ON THE NUMBER OF DIVISORS OF QUADRATIC POLYNOMIALS

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1. Introduction

The problem of determining the asymptotic behaviour, as $x \to \infty$, of the divisor sum

$$S(x) = \sum_{n \leqslant x} d(n^2 + a),$$

where $d(\mu)$ denotes the number of (positive) divisors of μ, has been mentioned by a number of writers [1], [2], [5], [8]. When we consider this problem it is not difficult to see that the case where $-a$ is a perfect square k^2, say, is exceptional, since then $n^2 + a$ can be factorized as $(n-k)(n+k)$. In this case the sum is almost identical with the sum

$$\sum_{n \leqslant x} d(n)\, d(n+2k),$$

which has been considered by Ingham [7]; in fact a slight adaptation of Ingham's method shews here that

$$S(x) = A_1(a)\, x \log^2 x + O(x \log x) \qquad (a = -k^2).$$

We shall not, therefore, refer to this case again. In the case when $-a$ is not a perfect square for some considerable time it has been commonly realized (see, for example, the remarks by Bellman [1] and the author [5]) that it is possible to deduce an asymptotic formula

$$S(x) = A_2(a)\, x \log x + O(x)$$

by a familiar elementary method; a proof of such a formula (with a less precise error term) has recently been supplied by Scourfield [8].

In this paper we shall examine the behaviour of $S(x)$ in more detail. Our primary object will be to replace the elementary formula for $S(x)$ by the formula

$$S(x) = A_2(a)\, x \log x + A_3(a)\, x + O(x^{\frac{8}{9}} \log^3 x).$$

We begin by transforming $S(x)$ so that it is expressed in terms of three sums $\sum_3$, $\sum_4$, and $\sum_5$. A fairly straightforward estimation then shews that $\sum_3$ and $\sum_4$ give rise to the explicit terms in the formula. The main difficulties are encountered in the estimation of $\sum_5$, which ultimately will be seen to be of a lower order of magnitude than $\sum_3$ and $\sum_4$. The sum $\sum_5$ is expressed in terms of a new type of exponential sum, which is defined in terms of a quadratic congruence. The theory of binary quadratic forms is used to obtain a non-trivial estimate for this exponential sum.

Similar but more complicated methods enable us to prove corresponding asymptotic formulae for the sums

$$\sum_{n \leqslant x} d(an^2 + bn + c),$$

$$\sum_{n \leqslant x} r(an^2 + bn + c),$$

where $r(\mu)$ denotes the number of representations of μ as the sum of two integral squares. The method, however, fails in more than one respect when applied to the conjugate sum

$$\sum_{\nu < \sqrt{n}} d(n - \nu^2).$$

The behaviour of the latter sum has in fact been determined by the author in a previous paper [5].

The theory of the exponential sums occurring in $\sum_5$ is related to another problem, which has been thought to be of sufficient interest to merit discussion here. The estimate obtained for these sums shews that there is a certain regularity in the distribution of the roots of the congruence

$$\nu^2 \equiv D \pmod{k}$$

for fixed D and variable k. At the end it is shewn that the ratio ν/k is distributed uniformly in the sense of Weyl.

2. Notation and conventions

The following notation and conventions will be adopted throughout.

Except in Sections 6 and 9, a denotes a non-zero integer such that $-a$ is not a perfect square. In Sections 6 and 9 the letter a is replaced by $-D$, where D is not a perfect square. In Section 6, in accordance with the classical notation, (a, b, c) denotes a binary quadratic form $ax^2+2bxy+cy^2$ with integral coefficients.

The letters $d, k, l, m, n, t, \lambda$, and N are positive integers; h, r, s, μ, v, ϱ, and σ are integers.

The meaning of x and y, when not occurring as indeterminates in a quadratic form, is as follows; x is a continuous real variable, which is to be regarded as tending to infinity; y is a real number not less than 1.

The positive highest common factor of r and s is denoted by (r, s); $d(h)$ is the number of positive divisors of h; $\sigma_\beta(h)$ is the sum of the βth powers of the positive divisors of h; moduli of congruences may be either positive or negative; $[u]$ is the greatest integer not exceeding u; $\|u\|$ is the function of period 1 which equals $|u|$ for $-\frac{1}{2}<u\leqslant\frac{1}{2}$.

The letters A, A_1, A_2, etc., are positive constants (not necessarily the same on each occurrence) that depend at most on a (or D); $A(h), A_1(h), A_2(h)$, etc., are constants that depend at most on h and a (or D). The equation $f=O(|g|)$ denotes an inequality of the type $|f|\leqslant A|g|$, true for all values of the variables consistent with stated conditions.

The symbol $\sum^+$ denotes a summation in which n is restricted to values for which n^2+a is positive.

3. Initial transformation of sum

We have
$$\sum_{n\leqslant x}^{+} d(n^2+a)=\sum_{\substack{kl=n^2+a\\ n\leqslant x}}^{+} 1.$$

It is clear that in the right-hand sum not more than one of k and l can exceed $(x^2+a)^{\frac{1}{2}}=X$, say. Hence

$$\sum_{n\leqslant x}^{+} d(n^2+a)=\sum_{k\leqslant X}^{+}+\sum_{l\leqslant X}^{+}-\sum_{k,l\leqslant X}^{+}=2\sum_{k\leqslant X}^{+}-\sum_{k,l\leqslant X}^{+}=2\sum\nolimits_1-\sum\nolimits_2, \text{ say.} \tag{1}$$

Next, let $a_1=\max(0, -a)$, and let $T_k(y)$ and $T_k^+(y)$ be defined for any $y\geqslant 1$ by

$$T_k(y)=\sum_{\substack{n^2\equiv -a \pmod k\\ n\leqslant y}} 1, \qquad T_k^+(y)=\sum_{\substack{n^2\equiv -a \pmod k\\ n\leqslant y}}^{+} 1,$$

so that, since $-a$ is not a perfect square, we have

$$T_k(y)-T_k^+(y)\begin{cases}\leqslant a_1^{\frac{1}{2}}, & \text{if } k\leqslant a_1,\\ =0, & \text{if } k>a_1.\end{cases}$$

Then we have
$$\sum\nolimits_1 = \sum_{k\leqslant X} T_k^+(x) = \sum_{k\leqslant X} T_k(x) + O(1), \tag{2}$$

and
$$\sum\nolimits_2 = \sum_{k\leqslant X} T_k^+(Y_k) = \sum_{k\leqslant X} T_k(Y_k) + O(1), \tag{3}$$

where $Y_k = (kX-a)^{\frac{1}{2}}$. Furthermore we have

$$T_k(y) = \sum_{\substack{\nu^2\equiv -a(\mathrm{mod}\, k)\\ 0<\nu\leqslant k}} \sum_{\substack{n\equiv\nu(\mathrm{mod}\, k)\\ n\leqslant y}} 1 = \sum_{\substack{\nu^2\equiv -a(\mathrm{mod}\, k)\\ 0<\nu\leqslant k}} \left(\left[\frac{y-\nu}{k}\right] - \left[\frac{-\nu}{k}\right]\right)$$
$$= \sum_{\substack{\nu^2\equiv -a(\mathrm{mod}\, k)\\ 0<\nu\leqslant k}} \left\{\frac{y}{k} + \psi\left(\frac{y-\nu}{k}\right) - \psi\left(\frac{-\nu}{k}\right)\right\},$$

where, for any real u, $\psi(u)$ denotes $[u]-u+\frac{1}{2}$. Let $\varrho(k)$ be the number of roots of the congruence $\nu^2 \equiv -a(\mathrm{mod}\, k)$, let

$$\Psi_k(y) = \sum_{\substack{\nu^2\equiv -a(\mathrm{mod}\, k)\\ 0<\nu\leqslant k}} \psi\left(\frac{y-\nu}{k}\right),$$

and let
$$\Phi_k(y) = \sum_{\substack{\nu^2\equiv -a(\mathrm{mod}\, k)\\ 0<\nu\leqslant k}} \psi\left(\frac{-\nu}{k}\right).$$

Then
$$T_k(y) = y\,\frac{\varrho(k)}{k} + \Psi_k(y) - \Phi_k(y). \tag{4}$$

Now, since to every root of $\nu^2 \equiv -a\,(\mathrm{mod}\, k)$ there corresponds a root $k-\nu$, it is evident that $\Phi_k(y)$ vanishes unless the congruence has a root congruent to 0 (mod k). Hence

$$\Phi_k(y) = \begin{cases} O(1), & \text{if } k\,|\,a, \\ 0, & \text{if } k\nmid a. \end{cases} \tag{5}$$

We deduce from (1), (2), (3), (4), and (5)

$$\sum_{n\leqslant x}{}^{+} d(n^2+a) = 2x\sum_{k\leqslant X}\frac{\varrho(k)}{k} - \sum_{k\leqslant X}\frac{\varrho(k)Y_k}{k}$$
$$+\sum_{k\leqslant X}\{2\Psi_k(x) - \Psi_k(Y_k)\} + O(1) = 2x\sum\nolimits_3 - \sum\nolimits_4 + \sum\nolimits_5 + O(1), \text{ say.} \tag{6}$$

4. The estimation of $\sum_3$ and $\sum_4$

The estimation of $\sum_3$ and $\sum_4$ is effected by considering the Dirichlet series

$$\sum_{\lambda=1}^{\infty} \frac{\varrho(\lambda)}{\lambda^s}.$$

An identity for this series has already been obtained by the author in Section 4 of [5]. Modifying slightly the notation of this paper, we write

$$K(s)=\left(1+\frac{1}{2^s}\right)^{-1}\sum_{\alpha=0}^{\infty}\frac{\varrho(2^\alpha)}{2^{\alpha s}}=\sum_{\alpha=0}^{\infty}\frac{b_\alpha}{2^{\alpha s}},$$

$$M(s)=\sum_{\substack{d^2|a\\(d,2)=1}}\frac{d}{d^{2s}}\sum_{\substack{l=1\\(l,2)=1}}^{\infty}\left(\frac{-a/d^2}{l}\right)\frac{1}{l^s}=\sum_{\substack{d^2|a\\(d,2)=1}}\frac{d}{d^{2s}}L_{(-a/d^2)}(s),$$

and

$$f_{-a}(s)=\frac{K(s)\,M(s)}{\zeta(2s)}=\sum_{m=1}^{\infty}\frac{\tau(m)}{m^s},\ \text{say}. \tag{7}$$

Then the following identity holds for $s>1$;

$$\sum_{\lambda=1}^{\infty}\frac{\varrho(\lambda)}{\lambda^s}=\zeta(s)\,f_{-a}(s). \tag{8}$$

Certain properties of the coefficients of the Dirichlet series defining $f_{-a}(s)$ will be needed, and are easily verified from [5]. Firstly, there is an identity of the form

$$\sum_{\alpha=0}^{\infty}\frac{\varrho(2^\alpha)}{2^{\alpha s}}=1+\frac{A_1}{2^s}+\dots+\frac{A_{t-1}}{2^{(t-1)s}}+\frac{A_t}{2^{ts}}\left(1+\frac{1}{2^s}+\frac{1}{2^{2s}}+\dots\right)$$
$$=1+\frac{A_1}{2^s}+\dots+\frac{A_{t-1}}{2^{(t-1)s}}+\frac{A_t}{2^{ts}}\left(1-\frac{1}{2^s}\right)^{-1},$$

where $t=t(a)$ is bounded. Hence

$$b_\alpha=O(1). \tag{9}$$

Secondly, it is plain from a consideration of the Euler product for

$$\frac{1}{\zeta(2s)}L_{(-a/d^2)}(s)$$

that the coefficients of

$$\frac{1}{\zeta(2s)}M(s)$$

are bounded. A straight forward argument then gives

$$\tau(m)=O(1). \tag{10}$$

A subsidiary lemma is required.

LEMMA 1. *For $y\geqslant 1$, we have*

$$\sum_{m\leqslant y}\tau(m)=O(\sqrt{y}).$$

It follows from (7), (8), and (9) that

$$\sum_{m\leqslant y}\tau(m)=\sum_{\substack{2^\alpha t^2d^2l\leqslant y\\ d^2\mid a;\,(d,2)=(l,2)=1}} b_\alpha\mu(t)\,d\cdot\left(\frac{-a/d^2}{l}\right)=\sum_{\substack{2^\alpha t^2d^2\leqslant y\\ d^2\mid a;\,(d,2)=1}} b_\alpha\mu(t)\,d\sum_{\substack{l\leqslant y/2^\alpha t^2d^2\\ (l,2)=1}}\left(\frac{-a/d^2}{l}\right)$$

$$=O\left(\sum_{\substack{2^\alpha t^2d^2\leqslant y\\ d^2\mid a}} b_\alpha d\right)=O\left(\sum_{2^\alpha t^2\leqslant y}1\right)$$

$$=O\left(\sum_{t^2\leqslant y}\log\left\{\frac{2y}{t^2}\right\}\right)=O(\sqrt{y}).$$

Starting with $\sum_3$ we have from (7) and (8)

$$\sum\nolimits_3=\sum_{lm\leqslant X}\frac{\tau(m)}{lm}=\sum_{m\leqslant X^{\frac12}}+\sum_{m>X^{\frac12}}=\sum\nolimits_6+\sum\nolimits_7,\text{ say.} \tag{11}$$

Then
$$\sum\nolimits_6=\sum_{m\leqslant X^{\frac12}}\frac{\tau(m)}{m}\sum_{l\leqslant X/m}\frac{1}{l}=\sum_{m\leqslant X^{\frac12}}\frac{\tau(m)}{m}\left\{\log\left(\frac{X}{m}\right)+\gamma+O\left(\frac{m}{X}\right)\right\}$$

$$=(\log X+\gamma)\sum_{m\leqslant X^{\frac12}}\frac{\tau(m)}{m}-\sum_{m\leqslant X^{\frac12}}\frac{\tau(m)\log m}{m}$$

$$+O\left(\frac{1}{X}\sum_{m\leqslant X^{\frac12}}|\tau(m)|\right).$$

Hence, by (10),

$$\sum\nolimits_6=(\log X+\gamma)f_{-a}(1)+f'_{-a}(1)+O\left(\log X\left|\sum_{m>X^{\frac12}}\frac{\tau(m)}{m}\right|\right)$$

$$+O\left(\left|\sum_{m>X^{\frac12}}\frac{\tau(m)}{m}\log m\right|\right)+O\left(\frac{1}{X}\sum_{m\leqslant X^{\frac12}}1\right),$$

and then, by Lemma 1 and partial summation,

$$\sum\nolimits_6=(\log X+\gamma)f_{-a}(1)+f'_{-a}(1)+O(X^{-\frac14}\log X)$$
$$=(\log x+\gamma)f_{-a}(1)+f'_{-a}(1)+O(x^{-\frac14}\log x). \tag{12}$$

Next
$$\sum\nolimits_7=\sum_{l\leqslant X^{\frac12}}\frac{1}{l}\sum_{X^{\frac12}<m\leqslant X/l}\frac{\tau(m)}{m}=\sum_{l\leqslant X^{\frac12}}\frac{1}{l}\cdot O(X^{-\frac14}),$$

by Lemma 1 and partial summation. Therefore

$$\sum\nolimits_7=O(x^{-\frac14}\log x). \tag{13}$$

Therefore, finally, by (11), (12), and (13),

$$\sum\nolimits_3 = (\log x + \gamma) f_{-a}(1) + f'_{-a}(1) + O(x^{-\frac{1}{4}} \log x). \tag{14}$$

The treatment of $\sum_4$ is very similar. We have

$$Y_k = k^{\frac{1}{2}} X^{\frac{1}{2}} + O(k^{-\frac{1}{2}} X^{-\frac{1}{2}}),$$

and hence

$$\sum\nolimits_4 = X^{\frac{1}{2}} \sum_{k \leqslant X} \frac{\varrho(k)}{k^{\frac{1}{2}}} + O\left(X^{-\frac{1}{2}} \sum_{k \leqslant X} \frac{\varrho(k)}{k^{\frac{3}{2}}}\right) = X^{\frac{1}{2}} \sum_{k \leqslant X} \frac{\varrho(k)}{k^{\frac{1}{2}}} + O(x^{-\frac{1}{2}}). \tag{15}$$

Next

$$\sum_{k \leqslant X} \frac{\varrho(k)}{k^{\frac{1}{2}}} = \sum_{lm \leqslant X} \frac{\tau(m)}{l^{\frac{1}{2}} m^{\frac{1}{2}}} = \sum_{m \leqslant X^{\frac{1}{2}}} + \sum_{m > X^{\frac{1}{2}}} = \sum\nolimits_8 + \sum\nolimits_9, \text{ say.} \tag{16}$$

We have

$$\begin{aligned}
\sum\nolimits_8 &= \sum_{m \leqslant X^{\frac{1}{2}}} \frac{\tau(m)}{m^{\frac{1}{2}}} \sum_{l \leqslant X/m} \frac{1}{l^{\frac{1}{2}}} = \sum_{m \leqslant X^{\frac{1}{2}}} \frac{\tau(m)}{m^{\frac{1}{2}}} \left[2\left(\frac{X}{m}\right)^{\frac{1}{2}} + \zeta\left(\frac{1}{2}\right) + O\left\{\left(\frac{m}{X}\right)\right\}^{\frac{1}{2}}\right] \\
&= 2X^{\frac{1}{2}} \sum_{m \leqslant X^{\frac{1}{2}}} \frac{\tau(m)}{m} + \zeta\left(\frac{1}{2}\right) \sum_{m \leqslant X^{\frac{1}{2}}} \frac{\tau(m)}{m^{\frac{1}{2}}} + O(1) \\
&= 2X^{\frac{1}{2}} f_{-a}(1) + O\left(X^{\frac{1}{2}} \left|\sum_{m > X^{\frac{1}{2}}} \frac{\tau(m)}{m}\right|\right) \\
&\qquad + O\left(\left|\sum_{m \leqslant X^{\frac{1}{2}}} \frac{\tau(m)}{m^{\frac{1}{2}}}\right|\right) + O(1) \\
&= 2X^{\frac{1}{2}} f_{-a}(1) + O(X^{\frac{1}{4}}) + O(\log X) + O(1),
\end{aligned}$$

by Lemma 1 and partial summation. Hence

$$\sum\nolimits_8 = 2x^{\frac{1}{2}} f_{-a}(1) + O(x^{\frac{1}{4}}) \tag{17}$$

Also

$$\sum\nolimits_9 = \sum_{l \leqslant X^{\frac{1}{2}}} \frac{1}{l^{\frac{1}{2}}} \sum_{X^{\frac{1}{2}} < m \leqslant X/l} \frac{\tau(m)}{m^{\frac{1}{2}}} = O\left(\sum_{l \leqslant X^{\frac{1}{2}}} \frac{1}{l^{\frac{1}{2}}} \log\left\{\frac{2X^{\frac{1}{2}}}{l}\right\}\right),$$

by Lemma 1 and partial summation. Hence

$$\sum\nolimits_9 = O(x^{\frac{1}{4}}). \tag{18}$$

We have, finally, by (15), (16), (17), and (18),

$$\sum\nolimits_4 = 2x f_{-a}(1) + O(x^{\frac{3}{4}}). \tag{19}$$

It is convenient at this point to state a lemma, which will be required later. It may be proved by methods similar to those used above.

Lemma 2. *For $y \geqslant 1$, we have*

$$\sum_{k \leqslant y} \varrho(k) = y f_{-a}(1) + O(y^{\frac{3}{4}}).$$

5. Estimation of $\sum_5$; first stage

We begin by considering the representation of the function $\psi(u)$ by the Fourier series

$$\psi_{(1)}(u)=\sum_{h=1}^{\infty}\frac{\sin 2\pi hu}{\pi h}.$$

We have the properties

(i) $\psi(u)=\psi_{(1)}(u)$, unless u is an integer,

(ii) $\psi_{(1)}(u)$ is boundedly convergent,

(iii) $\sum_{h>\omega}\frac{\sin 2\pi hu}{\pi h}=O\left(\frac{1}{\omega\|u\|}\right)$ for $\omega>1$,

from which it follows that, for all real values of u,

$$\psi(u)=\frac{1}{\pi}\sum_{1\leqslant h\leqslant\omega}\frac{\sin 2\pi hu}{h}+O\left\{\min\left(1,\frac{1}{\omega\|u\|}\right)\right\}=\psi_\omega(u)+O\{\theta_\omega(u)\}, \text{ say.} \tag{20}$$

The Fourier development of $\theta_\omega(u)$ will also be needed. Since $\theta_\omega(u)$ is an even function of u, we have for $\omega>2$,

$$\theta_\omega(u)=\tfrac{1}{2}C_0(\omega)+\sum_{h=1}^{\infty}C_h(\omega)\cos 2\pi hu, \tag{21}$$

where $$C_h(\omega)=4\int_0^{\frac{1}{2}}\theta_\omega(u)\cos 2\pi hu\,du.$$

Hence $$C_0(\omega)=4\int_0^{1/\omega}du+4\int_{1/\omega}^{\frac{1}{2}}\frac{du}{\omega u}=O\left(\frac{\log\omega}{\omega}\right), \tag{22}$$

and so, for $h\geqslant 0$, $$C_h(\omega)=O\left(\frac{\log\omega}{\omega}\right). \tag{22}$$

Also, for $h>0$,

$$\begin{aligned}C_h(\omega)&=4\int_0^{1/\omega}\cos 2\pi hu\,du+\frac{4}{\omega}\int_{1/\omega}^{\frac{1}{2}}\frac{\cos 2\pi hu\,du}{u}=\frac{4}{\omega}\int_{1/\omega}^{\frac{1}{2}}\frac{\sin 2\pi hu\,du}{2\pi hu^2}\\&=-\frac{4}{\omega}\left[\frac{\cos 2\pi hu}{4\pi^2h^2u^2}\right]_{1/\omega}^{\frac{1}{2}}-\frac{4}{\omega}\int_{1/\omega}^{\frac{1}{2}}\frac{\cos 2\pi hu\,du}{2\pi^2h^2u^3}=O\left(\frac{\omega}{h^2}\right).\end{aligned} \tag{23}$$

We use (20) and (21) to put $\Psi_k(y)$ into a form suitable for the estimation of $\sum_5$. We first introduce a notation for an important exponential sum. We denote

$$\sum_{\substack{\nu^2 \equiv -a \pmod k \\ 0<\nu\leqslant k}} e^{2\pi i h\nu/k} = \sum_{\substack{\nu^2 \equiv -a \pmod k \\ 0<\nu\leqslant k}} \cos\frac{2\pi h\nu}{k}$$

by $\varrho(h,k)$, where evidently $\varrho(0,k)=\varrho(k)$. Next(1) we define $\Psi_{k,\omega}(y)$ and $\Theta_{k,\omega}(y)$ by

$$\Psi_{k,\omega}(y) = \sum_{\substack{\nu^2 \equiv -a \pmod k \\ 0<\nu\leqslant k}} \psi_\omega\left(\frac{y-\nu}{k}\right), \qquad \Theta_{k,\omega}(y) = \sum_{\substack{\nu^2 \equiv -a \pmod k \\ 0<\nu\leqslant k}} \theta_\omega\left(\frac{y-\nu}{k}\right),$$

so that

$$\Psi_k(y) = \Psi_{k,\omega}(y) + O(\Theta_{k,\omega}(y)), \tag{24}$$

by (20). Now

$$\psi_\omega\left(\frac{y-\nu}{k}\right) = \frac{1}{\pi}\sum_{1\leqslant h\leqslant\omega}\frac{1}{h}\left(\sin\frac{2\pi hy}{k}\cos\frac{2\pi h\nu}{k} - \cos\frac{2\pi hy}{k}\sin\frac{2\pi h\nu}{k}\right).$$

Therefore

$$\Psi_{k,\omega}(y) = \frac{1}{\pi}\sum_{1\leqslant h\leqslant\omega}\frac{1}{h}\varrho(h,k)\sin\frac{2\pi hy}{k}, \tag{25}$$

since

$$\sum_{\substack{\nu^2 \equiv -a \pmod k \\ 0<\nu\leqslant k}} \sin\frac{2\pi h\nu}{k} = 0,$$

and similarly, by (21),

$$\Theta_{k,\nu}(y) = \frac{1}{2}C_0(\omega)\varrho(k) + \sum_{h=1}^{\infty} C_h(\omega)\varrho(h,k)\cos\frac{2\pi hy}{k}. \tag{26}$$

The treatment of $\sum_5$ through this form of $\Psi_k(y)$ requires estimates for sums of the type $\sum_k \varrho(h,k)$. These sums are considered in the next section.

6. The sum $\sum_k \varrho(h,k)$

We write

$$R(h,x) = \sum_{k\leqslant x}\varrho(h,k).$$

In this Section, as stated in Section 2, it is convenient to replace a by $-D$ in the definition of $\varrho(h,k)$.

The method depends on the theory of representation of numbers by binary quadratic forms. A very clear description of this theory in a form suitable for our purpose is to be found in either the "Disquisitiones Arithmeticae" [3] or in H. J. S. Smith's "Report on the Theory of Numbers" (incorporated in [9]).

We start from the fact that every primitive representation of k by a quadratic

(1) It is important to remember that $\theta_\omega(u)$ and hence $\Theta_{k,\omega}(y)$ are positive functions.

form of determinant D *appertains* to a residue class of solutions (which, for brevity, we refer to as a root) of the congruence

$$\nu^2 \equiv D \pmod{k},$$

two different representations which appertain to the same root being said to belong to the same *set*. Representations of k by non-equivalent forms cannot belong to the same set. Conversely, to every root of the congruence there corresponds a set of representations of k. There is thus a bi-unique correspondence between the roots of the congruence and the sets of representations of k by a system of representative forms of determinant D.

Let $ax^2+2bxy+cy^2$ be a form of determinant D. Then, if

$$k = ar^2 + 2brs + cs^2$$

is a primitive representation of k by the form, the root of the congruence appertaining to this representation is given by ([9], page 172)

$$\nu = ar\varrho + b(r\sigma + s\varrho) + cs\sigma,$$

where ϱ, σ satisfy

$$r\sigma - s\varrho = 1.$$

Hence a typical value of ν/k in $\varrho(h,k)$ is given by

$$\frac{\nu}{k} = \frac{ar\varrho + b(r\sigma+s\varrho)+cs\sigma}{ar^2+2brs+cs^2}.$$

This gives, for $r \neq 0$,

$$\frac{\nu}{k} = \frac{\varrho(ar^2+2brs+cs^2)+br+cs}{r(ar^2+2brs+cs^2)} = -\frac{\bar{s}}{r} + \frac{br+cs}{r(ar^2+2brs+cs^2)}, \tag{27}$$

where $\bar{s}$ is defined (modulo r) by the congruence $s\bar{s} \equiv 1 \pmod{r}$. It gives, similarly, for $s \neq 0$,

$$\frac{\nu}{k} = \frac{\bar{r}}{s} - \frac{ar+bs}{s(ar^2+2brs+cs^2)}, \tag{28}$$

where $r\bar{r} \equiv 1 \pmod{s}$.

Let $\vartheta_{r,s}$ denote the value of ν/k as given by (27) or (28). Then we have

$$R(h,x) = \sum_{a,b,c} \sum_{\substack{0<ar^2+2brs+cs^2\leq x \\ (r,s)=1 \\ (M)}} e^{2\pi i h\theta_{r,s}}, \tag{29}$$

where a, b, c indicates summation over a set of representative forms of determinant

D (positive forms, if $D<0$), and (M) indicates that only one representation from each possible set of representations is to be included. We formulate condition (M) by using the property that all representations in a set can be obtained from any one such representation by means of the proper automorphs of (a,b,c). The mode of formulation depends on whether (a,b,c) be definite or indefinite. We consider first the definite case in detail, and then indicate the modifications that are necessary in the argument for the indefinite case.

If D is negative, then the forms are positive. Here the number of representations in a set is constant for a given form. The number is in general two, but may in special cases be four or six. Hence

$$R(h,x)=\sum_{a,b,c}\varepsilon_{a,b,c}\sum_{\substack{ar^2+2brs+cs^2\leqslant x\\(r,s)=1}}e^{2\pi ih\theta_{r,s}},\tag{30}$$

where $\varepsilon_{a,b,c}$ is either $\frac{1}{2}$, $\frac{1}{4}$, or $\frac{1}{6}$. Plainly we may take a, b, and c as bounded by choosing the representative forms appropriately (as reduced forms, say). The inner sum may then be split up thus:

$$\sum_{\substack{ar^2+2brs+cs^2\leqslant x\\(r,s)=1}}e^{2\pi ih\theta_{r,s}}=\sum_{|s|<|r|}+\sum_{|r|<|s|}+\sum_{|r|=|s|=1}=\sum\nolimits_{10}+\sum\nolimits_{11}+O(1),\ \text{say.}\tag{31}$$

We must consider $\sum_{10}$ and $\sum_{11}$. The following lemmata will be required.

LEMMA 3. *If $h,r\neq 0$, and $0\leqslant\beta-\alpha\leqslant 2|r|$, we have*

$$\sum_{\substack{\alpha\leqslant s\leqslant\beta\\(r,s)=1}}\exp\left(-\frac{2\pi ih\bar{s}}{r}\right)=O[|r|^{\frac{1}{2}}\{(h,r)\}^{\frac{1}{2}}d(r)\log 2|r|].$$

This result on an "incomplete" Kloosterman sum may be deduced by a well-known method from Lemma 2 of the author's paper [6]. It depends on Weil's estimate for the Kloosterman sum.

LEMMA 4. *If $h\neq 0$ and $y\geqslant 1$, we have*

$$\sum_{l\leqslant y}\{(h,l)\}^{\frac{1}{2}}d(l)=O\{y\log 2y\cdot\sigma^2_{-\frac{1}{2}}(h)\}.$$

We have

$$\sum_{l\leqslant y}\{(h,l)\}^{\frac{1}{2}}d(l)=\sum_{\lambda|h}\sum_{\substack{(h,l)=\lambda\\l\leqslant y}}\{(h,l)\}^{\frac{1}{2}}d(l)=O\{\sum_{\lambda|h}\lambda^{\frac{1}{2}}d(\lambda)\sum_{l_1\leqslant y/\lambda}d(l_1)\}$$

$$=O\left(y\log 2y\sum_{\lambda|h}\frac{d(\lambda)}{\lambda^{\frac{1}{2}}}\right)=O\{y\log 2y\cdot\sigma^2_{-\frac{1}{2}}(h)\}.$$

The conditions of summation in $\sum_{10}$ imply, firstly, that

$$|r| < Ax^{\frac{1}{2}}, \tag{32}$$

and, secondly, that for given r

$$F_1(r) \leqslant s \leqslant F_2(r),$$

where $F_1(r)$ and $F_2(r)$ are defined by

$$F_2(r) = \min\left(|r|, -\frac{br}{c} + \frac{1}{c}(cx + Dr^2)^{\frac{1}{2}}\right),$$

$$F_1(r) = \max\left(-|r|, -\frac{br}{c} - \frac{1}{c}(cx + Dr^2)^{\frac{1}{2}}\right).$$

Let $\xi = \xi(x, h)$ be a suitable positive number to be chosen explicitly later. Then

$$\sum_{10} = \sum_{|r| \leqslant \xi} + \sum_{|r| > \xi} = \sum_{|r| > \xi} + O(\xi^2) = \sum_{12} + O(\xi^2), \text{ say.} \tag{33}$$

Next, if

$$\varphi(r, s) = \exp\left(\frac{2\pi i h(br + cs)}{r(ar^2 + 2brs + cs^2)}\right),$$

we have, by (27), (32), and (33),

$$\sum_{12} = \sum_{\substack{\xi < |r| < Ax^{\frac{1}{2}} \\ F_1(r) \leqslant F_2(r)}} \sum_{\substack{F_1(r) \leqslant s \leqslant F_2(r) \\ (r, s) = 1}} \exp\left(-\frac{2\pi i h\bar{s}}{r}\right) \cdot \varphi(r, s), \tag{34}$$

where the outer summation is interpreted so that $\sum_{12}$ is zero when $\xi \geqslant Ax^{\frac{1}{2}}$. The inner sum in (34) is transformed by partial summation into

$$\sum_{F_1(r) \leqslant \mu \leqslant F_2(r)} g(\mu)\{\varphi(r, \mu) - \varphi(r, \mu + 1)\} + g\{[F_2(r)]\}\, \varphi\{r, [F_2(r)] + 1\}, \tag{35}$$

where for any integer μ (with r fixed)

$$g(\mu) = \sum_{\substack{F_1(r) \leqslant s \leqslant \mu \\ (r, s) = 1}} \exp\left(-\frac{2\pi i h\bar{s}}{r}\right).$$

Since in (35)

$$\varphi(r, \mu) - \varphi(r, \mu + 1) = O\left(\frac{|h|}{|r|^3}\right), \tag{36}$$

we have, by Lemma 3, that expression (35) is

$$O\left(\frac{|h|}{|r|^3} |r|^{\frac{1}{2}} \{(h, r)\}^{\frac{1}{2}} d(r) \log 2|r| \sum_{F_1(r) \leqslant \mu \leqslant F_2(r)} 1\right) + O(|r|^{\frac{1}{2}} \{(h, r)\}^{\frac{1}{2}} d(r) \log 2|r|)$$

$$= O(|h|\, |r|^{-\frac{3}{2}} \{(h, r)\}^{\frac{1}{2}} d(r) \log 2|r|) + O(|r|^{\frac{1}{2}} \{(h, r)\}^{\frac{1}{2}} d(r) \log 2|r|).$$

Hence, whatever the value of ξ, we have from (34)

$$\sum_{12} = O\left(|h| \log x \sum_{l>\xi} \frac{\{(h,l)\}^{\frac{1}{2}} d(l)}{l^{\frac{3}{2}}}\right) + O\left(x^{\frac{1}{2}} \log x \sum_{l<Ax^{\frac{1}{2}}} \{(h,l)\}^{\frac{1}{2}} d(l)\right),$$

from which we deduce, by Lemma 4,

$$\sum_{12} = O\{|h| \, \sigma^2_{-\frac{1}{2}}(h) \, \xi^{-\frac{1}{2}} \log^2 x\} + O\{\sigma^2_{-\frac{1}{2}}(h) \, x^{\frac{3}{4}} \log^2 x\}.$$

Therefore from this and (33)

$$\sum_{10} = O\{|h| \, \sigma^2_{-\frac{1}{2}}(h) \, \xi^{-\frac{1}{2}} \log^2 x\} + O(\xi^2) + O\{\sigma^2_{-\frac{1}{2}}(h) \, x^{\frac{3}{4}} \log^2 x\}. \tag{37}$$

If ξ is chosen so that $|h| \, \xi^{-\frac{1}{2}} = \xi^2$, then $\xi = |h|^{\frac{2}{5}}$ and the first inequality in

$$\left.\begin{array}{l}\sum_{10} \\ \sum_{11}\end{array}\right\} = O\{|h|^{\frac{4}{5}} \sigma^2_{-\frac{1}{2}}(h) \log^2 x\} + O\{\sigma^2_{-\frac{1}{2}}(h) x^{\frac{3}{4}} \log^2 x\} \tag{38}$$

must hold. The second inequality can be proved similarly by starting from (28).

We deduce from (30), (31), and (38) that, if D is negative, then

$$R(h,x) = O\{|h|^{\frac{4}{5}} \sigma^2_{-\frac{1}{2}}(h) \log^2 x\} + O\{\sigma^2_{-\frac{1}{2}}(h) x^{\frac{3}{4}} \log^2 x\}. \tag{39}$$

We pass on to the case where D is positive and the forms are indefinite. We must consider condition (M) for each form (a,b,c) that appears in (29). We take, for simplicity, the case when the form appearing is primitive, since the derived forms that may appear can be considered similarly. It is plain that we may take a to be positive and c to be negative by choosing an appropriate representative form. Let m be the highest common factor of a, b, c; let also T, U be the least positive solution of the Pellian equation

$$T^2 - DU^2 = m^2.$$

Then (see [9], page 213) each set of representations of a positive number k by $ax^2 + 2bxy + cy^2$ contains one and only one representation which satisfies the inequalities

$$x > 0, \quad y > 0, \quad y \leqslant \frac{aU}{T - bU} x.$$

Moreover, when these inequalities hold, the form takes positive values only. Hence any inner sum in (29) corresponding to a primitive form (a,b,c) may be expressed in the form

$$\sum_{\substack{ar^2+2brs+cs^2\leqslant x\\ r,s>0,\, s\leqslant aUr/(T-bU)\\ (r,s)=1}} e^{2\pi i h\theta_{r,s}}.$$

This may be considered in a somewhat similar way to (31), since it may be verified that the conditions of summation imply, firstly,

$$s\leqslant (x/m)^{\frac{1}{2}}\,U,\quad 0\leqslant ar+bs\leqslant (x/m)^{\frac{1}{2}}\,T,$$

and, secondly,

$$a(ar^2+2brs+cs^2)\geqslant \begin{cases} m^2s^2/U^2\\ m^2(ar+bs)^2/T^2.\end{cases}$$

In particular the treatment of the function corresponding to $\varphi(r,s)$ will not present any difficulty. Finally, we obtain the same estimate for $R(h,x)$ as in (39). We thus have

THEOREM 1. *Let D be any fixed integer that is not a perfect square, and let*

$$R(h,x)=\sum_{k\leqslant x}\sum_{\substack{\nu^2\equiv D\ (\mathrm{mod}\ k)\\ 0<\nu\leqslant k}} e^{2\pi i h\nu/k}.$$

Then, for $h\neq 0$,

$$R(h,x)=O\{|h|^{\frac{4}{5}}\sigma^2_{-\frac{1}{2}}(h)\log^2 x\}+O\{\sigma^2_{-\frac{1}{2}}(h)x^{\frac{3}{4}}\log^2 x\},$$

and, in particular,

$$|R(h,x)|\leqslant A(h)\,x^{\frac{3}{4}}\log^2 x.$$

Theorem 1 will be used for the proof of Theorem 3. It cannot, however, be applied directly in an advantageous manner to the estimation of $\sum_5$, since here the sums $\varrho(h,k)$ appear with trigonometrical factors depending on k. It is better to consider sums of the form

$$R_1^{\pm}(h,X)=\sum_{k\leqslant X}\varrho(h,k)\,e^{\pm 2\pi i h x/k},$$

$$R_2^{\pm}(h,X)=\sum_{k\leqslant X}\varrho(h,k)\,e^{\pm 2\pi i h Y_k/k},$$

where we recall that Y_k is defined as in (3). These can be estimated exactly as $R(h,X)$, except that it is necessary to modify the definitions of $\theta_{r,s}$ and $\varphi(r,s)$, with a consequent change in the value of ξ, as follows:

(i) include the additional term

$$\pm\frac{2\pi i h x}{ar^2+2brs+cs^2}\quad\text{or}\quad \pm\frac{2\pi i h Y_k}{ar^2+2brs+cs^2}$$

in $\theta_{r,s}$:

(ii) include the factor

$$\exp\left(\pm\frac{2\pi i h x}{ar^2+2brs+cs^2}\right) \quad \text{or} \quad \exp\left(\pm\frac{2\pi i h Y_k}{ar^2+2brs+cs^2}\right)$$

in $\varphi(r,s)$. Then, now, instead of (36), we have

$$\varphi(r,\mu)-\varphi(r,\mu+1)=O\left(\frac{|h|\,x}{|r|^3}\right),$$

which gives $\qquad O\{|h|\,\sigma^2_{-\frac{1}{2}}(h)\,\xi^{-\frac{1}{2}}x\log^2 x\}+P(\xi^2)+O\{\sigma^2_{-\frac{1}{2}}(h)x^{\frac{3}{4}}\log^2 x\}$

in place of the estimate (37). We then obtain the following lemma after choosing $\xi=|h|^{\frac{2}{5}}x^{\frac{2}{5}}$.

LEMMA 5. *We have, for* $h\neq 0$,

$$\left.\begin{matrix} R_1^{\pm}(h,X) \\ R_2^{\pm}(h,X) \end{matrix}\right\} = O\{\sigma^2_{-\frac{1}{2}}(h)\,|h|^{\frac{4}{5}}x^{\frac{4}{5}}\log^2 x\}.$$

7. Estimation of $\sum_5$: second stage

The estimation of $\sum_5$ can now be concluded by collecting together the results from previous sections.

Firstly, now interpreting ω (>2) as a suitable real number depending only on x, we have, by (6), and (24),

$$\sum\nolimits_5=\sum_{k\leq X}\{2\Psi_{k,\omega}(x)-\Psi_{k,\omega}(Y_k)\}+O(\sum_{k\leq X}\Theta_{k,\omega}(x))+O(\sum_{k\leq X}\Theta_{k,\omega}(Y_k)),$$

since $\Theta_{k,\omega}(y)$ is positive. Therefore

$$\begin{aligned}\sum\nolimits_5&=O(\sum_{k\leq X}\Psi_{k,\omega}(x))+O(\sum_{k\leq X}\Psi_{k,\omega}(Y_k))+O(\sum_{k\leq X}\Theta_{k,\omega}(x))+O(\sum_{k\leq X}\Theta_{k,\omega}(Y_k))\\ &=O(\sum\nolimits_{13})+O(\sum\nolimits_{14})+O(\sum\nolimits_{15})+O(\sum\nolimits_{16}),\ \text{say.}\end{aligned} \tag{40}$$

We have, by (25),

$$\sum\nolimits_{13}=\frac{1}{\pi}\sum_{k\leq X}\sum_{1\leq h\leq\omega}\frac{1}{h}\varrho(h,k)\sin\frac{2\pi hx}{k}=\frac{1}{\pi}\sum_{1\leq h\leq\omega}\frac{1}{h}\sum_{k\leq X}\varrho(h,k)\sin\frac{2\pi hx}{k}.$$

Thus, on recalling the definition of $R_1^{\pm}(h,X)$, we have

$$\sum\nolimits_{13}=\frac{1}{2\pi i}\sum_{1\leq h\leq\omega}\frac{1}{h}\{R_1^{+}(h,X)-R_1^{-}(h,X)\}.$$

Therefore, by Lemma 5,

$$\sum\nolimits_{13} = O\left(x^{\frac{4}{5}} \log^2 x \sum_{1 \leqslant h \leqslant \omega} \frac{\sigma^2_{-\frac{1}{2}}(h)}{h^{\frac{1}{5}}}\right),$$

which gives by a simple calculation the first part of

$$\left.\begin{matrix}\sum_{13}\\ \sum_{14}\end{matrix}\right\} = O(x^{\frac{4}{5}} \omega^{\frac{4}{5}} \log^2 x). \tag{41}$$

The second part may be proved in a similar manner.

Next, since $C_h(\omega)$ does not depend on k, we have, by (26),

$$\sum\nolimits_{15} = \tfrac{1}{2}\, C_0(\omega) \sum_{k \leqslant X} \varrho(k) + \tfrac{1}{2} \sum_{h=1}^{\infty} C_h(\omega) \,\{R_1^+(h, X) + R_1^-(h, X)\}.$$

Therefore, by Lemmata 2 and 5, (22), and (23),

$$\begin{aligned}\sum\nolimits_{15} &= O\left(\frac{x \log \omega}{\omega}\right) + O\left(\frac{x^{\frac{4}{5}} \log^2 x \log \omega}{\omega} \sum_{1 \leqslant h \leqslant \omega} \sigma^2_{-\frac{1}{2}}(h)\, h^{\frac{4}{5}}\right) + O\left(x^{\frac{4}{5}} \omega \log^2 x \sum_{h > \omega} \frac{\sigma^2_{-\frac{1}{2}}(h)}{h^{\frac{6}{5}}}\right)\\ &= O\left(\frac{x \log \omega}{\omega}\right) + O(x^{\frac{4}{5}} \omega^{\frac{4}{5}} \log^2 x \log \omega) + O(x^{\frac{4}{5}} \omega^{\frac{4}{5}} \log^2 x).\end{aligned}$$

Therefore the first part of the inequality

$$\left.\begin{matrix}\sum_{15}\\ \sum_{16}\end{matrix}\right\} = O\left(\frac{x \log \omega}{\omega}\right) + O(x^{\frac{4}{5}} \omega^{\frac{4}{5}} \log^2 x \log \omega) \tag{42}$$

holds; the second may be verified similarly.

We have from (40), (41), and (42)

$$\sum\nolimits_{5} = O\left(\frac{x \log \omega}{\omega}\right) + O(x^{\frac{4}{5}} \omega^{\frac{4}{5}} \log^2 x \log \omega).$$

Hence finally choosing ω so that $x\omega^{-1} = x^{\frac{4}{5}} \omega^{\frac{4}{5}}$ and thus $\omega = x^{\frac{1}{9}}$, we deduce

$$\sum\nolimits_{5} = O(x^{\frac{8}{9}} \log^3 x). \tag{43}$$

8. The asymptotic formula

The asymptotic formula for the divisor sum is now immediate from (6), (14), (19), and (43).

THEOREM 2. *Let a be a non-zero integer such that $-a$ is not a perfect square. Then, as $x \to \infty$, we have*

$$\sum_{n \leqslant x}^{+} d(n^2+a) = 2f_{-a}(1)\, x \log x + \{2(\gamma-1) f_{-a}(1) + 2f'_{-a}(1)\} x + O(x^{\frac{8}{9}} \log^3 x),$$

where

$$f_{-a}(s) = \frac{K(s)}{\zeta(2s)} \sum_{\substack{d^2 | a \\ (d,2)=1}} \frac{d}{d^{2s}} \cdot L_{(-a/d^2)}(s).$$

We observe that the divisor sum is in fact asymptotically equivalent to $2f_{-a}(1)\, x \log x$, since it is easy to verify from the properties of Dirichlet's L functions that $f_{-a}(s)$ does not vanish at $s = 1$.

9. The distribution of the roots of the congruence

Our result on the distribution of the roots of the congruence $\nu^2 \equiv D \pmod{k}$ is a corollary of Theorem 1. We take all numbers of the type ν/k, where $\nu^2 \equiv D \pmod{k}$ and $0 < \nu \leqslant k$, and arrange them as a sequence $p_1, p_2, \ldots, p_m, \ldots$, so that the corresponding denominators k are in ascending order. (The arrangement of the numbers in a group corresponding to a fixed value of k is immaterial.)

We then adopt a method due to Weyl (see [4]), and consider the sum

$$\sum_{m \leqslant N} e^{2\pi i h p_m}$$

for each non-zero value of h, as $N \to \infty$. If M is the denominator (before cancellation) in p_N, then

$$N > \sum_{k < M} \varrho(k) > AM, \tag{44}$$

by Lemma 2. Next

$$\sum_{m \leqslant N} e^{2\pi i h p_m} = \sum_{k < M} \sum_{\substack{\nu^2 \equiv D \pmod{k} \\ 0 < \nu \leqslant k}} e^{2\pi i h \nu / k} + O\{\varrho(M)\}.$$

Therefore, since $\varrho(M) = O\{d(M)\}$, we have, by Theorem 1 and then by (44),

$$\Big| \sum_{m \leqslant N} e^{2\pi i h p_m} \Big| < A_1(h)\, M^{\frac{3}{4}} \log^2 M$$
$$< A_2(h)\, N^{\frac{3}{4}} \log^2 N,$$

and so, for any $h \neq 0$,

$$\frac{1}{N} \Big| \sum_{m \leqslant N} e^{2\pi i h p_m} \Big| \to 0 \quad \text{as} \quad N \to \infty.$$

This gives our final theorem.

THEOREM 3. *The sequence $p_1, p_2, \ldots, p_m, \ldots$, as defined above, is uniformly distributed in the interval* $(0, 1)$.

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