(X^* - Y^*)), \quad -\pi/2 \leqslant \theta_2 < \pi/2. \tag{5.15}$$

This decomposition is unique, and we set

$$\chi(t) = \chi(t_1). \tag{5.16}$$

We now define, for $t_0 \in T$,

$$G_\chi^\pm(w:\nu:t_0) = \mp \overline{w\chi(t_0)}\, [e^{\mp\nu\pi/2}(w\xi_{\alpha_t}(t_0))^{\sqrt{-1}\nu/2} + e^{\pm\nu\pi/2}(w\xi_{\alpha_t}(t_0))^{-\sqrt{-1}\nu/2}] \tag{5.17}$$

where $w^{-1}t_0$, as usual, decomposes according to (5.15) and (5.16). Take a regular element t_0 for which $0 < |\theta_w| < \pi/2$, $w \in W(G,T)$, apply the differential operator Π_y^T to $G_\chi^\pm(w:\nu:t_0)$, and take the limit as t_0 approaches y. This yields

$$G_\chi^\pm(w:\nu:y;\Pi_y^T) = \pm \overline{w\chi(y)} \left\{ e^{\mp\nu\pi/2}(w\xi_{\alpha_t}(y))^{\sqrt{-1}\nu/2} \left[\prod_{\alpha \in P_y^+} \left(w\left(\log\bar{\chi} + \frac{\sqrt{-1}\,\nu}{2}\alpha_t\right), \alpha\right)\right]\right.$$
$$\left. + e^{\pm\nu\pi/2}(w\xi_{\alpha_t}(y))^{-\sqrt{-1}\nu/2} \left[\prod_{\alpha \in P_y^+} \left(w\left(\log\bar{\chi} - \frac{\sqrt{-1}\,\nu}{2}\alpha_t\right), \alpha\right)\right]\right\}, \tag{5.18}$$

and, as in the case of $F_\chi^\pm$ (5.10), we obtain

$$(\tfrac{1}{2}) \sum_{w \in W_y(G,T) w_i} \det(w)\, G_\chi^{\pm}(w:\nu:y;\Pi_y^T)$$
$$= \pm(\tfrac{1}{2})[W_y(G,T)] \det(w_i)\, \overline{w_i\chi(y)} \left\{ \left[e^{\mp\nu\pi/2} e^{\nu\theta_{w_i}} \prod_{\alpha \in P_y^+} \left(w_i \left(\log\bar{\chi} + \frac{\sqrt{-1}\,\nu}{2}\alpha_t \right), \alpha \right) \right] \right.$$
$$\left. + e^{\pm\nu\pi/2} e^{-\nu\theta_{w_i}} \left[\prod_{\alpha \in P_y^+} \left(w_i \left(\log\bar{\chi} - \frac{\sqrt{-1}\,\nu}{2}\alpha_t \right), \alpha \right) \right] \right\}, \tag{5.19}$$

where θ_{w_i} is define by (5.8).

In the formula for $\Phi_f^T(y;\Pi_y^T)$, we use G_χ^+ when $0 \leqslant \theta_{w_i} < \pi/2$ and G_χ^- when $-\pi/2 \leqslant \theta_{w_i} \leqslant 0$. The ambiguity when $\theta_{w_i}=0$ is again handled by Lemma 5.11. We have the following analogue of Corollary 5.12.

COROLLARY 5.20. *Suppose that* $\chi \in \hat{A}_K^-$ *and* $w_j^{-1}y \in A_K$ *for some* j, $1 \leqslant j \leqslant N$. *Then* $\theta_{w_j}=0$ *and*

$$(\tfrac{1}{2}) \sum_{w \in W_y(G,T) w_j} \det(w)\, G_\chi^{\pm}(w:\nu:y;\Pi^T)$$
$$= [W_y(G,T)] \det(w_j)\, \overline{w_j\chi(y)} \sinh(\nu\pi/2) \prod_{\alpha \in P_y^+} \left(w_j \left(\log\bar{\chi} + \frac{\sqrt{-1}\,\nu}{2}\alpha \right), \alpha \right).$$

There is also an obvious analogue for Corollary 5.14. We now give the general formula for $\Phi_f^T(y:\Pi_y^T)$.

THEOREM 5.21. *Suppose that* y *is a non-regular, non-central element in* T *and that* $w^{-1}y$, $1 \leqslant i \leqslant N$, *is decomposed according to (5.7) and (5.8). Then*

$$\Phi_f^T(y;\Pi_y^T) = (-1)^{r+r_y} \sum_{\tau \in L_T} \left[\prod_{\alpha \in P_y^+} (\tau,\alpha) \right] \Theta_\tau(f)\, \overline{\xi_\tau(y)}$$
$$+ (\sqrt{-1}/4)(-1)^{r_I}([W_y(G,T)]/[W(G,A)]) \sum_{\chi \in \hat{A}_K^+} \varepsilon(\chi)$$
$$\times \left\{ \sum_{\substack{w_i \\ 0 \leqslant \theta_{w_i} < \pi/2}} \det(w_i) \int_{-\infty}^{\infty} T^{(\chi,\nu)}(f)\, [F^+(w_i:\nu:y;\Pi_y^T)/\sinh(\nu\pi/2)]\, d\nu \right.$$
$$\left. + \sum_{\substack{w_i \\ -\pi/2 \leqslant \theta_{w_i} < 0}} \det(w_i) \int_{-\infty}^{\infty} T^{(\chi,\nu)}(f)\, [F^-(w_i:\nu:y;\Pi_y^T)/\sinh(\nu\pi/2)]\, d\nu \right\}$$
$$+ (\sqrt{-1}/4)(-1)^{r_I}([W_y(G,T)]/[W(G,A)]) \sum_{\chi \in \hat{A}_K^-} \varepsilon(\chi)$$
$$\times \left\{ \sum_{\substack{w_i \\ 0 \leqslant \theta_{w_i} < \pi/2}} \det(w_i) \int_{-\infty}^{\infty} T^{(\chi,\nu)}(f)\, [G_\chi^+(w:\nu:y;\Pi_y^T)/\cosh(\nu\pi/2)]\, d\nu \right.$$

$$+ \sum_{\substack{w_i \\ -\pi/2 \leqslant \theta_{w_i} < 0}} \det(w_i) \int_{-\infty}^{\infty} T^{(\chi, \nu)}(f) [G_\chi^-(w:\nu:y; \Pi_y^T)/\cosh(\nu\pi/2)] \, d\nu \Bigg\}.$$

The proof of Theorem 5.21 follows from the preceding discussion. For particular y, the formula for $\Phi_f^T(y, \Pi_y^T)$ can be simplified by Corollaries 5.12, 5.14 and 5.20.

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