PHYSICAL STATES ON A C^*-ALGEBRA

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1. Introduction

Let A be a C^*-algebra with identity $\mathbf{1}$. A *physical state* is a function ϱ: $A \to \mathbf{C}$ which is a state on each singly generated C^*-subalgebra of A. Here "singly generated" means generated by $\mathbf{1}$ and a single self-adjoint element $a \in A$. The present paper is devoted to a discussion of whether a physical state ϱ on A is *linear*, i.e. whether it is a state in the ordinary sense. In the proper physical interpretation, this is the problem of linearity of the expectation functional on the algebra of observables in quantum mechanics, cf. Mackey [8] and Kadison [6].

Mathematically, the problem is also closely related to the following problem: Let R be a von Neumann algebra, and let P be the lattice of orthogonal projections in R. A function μ: $P \to \mathbf{R}^+$ such that $\mu(0) = 0$ is called a *completely additive measure* on P if

$$\mu\Big(\sum_{i \in I} e_i\Big) = \sum_{i \in I} \mu(e_i)$$

for any family $\{e_i\}_{i \in I}$ of mutually orthogonal projections in P. μ is a *probability measure* if $\mu(\mathbf{1}) = 1$. Given a probability measure μ on P one may ask whether there exists a positive normal state ϱ on R such that $\varrho | P = \mu$. This question, which poses what we may call the extension problem for measures (in non-commutative setting), was first suggested by Mackey. An affirmative solution for the special case where $R = \mathcal{L}(H) =$ all bounded linear operators on a separable Hilbert space H, with $\dim H \geqslant 3$, was given in an ingenious paper by Gleason [5]. In the case where the measure is the dimension-function on the projections of a type II_1-factor, the problem of extension is precisely the problem of the additivity of the trace [7], [9].

The connection between the extension problem for measures and the linearity problem

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for physical states, is established in this way: Let μ be a probability measure on the projections P in a von Neumann algebra R.

For each self-adjoint element $a \in R$ we may define

$$\varrho(a) = \int_{\sigma(a)} \lambda d\mu(e_\lambda)$$

where $\{e_\lambda\}$ is the spectral resolution of a, and $\sigma(a)$ is the spectrum of a. Since every element $x \in R$ may be written $x = a_1 + ia_2$ in a canonical way where a_1, a_2 are self-adjoint elements of R, we may extend ϱ to all of R by

$$\varrho(x) = \varrho(a_1) + i\varrho(a_2).$$

ϱ then becomes a physical state on R, and we have $\varrho | P = \mu$. ϱ is linear on R if and only if it solves the extension problem for μ. Hence the extension problem is a special case of the linearity problem for physical states. The latter, in the form it is given above, is due to R. V. Kadison.

In § 3 of the present paper we give a complete solution for the case of a physical state ϱ on a commutative C^*-algebra A. When A is non-commutative, the problem remains unsolved in general. However, as Gleason's result indicates, solutions in particular cases may be found. In § 4 we give a brief outline of methods and results in this direction. A detailed exposition will be published elsewhere.

We are indebted to R. V. Kadison for calling our attention to these questions, for his helpfulness through several discussions on the subject, and for his steady encouragement. We also wish to express our gratitude to J. M. G. Fell, E. G. Effros and C. Akemann for valuable conversations.

Throughout this paper concepts and results from the theory of C^*-algebras will be used quite freely. Our general reference is the book by Dixmier [4].

2. Physical states and quasi-states

Let A be a C^*-algebra with an identity $\mathbf{1}$. A *physical state* is a function ϱ: $A \to C$ which is a state on each singly generated C^*-subalgebra of A. Here "singly generated" means generated by $\mathbf{1}$ and a single selfadjoint element $a \in A$.

We start with two simple examples which show how much linearity it is reasonable to expect from a physical state. First, let A be non-abelian, and let a be a non-normal element in A (if each element in A is normal, then A is abelian, so non-normal elements exist). Let ϱ be a state on A such that $\varrho(a^*a - aa^*) \neq 0$. Define, for any $x \in A$

$$\varrho_1(x) = \varrho(x) + \varrho(x^*x - xx^*).$$

Clearly $\varrho_1 = \varrho$ on A_h (= the self-adjoint elements in A) and on any abelian C^*-subalgebra of A, so ϱ_1 is a physical state on A. But ϱ_1 is not linear on A.

Next, let A be abelian with two generators $a, b \in A_h$. Then $a + ib$ is not contained in any singly generated C^*-subalgebra of A, so the set N of elements not contained in any singly generated C^*-subalgebra is not empty. There is a state ϱ on A which does not vanish identically on N, and we define

$$\varrho_1(x) = \begin{cases} \varrho(x) & \text{if} \quad x \in A \setminus N \\ 0 & \text{if} \quad x \in N \end{cases}$$

Then $\varrho = \varrho_1$ on A_h and on any singly generated C^*-subalgebra of A, but ϱ_1 is not linear on A. Hence, even in the commutative case, non-linear physical states exist. This kind of non-linearity is not a serious deficiency. What we want, is to show that a physical state is linear on A_h, since the bounded observables are supposed to correspond to the self-adjoint elements of A. However, to avoid trivial technical complications, we introduce the following

Definition. Let A be a C^*-algebra. A *positive quasi-linear functional* is a function $\varrho: A \to \mathbf{C}$ such that

(i) $\varrho \mid B$ is a positive linear functional for each singly generated C^*-subalgebra B of A.

(ii) $\varrho(a) = \varrho(a_1) + i\varrho(a_2)$, when $a = a_1 + ia_2$ is the canonical decomposition of a in self-adjoint parts a_1, a_2.

If in addition

(iii) $\sup \{\varrho(a) : a \in A;\ \|a\| \leq 1;\ a \geq 0\} = 1$ then we say that ϱ is a *quasi-state* on A.

Observe that if A has an identity, then (iii) is equivalent to the condition $\varrho(1) = 1$. We may also note that if two positive quasi-linear functionals ϱ and γ coincide on each singly generated C^*-subalgebra of A, then $\varrho = \gamma$ by (ii). Clearly (i) implies that ϱ is real on self-adjoint elements, so by (ii) it follows that $\overline{\varrho(a^*)} = \varrho(a)$ for all $a \in A$. Let us use the notation

$$\|\varrho\| = \sup \{\varrho(a) : a \in A;\ \|a\| \leq 1;\ a \geq 0\}$$

It is easily seen (by [7], 2.1.5 (vi)) that if A does not have an identity, then a positive quasi-linear functional ϱ may be extended to a positive quasi-linear functional $\tilde{\varrho}$ on $\tilde{A}$ (= the C^*-algebra obtained by adjoining an identity to A) by defining $\tilde{\varrho}(1) = K$; if $K \geq \|\varrho\|$. In particular a quasi-state on A extends to a quasi-state on $\tilde{A}$.

We shall use the following notation: If $x \in A_h$, $A(x)$ is the C^*-subalgebra generated by x (and 1 if A has an identity).

3. Quasi-states on abelian C^*-algebras

THEOREM 1. *Any positive quasi-linear functional ϱ on an abelian C^*-algebra A is linear.*

The proof of this theorem is rather lengthy and essentially measure theoretic. It is obtained through a sequence of lemmas. Without loss of generality we may assume that A has an identity $\mathbf{1}$, and that $\varrho(\mathbf{1})=1$.

LEMMA 1. *If $0 \leqslant a$, $0 \leqslant b$; $a, b \in A$ and $ab=0$, then there is $c \in A_h$ such that $a, b \in A(c)$.*

Proof. Let B be the C^*-subalgebra of A generated by a, b and $\mathbf{1}$, and let Y be the set of pure states on B, regarded as a compact Hausdorff space. It is sufficient to show that Y is homeomorphic to a compact subset of $\mathbf{R}$. For $p \in Y$ we have

$$0 = p(ab) = p(a)p(b),$$

so we may define

$$\Phi(p) = \begin{cases} p(a) & \text{if} \quad p(b)=0 \\ -p(b) & \text{if} \quad p(a)=0. \end{cases}$$

Clearly $\Phi(p)=0$ if $p(a)=p(b)=0$. Let $p \neq q$, $p, q \in Y$. Since $p(\mathbf{1})=q(\mathbf{1})=1$, we must have $p(a) \neq q(a)$ or $p(b) \neq q(b)$, which shows that Φ is one-to-one. The range of Φ is compact. Let $p_\nu \to p$ in Y; we may assume $p(a) \neq 0$, $p(b)=0$. Then $p_\nu(a) \to p(a)$, so there is an index ν_0 such that $\nu \geqslant \nu_0$ implies $p_\nu(a) \neq 0$ and hence also $p_\nu(b)=0$. It follows that $\Phi(p_\nu) \to \Phi(p)$, so Φ is continuous and therefore a homeomorphism.

Some notation. Let X be the set of pure states on A, regarded as a compact Hausdorff space. We identify A and $C(X)=$ the space of all continuous complex functions on X. For each $a \in A_h$, let $\sigma(a)=a(X)$ be the spectrum of a. $\sigma(a)$ is a compact subset of $\mathbf{R}$ and there is an isometric isomorphism of $C(\sigma(a))$ into A with range $A(a)$. If $f \in C(\sigma(a))$, the map $f \to f(a) \in A(a)$ is given by

$$f(a)(s) = f(a(s)), \quad s \in X.$$

Let $a \in A_h$, and suppose that ϱ is a quasi-state on A. Then $\varrho \mid A(a)$ is a state, and determines a state ϱ_a on $C(\sigma(a))$ by

$$\varrho_a(f) = \varrho(f(a)), \quad f \in C(\sigma(a)).$$

By Riesz' theorem ϱ therefore gives rise to a unique regular Borel-measure μ_a on $\sigma(a)$ such that

$$\varrho(f(a)) = \int_{\sigma(a)} f(\lambda)\, d\mu_a(\lambda), \quad \lambda \in \sigma(a),$$

for all $f \in C(\sigma(a))$.

In the sequel, we let C, F, K denote compact sets, and we let U, V, W denote open sets without necessarily further mentioning that they are compact or open respectively. For any set A, the interior of A is denoted by A^o.

LEMMA 2. *Let $a \in A_h$, $K \subseteq \sigma(a)$. Suppose $a^{-1}(K) \subseteq U \subseteq X$. Then there is an open set $V \subseteq \sigma(a)$; $V \supseteq K$, such that $a^{-1}(K) \subseteq a^{-1}(V) \subseteq U$.*

Proof. $K' = a(X \setminus U)$ is compact and disjoint from K. Hence $V = \sigma(a) \setminus K'$ satisfies the conditions of the lemma.

LEMMA 3. *Let ϱ be a quasi-state on A and φ a state on A. For each $a \in A_h$, let μ_a be the measure on $\sigma(a)$ determined by ϱ, and let μ be the regular Borel measure on X determined by φ. Then $\varrho = \varphi$ if and only if*

$$\mu_a(K) = \mu(a^{-1}(K)) \tag{*}$$

for each $a \in A_h$ and each compact set $K \subseteq \sigma(a)$.

Proof. Suppose $\varrho = \varphi$. Take $a \in A_h$ and $K \subseteq \sigma(a)$. Choose $\varepsilon > 0$. There is a continuous function f: $\sigma(a) \to [0, 1]$ such that $f(\lambda) \equiv 1$ for $\lambda \in K$, satisfying $\varrho_a(f) = \int_{\sigma(a)} f d\mu_a \leqslant \mu_a(K) + \varepsilon$. Now $f(a)$: $X \to [0, 1]$ and $f(a)(s) \equiv 1$ for $s \in a^{-1}(K)$, so $\mu(a^{-1}(K)) \leqslant \varphi(f(a)) = \varrho(f(a)) = \varrho_a(f) \leqslant \mu_a(K) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, it follows that $\mu(a^{-1}(K)) \leqslant \mu_a(K)$. On the other hand, for $\varepsilon > 0$ we may choose $U \supseteq a^{-1}(K)$ such that $\mu(U \setminus a^{-1}(K)) < \varepsilon$. We then take V as in Lemma 2, and a continuous function g on $\sigma(a)$ satisfying $\chi_K \leqslant g \leqslant \chi_V$. Then

$$\mu_a(K) \leqslant \varrho_a(g) = \varrho(g(a)) = \varphi(g(a)) \leqslant \mu(U) < \mu(a^{-1}(K)) + \varepsilon.$$

It follows that $\mu_a(K) \leqslant \mu(a^{-1}(K))$ and (*) holds.

Conversely, suppose that (*) is valid for all $a \in A_h$ and $K \subseteq \sigma(a)$. Take an $a \in A_h$ and let φ_a be the restriction of φ to $A(a)$, and let ν_a be the corresponding measure on $\sigma(a)$. Then, by the first part of the proof, $\nu_a(K) = \mu(a^{-1}(K))$ for each $K \subseteq \sigma(a)$. By (*) it follows that $\nu_a(K) = \mu_a(K)$ for all K, and hence by regularity $\nu_a = \mu_a$. But then $\varphi_a = \varrho \,|\, A(a)$, so φ coincides with ϱ on each singly generated C^*-subalgebra, i.e. $\varphi = \varrho$. The proof is complete.

Hence, to prove that the quasi-state ϱ is linear on A, it is sufficient to construct a regular Borel measure μ on X satisfying (*) of Lemma 3. Let $\mathcal{K}$ denote the collection of compact subsets of X, and let $\mathcal{K}'$ denote the sub-collection of compact sets of the form $a^{-1}(K)$; $a \in A_h$, $K \subseteq \sigma(a)$. We start by defining μ on $\mathcal{K}$ by

$$\mu(K) = \inf \{\varrho(a) \colon a \in A_h;\ a \geqslant \chi_K\}.$$

Then μ is a set-function on $\mathcal{K}$, $\mu(\emptyset) = 0$, $\mu(X) = 1$ and $K_1 \subseteq K_2$ implies $\mu(K_1) \leqslant \mu(K_2)$. It is clear that in taking the inf above for a given K, we may assume $a \leq 1$. Indeed, since $a \in A_h$, $\min(a, 1) \in A(a)$ so $\varrho(\min(a, 1)) \leq \varrho(a)$.

LEMMA 4. *For $K \in \mathcal{K}$ and $\varepsilon > 0$ given, there is $F \in \mathcal{K}'$ such that $K \subseteq F^o$ and $\mu(F) < \mu(K) + \varepsilon$.*

Proof. We first note that if $a \in A_h$ and $C \subseteq \sigma(a)$, then $\mu(a^{-1}(C)) \leq \mu_a(C)$. Indeed, choose $f \in \mathcal{C}(\sigma(a))$; $f \geq \chi_C$ such that $\varrho_a(f) \leq \mu_a(C) + \varepsilon$. Then $f(a) \geq \chi_{a^{-1}(C)}$, so $\mu(a^{-1}(C)) \leq \varrho(f(a)) = \varrho_a(f) \leq \mu_a(C) + \varepsilon$ which proves the claim. Now let $K \in \mathcal{K}$ and let $\varepsilon > 0$ be given. Choose $a \in A_h$; $\chi_K \leq a \leq 1$ such that $\varrho(a) < \mu(K) + \varepsilon$. Then choose $\delta > 0$ such that $\mu_a([1-\delta, 1] \cap \sigma(a)) < \mu_a(\{1\}) + \varepsilon$, and put $F = a^{-1}([1-\delta, 1] \cap \sigma(a))$. Then $K \subseteq F^o$ and

$$\mu(F) \leq \mu_a([1-\delta, 1] \cap \sigma(a)) < \mu_a(\{1\}) + \varepsilon \leq \int_{\sigma(a)} \lambda \, d\mu_a(\lambda) + \varepsilon = \varrho(a) + \varepsilon < \mu(K) + 2\varepsilon$$

from which the lemma follows.

LEMMA 5. *For any $a \in A_h$ and $K \subseteq \sigma(a)$ we have $\mu_a(K) = \mu(a^{-1}(K))$.*

Proof. By the first part of the proof of the preceeding lemma, we already know that that $\mu(a^{-1}(K)) \leq \mu_a(K)$. Now let $K_1 = a^{-1}(K)$ and choose, for $\varepsilon > 0$ given, $F \subseteq X$ such that $K_1 \subseteq F^o$ and $\mu(F) < \mu(K_1) + \varepsilon$ (Lemma 4). By Lemma 2 there is an open set $V \subseteq \sigma(a)$ such that $K_1 \subseteq a^{-1}(V) \subseteq F^o$. Let $f \in \mathcal{C}(\sigma(a))$ satisfy $\chi_K \leq f \leq \chi_V$. Then $\chi_{K_1} \leq f(a) \leq \chi_{F^o}$. Let b be any element of A_h satisfying $\chi_F \leq b \leq 1$. Then $(1-b)f(a) = 0$, so by Lemma 1, $f(a)$ and b belong to the same singly generated C^*-subalgebra. Since $0 \leq f(a) \leq b$ it follows that $\varrho(f(a)) \leq \varrho(b)$. Thus $\varrho(f(a)) \leq \mu(F)$ by the definition of μ. Hence $\mu_a(K) \leq \varrho_a(f) = \varrho(f(a)) \leq \mu(F) < \mu(a^{-1}(K)) + \varepsilon$, so $\mu_a(K) \leq \mu(a^{-1}(K))$. This, together with the opposite inequality, finishes the proof.

LEMMA 6. *Let $K \in \mathcal{K}$. For each $U \supseteq K$ and each $\varepsilon > 0$ there is $a \in A_h$ satisfying $\chi_K \leq a \leq \chi_U$ and $\varrho(a) < \mu(K) + \varepsilon$.*

Proof. Let $K \subseteq U$ and $\varepsilon > 0$ be given. By Lemma 4 choose $F \in \mathcal{K}'$ such that $K \subseteq F^o$ and $\mu(F) < \mu(K) + \varepsilon$. Then pick an $a \in A_h$ such that $\chi_K \leq a \leq \chi_{U \cap F^o}$. The same argument as in the proof of Lemma 5 shows that $\varrho(a) \leq \varrho(b)$ for any b satisfying $\chi_F \leq b \leq 1$, so that $\varrho(a) \leq \mu(F) < \mu(K) + \varepsilon$, and the proof is complete.

LEMMA 7. *If $K_1, K_2 \in \mathcal{K}$, and $K_1 \cap K_2 = \emptyset$, then $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$.*

Proof. Since $K_1 \cap K_2 = \emptyset$, there are open sets $U_1 \supseteq K_1$, $U_2 \supseteq K_2$ such that $U_1 \cap U_2 = \emptyset$. By Lemma 6 we may choose $a, b \in A_h$ satisfying $\chi_{K_1} \leq a \leq \chi_{U_1}$, $\chi_{K_2} \leq b \leq \chi_{U_2}$, and $\varrho(a) < \mu(K_1) + \varepsilon$, $\varrho(b) < \mu(K_2) + \varepsilon$. Now $ab = 0$, so by Lemma 1

$$\mu(K_1 \cup K_2) \leq \varrho(a+b) = \varrho(a) + \varrho(b) < \mu(K_1) + \mu(K_2) + \varepsilon$$

proving that $\mu(K_1 \cup K_2) \leqslant \mu(K_1) + \mu(K_2)$. On the other hand, Lemma 6 tells us that $\mu(K_1 \cup K_2) = \inf \{\varrho(c): \chi_{K_1 \cup K_2} \leqslant c \leqslant \chi_U\}$ for any open set $U \supseteq K_1 \cup K_2$. We may therefore take $U = U_1 \cup U_2$ as above, and choose $c \in A_h$ such that $\varrho(c) < \mu(K_1 \cup K_2) + \varepsilon$, $\chi_{K_1 \cup K_2} \leqslant c \leqslant \chi_{U_1 \cup U_2}$. Then $c = a + b$, with $ab = 0$ and $\chi_{K_1} \leqslant a \leqslant \chi_{U_1}$, $\chi_{K_2} \leqslant b \leqslant \chi_{U_2}$, so

$$\mu(K_1) + \mu(K_2) \leqslant \varrho(a) + \varrho(b) = \varrho(c) < \mu(K_1 \cup K_2) + \varepsilon,$$

which gives us the opposite inequality and finishes the proof.

Now defined, for U open $\subseteq X$

$$\mu(U) = 1 - \mu(X \setminus U)$$

If U is also compact, this is consistent with the previous definition of μ on $\mathcal{K}$, by Lemma 7.

LEMMA 8. *For each* U,

$$\mu(U) = \sup_{K \subseteq U} \mu(K).$$

Proof. Let $K' = X \setminus U$. If $K \subseteq U$, $K \cap K' = \emptyset$, so $\mu(K) + \mu(K') = \mu(K \cup K') \leqslant 1$. Hence $\mu(K) \leqslant 1 - \mu(K') = \mu(U)$ for each $K \subseteq U$.

By Lemma 4 there is $F \in \mathcal{K}'$ such that $F \supseteq K'$ and $\mu(F) < \mu(K') + \varepsilon$ for any preassigned $\varepsilon > 0$. We have $F = a^{-1}(C)$ for some $a \in A_h$, $C \subseteq \sigma(a)$. Let $W = \sigma(a) \setminus C$, and put $V = a^{-1}(W)$, so $V \subseteq U$. μ_a is regular, so there is a compact set $C_1 \subseteq W$ such that $\mu_a(C_1) > \mu_a(W) - \varepsilon$. Then, by Lemma 5

$$\begin{aligned} \mu(a^{-1}(C_1)) &= \mu_a(C_1) > \mu_a(W) - \varepsilon \\ &= 1 - \mu_a(C) - \varepsilon = 1 - \mu(F) - \varepsilon \\ &> 1 - \mu(K') - 2\varepsilon = \mu(U) - 2\varepsilon \end{aligned}$$

which proves the lemma.

We observe that if $a \in A_h$ and $f \in C(\sigma(a))$, then for any set $B \subseteq \mathbf{R}$ we have $f(a)^{-1}(B) = a^{-1}(f^{-1}(B))$. Indeed

$$\begin{aligned} f(a)^{-1}(B) &= \{s \in X: f(a)(s) \in B\} \\ &= \{s \in X: f(a(s)) \in B\} \\ &= a^{-1}(f^{-1}(B)). \end{aligned}$$

LEMMA 9. *Let* $K_1, K_2 \in \mathcal{K}$ *with* $K_1 \subseteq K_2$. *Then* $\mu(K_2) - \mu(K_1) = \inf_{U \supseteq K_2 \setminus K_1} \mu(U)$.

Proof. Let $\varepsilon > 0$ be given. We first choose $U \supseteq K_2$ such that $\mu(U) < \mu(K_2) + \varepsilon$. This choice can be made by virtue of Lemma 4 and the observation that if $V \subseteq C$ for any V and C, then $\mu(V) \leqslant \mu(C)$ (from Lemma 8). Now choose $a \in A_h$ such that $\chi_{K_2} \leqslant a \leqslant \chi_U$, and let

$F=\{s\in X: a(s)\geqslant \frac{1}{2}\}$. Take $b\in A_h$ such that $\chi_F\leqslant b\leqslant \chi_U$, and let $W=\{s\in X: b(s)>\frac{1}{2}\}$. Then we have

$$K_2\subseteq F^o\subseteq F\subseteq W\subseteq U.$$

For $n=1, 2, \ldots$, let $b_n\in A_h$ be chosen such that $\chi_{K_1}\leqslant b_n\leqslant \chi_{F^o}$, and $\varrho(b_n)<\mu(K_1)+1/n$ (Lemma 6). Put $F_n=\{s\in X: b_n(s)=1\}$. Then $K_1\subseteq F_n\subseteq F^o$ for all n; and $\mu(K_1)\leqslant \mu(F_n)\leqslant \varrho(b_n)<\mu(K_1)+1/n$. Thus $\mu(F_n)\to\mu(K_1)$ as $n\to\infty$. With $V_n=(X\setminus F_n)\cap W$, V_n is open, $V_n\cap F_n=\emptyset$ and $V_n\cup F_n=W$ for all n. We observe that $b_n(1-b)=0$ for all n. Hence, by Lemma 1, there is an $a_n\in A_h$ such that b, $b_n\in A(a_n)$; $n=1, 2, \ldots$. By the observation preceding this lemma, this implies that, for each n, there are sets V'_n, F'_n and $W'\subseteq\sigma(a_n)$ such that $V_n=a_n^{-1}(V'_n)$, $F_n=a_n^{-1}(F'_n)$ and $W=a_n^{-1}(W')$. We have $V'_n\cap F'_n=\emptyset$ and $V'_n\cup F'_n=W'$. Thus, by Lemma 5 and the definition of $\mu(W)$, we get

$$\mu(W)=\mu_{a_n}(W')=\mu_{a_n}(V'_n)+\mu_{a_n}(F'_n)=\mu(V_n)+\mu(F_n).$$

With $V=(X\setminus K_1)\cap W$, V is open and contains $K_2\setminus K_1$. Since $F_n\supseteq K_1$, we get $V_n\subseteq V$; $n=1,2,\ldots$. So $\mu(V)\geqslant\mu(V_n)=\mu(W)-\mu(F_n)\to\mu(W)-\mu(K_1)$, so $\mu(W)\leqslant\mu(V)+\mu(K_1)$. On the other hand, if $K\subseteq V$ it follows that $K\cap K_1=\emptyset$ and $K\cup K_1\subseteq W$, so $\mu(W)\geqslant\mu(K_1\cup K_2)=\mu(K_1)+\mu(K)$ by Lemma 7. By Lemma 8 we then get $\mu(W)\geqslant\mu(K_1)+\mu(V)$ which combined with the opposite inequality above gives $\mu(V)=\mu(W)-\mu(K_1)$. Consequently

$$\mu(K_2)-\mu(K_1)\leqslant\mu(W)-\mu(K_1)=\mu(V)\leqslant\mu(U)-\mu(K_1)<\mu(K_2)-\mu(K_1)+\varepsilon,$$

since $W\subseteq U$. As $\varepsilon>0$ was arbitrary, this inequality completes the proof.

LEMMA 10. *Let $K_1, K_2\in\mathcal{K}$ with $K_1\subseteq K_2$. Then* $\mu(K_2)-\mu(K_1)=\sup_{K\subseteq K_2\setminus K_1}\mu(K)$.

Proof. First, if $K\subseteq K_2\setminus K_1$, then $K_1\cap K=\emptyset$ and $K_1\cup K\subseteq K_2$, so $\mu(K_1)+\mu(K)\leqslant\mu(K_2)$, i.e. $\mu(K)\leqslant\mu(K_2)-\mu(K_1)$. Now choose, by Lemma 4, $F\supseteq K_1$ such that $F^o\supseteq K_1$ and $\mu(F)\leqslant\mu(K_1)+\varepsilon$. Put $K=(X\setminus F^o)\cap K_2$ so $K\subseteq K_2\setminus K_1$. We observe that $(F\cap K_2)\setminus K_1\supseteq K_2\setminus(K\cup K_1)$, so, by the preceding lemma, we get $\mu(F\cap K_2)-\mu(K_1)\geqslant\mu(K_2)-\mu(K\cup K_1)$. Hence we have

$$\varepsilon\geqslant\mu(F)-\mu(K_1)\geqslant\mu(F\cap K_2)-\mu(K_1)\geqslant\mu(K_2)-\mu(K\cup K_1)=\mu(K_2)-\mu(K_1)-\mu(K),$$

so $\mu(K)>\mu(K_2)-\mu(K_1)-\varepsilon$ which proves the lemma.

Let
$$\Gamma=\{A\subseteq X: A=K_2\setminus K_1;\ K_2\supseteq K_1\}$$

We claim that Γ is a semi-ring of sets in the sense of Zaanen [10]. That is, we must check that

(a) $A, B \in \Gamma \Rightarrow A \cap B \in \Gamma$

(b) $A, B \in \Gamma$ and $B \subseteq A \Rightarrow A \setminus B = \bigcup_{n=1}^{\infty} C_n$, where $C_n \in \Gamma$ for all n, and the sets C_n are mutually disjoint

(c) $\emptyset \in \Gamma$.

LEMMA 11. *Γ is a semi-ring.*

Proof. (c) is evidently satisfied, $\emptyset = X \setminus X$. (a): If $A = K_1 \setminus K_2$, $B = C_1 \setminus C_2$ with $K_2 \subseteq K_1$, $C_2 \subseteq C_1$, then $A \cap B = K_1 \cap C_1 \setminus ((K_2 \cap C_1) \cup (C_2 \cap K_1))$ so $A \cap B \in \Gamma$. (b): Let A, $B \in \Gamma$ with $A \supseteq B$. If $A = K_1 \setminus K_2$, then $B \subseteq K_1$ so $\bar{B} \subseteq K_1$. Since $B = C_1 \setminus C_2$ for some pair of compact sets $C_1 \supseteq C_2$, it follows that $\bar{B} \setminus B$ is compact. Let $C_1' = \bar{B} \cup K_2$, $C_2' = (\bar{B} \setminus B) \cup K_2$. Then $C_1' \supseteq C_2'$ and both sets are compact, and $B = C_1' \setminus C_2'$ since $B \cap K_2 = \emptyset$. Furthermore, $K_1 \supset C_1' \supseteq C_2' \supseteq K_2$, so we can write $A \setminus B = (K_1 \setminus C_1') \cup (C_2' \setminus K_2)$ which is a disjoint union of sets in Γ.

Hence Γ is a semi-ring which contains all open and all compact subsets of X. If $A \in \Gamma$, $A = K_2 \setminus K_1$ with $K_2 \supseteq K_1$, we define

$$\mu(A) = \mu(K_2) - \mu(K_1).$$

By Lemma 9 and 10 this definition is unambiguous, and we have

$$\mu(A) = \sup_{K \subseteq A} \mu(K) = \inf_{U \supseteq A} \mu(U).$$

The definition of μ on Γ is clearly consistent with the previous definitions of μ on open and compact sets. We are going to show that μ is a measure on Γ. First we need:

LEMMA 12. (a) *If $U_1 \subseteq U_2$, then $U_2 \setminus U_1 \in \Gamma$ and $\mu(U_2 \setminus U_1) = \mu(U_2) - \mu(U_1)$.*
(b) *For U_1, U_2 arbitrary (open) $\mu(U_1 \cup U_2) \leqslant \mu(U_1) + \mu(U_2)$.*

Proof. (a) $U_2 \setminus U_1 = (X \setminus U_1) \setminus (X \setminus U_2) \in \Gamma$, so

$$\mu(U_2 \setminus U_1) = \mu(X \setminus U_1) - (X \setminus U_2) = 1 - \mu(U_1) - 1 + \mu(U_2) = \mu(U_2) - \mu(U_1).$$

(b) Put $U = U_1 \cup U_2$. Then, by (a), $\mu(U) = \mu(U \setminus U_1) + \mu(U_1) \leqslant \mu(U_2) + \mu(U_1)$, from Lemma 9, since $U \setminus U_1 \subseteq U_2$ and $U \setminus U_1 \in \Gamma$.

LEMMA 13. *μ is a measure on Γ.*

Proof. We know from the definition of μ and Lemma 9 that $\mu(\emptyset) = 0$, $\mu(A) \geqslant 0$ for all $A \in \Gamma$ and that $A \subseteq B \Rightarrow \mu(A) \leqslant \mu(B)$, for A, $B \in \Gamma$. The only thing left to verify is therefore that μ is countably additive on Γ.

Let $\{A_n\} \subseteq \Gamma$ with the A_n pairwise disjoint, and suppose $A = \bigcup_{n=1}^{\infty} A_n$ belongs to Γ. For $\varepsilon > 0$, there is $K \subseteq A$ such that $\mu(K) > \mu(A) - \varepsilon$, and there is $U_n \supseteq A_n$ such that $\mu(U_n) < \mu(A_n) + \varepsilon/2^n$ for each n. Now $\bigcup_{n=1}^{\infty} U_n \supseteq \bigcup_{n=1}^{\infty} A_n = A \supseteq K$, so there is an m such that $\bigcup_{n=1}^{m} U_n \supseteq K$. Hence

$$\sum_{n=1}^{\infty} \mu(A_n) \geqslant \sum_{n=1}^{\infty} \mu(U_n) - \varepsilon \geqslant \sum_{n=1}^{m} \mu(U_n) - \varepsilon \geqslant \mu(\bigcup_{n=1}^{m} U_n) - \varepsilon \geqslant \mu(K) - \varepsilon > \mu(A) - 2\varepsilon,$$

where we have used Lemma 12(b). So $\mu(A) \leqslant \sum_{n=1}^{\infty} \mu(A_n)$. On the other hand, let $K_n \subseteq A_n$ be chosen in such a way that $\mu(K_n) > \mu(A_n) - \varepsilon/2^n$, $n = 1, 2, \ldots$. Let p be any positive integer. Then $\bigcup_{n=1}^{p} K_n$ is compact and contained in A. The sets K_n are clearly disjoint, so

$$\mu(A) \geqslant \mu(\bigcup_{n=1}^{p} K_n) = \sum_{n=1}^{p} \mu(K_n) > \sum_{n=1}^{p} \mu(A_n) - \varepsilon$$

from Lemma 7. Hence $\mu(A) \geqslant \sum_{n=1}^{p} \mu(A_n)$ for any p, which, combined with the inequality already proved, gives $\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$.

By standard arguments (see for instance [10]) it now follows that μ extends to a regular measure on the σ-field of Borel-sets in X. By Lemma 5 μ satisfies (*) of Lemma 3, so if we take φ to be the state on A given by $\varphi(a) = \int_X a(s)\, d\mu(s)$; $a \in A$, $s \in X$, then $\varphi = \varrho$. The proof of the theorem is complete.

4. Quasi-states on general C^*-algebras

Let ϱ be a quasi-state on $\mathcal{L}(H)$ (=all bounded linear operators on a Hilbert-space H), and let P denote the set of orthogonal projections on H. If we assume that $\dim H \geqslant 3$ and that $\varrho | P$ is completely additive, then Gleason's theorem [5] states that there is a positive, normal state φ on $\mathcal{L}(H)$ such that $\varphi | P = \varrho | P$. It is now simple to show that $\varrho = \varphi$, so that ϱ in fact is linear on H. Without the assumption that $\varrho | P$ is completely additive, i.e. that it is merely finitely additive, the problem remains unsolved. However, by a slight modification and extension of Gleason's methods, one may show that *any quasi-state on* $\mathcal{LC}(H)$ (=all compact linear operators on H) *is linear if* $\dim H \geqslant 4$. The details of this proof, and the proofs of the other results mentioned in this section will be published in a forthcoming paper [1].

Let A be a C^*-algebra and let Q be the set of all positive quasi-linear functionals ϱ on A with $\|\varrho\| \leqslant 1$. Q is clearly a convex set under the pointwise operations, and if we give Q the topology of pointwise convergence on A, we can show that Q is compact. Let us say that a quasi-state ϱ is *pure* if each element $\gamma \in Q$ such that $\gamma \leqslant \varrho$ is of the form $\gamma = \lambda \varrho$, with $0 \leqslant \lambda \leqslant 1$, $\lambda \in \mathbf{R}$. As for ordinary states it turns out that a quasi-state ϱ is pure if and only if

ϱ is an extreme point of Q different from 0. On the basis of the Krein-Milman theorem it is therefore clear that to show that an arbitrary $\varrho \in Q$ is linear, it is sufficient to show that any pure quasi-state on A is linear.

The introduction of pure quasi-states makes a subdivision of the problem possible. Let ϱ be a pure quasi-state on the C^*-algebra A. Suppose for a moment that ϱ is actually linear. Then there is a standard way of associating a representation π_ϱ of A to ϱ. π_ϱ is irreducible and $I = \ker \pi_\varrho$ is a primitive ideal of A. ϱ vanishes on I so it lifts to a pure state $\bar{\varrho}$ on A/I, and if φ: $A \to A/I$ is the quotient homomorphism, we have $\varrho = \bar{\varrho} \circ \varphi$. This procedure suggests the following subdivision of the linearity problem for a pure quasi-state ϱ on A.

(I) Find a primitive ideal I of A such that $\varrho(I) = 0$.

(II) Lift ϱ to a pure quasi-state $\bar{\varrho}$ on A/I, such that $\varrho = \bar{\varrho} \circ \varphi$, where φ is the quotient map.

(III) Show that $\bar{\varrho}$ is linear on A/I.

Since φ is linear, a positive solution of (I), (II) and (III) will imply that $\varrho = \bar{\varrho} \circ \varphi$ is linear.

With respect to (I), one may show that this is true if Prim A (= the set of all primitive ideals of A) is a Hausdorff-space in the hull-kernel topology. The basis of the proof of this fact is the following recent result by Dauns and Hofmann [2]. If A is a C^*-algebra, $\mathcal{C}^b(\text{Prim}\, A)$ is the space of all bounded continuous complex-valued functions on Prim A, then for each $x \in A$, $h \in \mathcal{C}^b(\text{Prim}\, A)$ there is an element $y \in A$ such that

$$y(\text{mod}\, I) = h(I) \cdot (x(\text{mod}\, I))$$

for all $I \in \text{Prim}\, A$. More conveniently, this result may be reformulated as follows. For each $I \in \text{Prim}\, A$, let A/I be the quotient C^*-algebra, and let φ_I: $A \to A/I$ be the canonical quotient homomorphism. Let $\Gamma = \sum_{\oplus, I \in \text{Prim}\, A} A/I$ be the C^*-direct product of the A/I. We define a map φ: $A \to \Gamma$ by

$$\varphi(x) = \{\varphi_I(x)\}.$$

φ then becomes an isometric isomorphism of A onto a closed C^*-subalgebra of Γ. We may therefore identify A with its image under φ, and write x for $\varphi(x)$. In this way x becomes a vector-field on Prim A with values in the various A/I, and we write $x(I) = \varphi_I(x)$; $x \in A$, $I \in \text{Prim}\, A$. With these conventions the Dauns–Hofmann result takes the form: For each $x \in A$, and $h \in \mathcal{C}^b(\text{Prim}\, A)$, there is $y \in A$ such that

$$y(I) = h(I)\, x(I); \quad I \in \text{Prim}\, A.$$

Now suppose that Prim A is Hausdorff, and let ϱ be a pure quasi-state on A. The positive solution of (I) then takes the form: There is an element $I_0 \in \text{Prim } A$ such that if $x \in A$, then $x(I_0) = 0 \Rightarrow \varrho(x) = 0$.

The lifting problem (II) now takes the form: if $x, y \in A$ and $x(I_0) = y(I_0)$, can we show that $\varrho(x) = \varrho(y)$? A simple calculation shows that if $h = h^* \in \mathcal{C}^b(\text{Prim } A)$, and $h(I_0) = 1$, then $\varrho(x) = \varrho(hx)$ for all $x \in A$. From this it follows that if $x(I) = y(I)$ on some neighbourhood W of I_0 in Prim A, then $\varrho(x) = \varrho(y)$. Indeed, choose $h \in \mathcal{C}^b(\text{Prim } A)$ such that $h \geqslant 0$, $h(I_0) = 1$ and h vanishes outside W. Then $hx = hy$, so $\varrho(x) = \varrho(hx) = \varrho(hy) = \varrho(y)$.

In the special case where Prim A is discrete, it therefore follows that (II) has a positive solution.

A positive linear functional on a C^*-algebra is automatically continuous. This is far from obvious for quasi-states. However, if we *assume* that the pure quasi-state ϱ on A is continuous, then ϱ can indeed by lifted to a pure quasi-state $\tilde{\varrho}$ on A/I_0.

With respect to part (III) of the problem, the situation is less encourageing. Only very special solutions exist. However, on the basis of the fact mentioned previously, that any quasi-state on $\mathcal{LC}(H)$ is linear if $\dim H \geqslant 3$, one may show the following results:

(a) Any quasi-state on a dual C^*-algebra A is linear if $\dim \pi \neq 2$ for all irreducible representations π of A.

(b) If A is CCR with Prim A Hausdorff and $\dim \pi \neq 2$ for all irreducible representations π of A, then each pure, continuous quasi-state on A is linear.

Some other cases will be treated in [1].

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