THE VALUATION THEORY OF MEROMORPHIC FUNCTION FIELDS OVER OPEN RIEMANN SURFACES

BY

NORMAN L. ALLING

M. I. T., Cambridge, Mass., U.S.A.(1)

Introduction

With the advent of the generalization of the Weierstrass (product) theorem and the Mittag–Leffler theorem to arbitrary open Riemann surfaces X (due to Florack [6]), the analysis, made by Henriksen [10] for the plane and Kakutani [13] for schlicht domains of the plane, of the maximal ideals in the algebra A of all analytic functions on X can be carried out in general; this will be done in § 1. The residue class field K, associated with free maximal ideals M in A, has been considered by Henriksen [10]. That K has a natural valuation whose residue class field is the complex field C does not seem to have been noticed before. It will be shown in § 1 that the value group of K is a divisible η_1-group and that every countable pseudo-convergent sequence in K has a pseudo-limit in K: i.e., K is 1-maximal.

Let A_M be the quotient ring of A with respect to M in F, the field of meromorphic functions on X. It will be shown in § 2 that A_M is a valuation ring of F. The value group of A_M will be shown to be a non-divisible near η_1-group with a smallest non-zero convex subgroup, which is discrete; thus the structure of the prime ideals in A that contain M can be analyzed. It is also shown in § 2 that this valuation on F is 1-maximal.

In § 3 the composite of the place of F, whose valuation ring is A_M and of the place of K, will be shown to be a place of F over C onto C whose valuation is 1-maximal, and whose value group is a non-divisible η_1-group.

In § 4 the space S of all places of F over C onto C will be considered. Under the weak topology S is compact. Let T be the closure of X in S and let S_A be the places that arise

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from maximal ideals in A. It will be shown that $X < S_A \subset T < S$. There is a continuous mapping of βX, the Stone-Čech compactification of X, onto T which maps $\delta X = \{p \in \beta X$, p an adherence point of a discrete subset of $X\}$ one-to-one onto S_A.

In § 5 a few open questions raised by these researches will be stated.

Acknowledgements. Thanks are due to Professor Tate for observing that the total order in the prime ideals contained in a free maximal ideal suggested the presence of a valuation on A_M. This is indeed so, and was of great importance in the researches leading up to this paper. I am indebted to Professor Zariski, who suggested Lemma 4.5 in a conversation. Thanks are also due to Professor Röhrl for suggesting that Narasimhan's imbedding theorem [17] might aid in providing the function produced in Lemma 4.7, and for allowing me to discuss these researches with him at considerable length.

1. Ideal theory

Let X be an open (connected) Riemann surface and let A be the set of all analytic functions on X. A is, of course, an algebra over the field C of complex numbers under pointwise operations. Let $f \in A$ and let $Z(f) = \{x \in X: f(x) = 0\}$. Clearly $Z(f) = \emptyset$ if, and only if, f is a unit in A and $Z(fg) = Z(f) \cup Z(g)$, for all $f, g \in A$. Helmer [9] has proved the following lemma in case X is the plane.

LEMMA 1.1. (Helmer.) *Let f, $g \in A$ such that $Z(f) \cap Z(g) = \emptyset$; then there exists a, $b \in A$ such that $af + bg = 1$.*

Helmer used the classical Mittag–Leffler theorem in his proof. We will use Florack's [6] generalization of the Mittag-Leffler theorem to prove Lemma 1.1 much as Helmer does. Thus it seems desirable to state the Weierstrass (product) theorem and the Mittag–Leffler theorem in this setting; it will frequently be resorted to.

Background. Let F be the field of meromorphic functions on X. For $x \in X$ let O_x be the set of all $f \in F$ such that $f(x) \in C$ and let P_x be the set of all $f \in F$ such that $f(x) = 0$. Then O_x is a valuation ring of F and P_x is its maximal ideal. Let the valuation associated with O_x be denoted by V_x. The value group of V_x is, of course, the integers. For each $x \in X$ choose $t_x \in F$ such that $V_x(t_x) = 1$. t_x is called a *local uniformizer* at x. Let $m = V_x(f)$ for a non-zero $f \in F$. Thus $V_x(ft_x^{-m}) = 0$, and there exists a unique non-zero complex number a_m such that $V_x(ft_x^{-m} - a_m) > 0$. Thus given $k \geq m$, there exists $a_n \in C$ such that $V_x(f - \sum_{n=m}^{k} a_n t_x^n) > k$. $\sum_{n=m}^{k} a_n t_x^n$ will be called the *k-th partial sum of f at x*. Let $f \in F$. Clearly $f \in A$ if and only if $V_x(f) \geq 0$ for all $x \in X$. Further f is a unit in A if and only if $V_x(f) = 0$ for all $x \in X$. Finally, given a non-zero element f of F, the zeros and poles of f are disjoint, discrete subsets of X.

PROPOSITION 1.2. *Let* $b=\sum_{m=0}^{r} b_m t^m$ *and* $a=\sum_{k=0}^{r} a_k t^k \in C[t]$, $b_0 \neq 0$. *There exists* $c=\sum_{n=0}^{r} c_n t^n \in C[t]$ *such that* $cb-a$ *is either zero or is divisible by* t^{r+1}.

Proof. We must solve the following system of linear equations for $c_0, \ldots, c_r$:

$$c_0 b_0 = a_0,$$
$$c_0 b_1 + c_1 b_0 = a_1,$$
$$c_0 b_r + \ldots + c_r b_0 = a_r.$$

Since the determinate of the system of equations, b_0^{r+1}, is not zero, such numbers $c_0, \ldots, c_r$ exist in C, proving the proposition.

Employing first Florack's generalization of the Weierstrass theorem [6], Proposition 1.2, and then Florack's generalization of the Mittag-Leffler theorem [6], we get the following.

THEOREM 1.3. *Let* D *be a discrete subset of* X. *For each* $x \in D$ *choose integers* $m(x) \leqslant k(x)$ *and complex numbers* $a_{n,x}$, $m(x) \leqslant n \leqslant k(x)$. *There exists* $u \in F$ *such that* $V_x(u-\sum_{n=m(x)}^{k(x)} a_{n,x} t_x^n) > k(x)$ *for all* $x \in D$, *and* $V_x(u) \geqslant 0$ *for all* $x \in X-D$.

Further, we can get the following.

COROLLARY 1.4. *Let* D *be a discrete subset of* X. *For each* $x \in D$ *choose* $f_x \in F$ *and an integer* $k(x)$ *such that* $V_x(f_x) \leqslant k(x)$. *There exists* $u \in F$ *such that* $V_x(u-f_x) > k(x)$ *for all* $x \in D$ *and* $V_x(u) \geqslant 0$ *for all* $x \in X-D$.

We now return to the proof of Helmer's lemma.

Proof. If g is a unit let $a=0$ and $b=1/g$. Assume now that g is not a unit in A; then $D=Z(g) \neq \emptyset$. For $x \in D$ let $m(x)=V_x(gf)$. Since $Z(f) \cap Z(g)=\emptyset$, $m(x)=V_x(g)$. Let B_x be the $(2m(x)-1)$-th partial sum of gf at x. By Proposition 1.2 there exists $C_x=\sum_{n=-m(x)}^{-1} c_{n,x} t_x^n$ such that $V_x(C_x B_x - 1) \geqslant m(x)$. By Theorem 1.3 there exists $u \in F$ such that $V_x(u-C_x) > -1$ for all $x \in D$ and $V_x(u) \geqslant 0$ for all $x \in X-D$. Note: $V_x(ug) \geqslant 0$ for all $x \in X$; thus $ug=a \in A$. $af-1=C_xB_x-1+C_x(gf-B_x)+(u-C_x)B_x+(u-C_x)(gf-B_x)$. By construction, the value at x of each term in this summation is not less than $m(x)$, for $x \in D$; thus $(af-1)/g=-b \in A$, proving the lemma.

The following is an immediate consequence of Helmer's lemma. (See Henriksen [10] for details.)

COROLLARY 1.5. *All finitely generated ideals in* A *are principal.*

In these considerations the following corollary is of great importance. (In this paper all ideals are assumed to be proper.)

COROLLARY 1.6. *Let I be an ideal in A and let $Z(I)=\{Z(a): a\in I\}$. ($Z(I)$ has the finite intersection property: i.e., the intersection of a finite number of elements of $Z(I)$ is non-empty.*

Let Δ be the set of all discrete subsets of X together with X itself. By the generalized Weierstrass theorem, $\Delta=Z(A)$. A subset δ of Δ will be called a *Δ-filter* if

(a) $\emptyset\notin\delta$,

(b) if $D\in\delta$ and $D'\in\Delta$ such that $D\subset D'$ implies $D'\in\delta$, and

(c) δ is closed under finite intersection.

Let the Δ-filters be ordered by inclusion and let maximal Δ-filters be called *Δ-ultrafilters*. Let δ be a Δ-ultrafilter and let $D_0\in\Delta$. $D_0\in\delta$ if and only if $D\cap D_0\neq\emptyset$ for all $D\in\delta$. A Δ-filter δ will be called *fixed* or *free* according as $\bigcap_{D\in\delta} D$ is non-empty or is empty. We then have the following.

THEOREM 1.7. *If I is an ideal in A then $Z(I)$ is a Δ-filter. If δ is a Δ-filter then $Z^{-1}(\delta)$ is an ideal in A; further $I\subset Z^{-1}Z(I)$. Thus Z is a one-to-one correspondence between the maximal ideals of A and the Δ-ultrafilters. If δ is a fixed Δ-ultrafilter then $\bigcap_{D\in\delta} D$ consists of a single point x, $\delta=\{D\in\Delta: x\in D\}$, and $Z^{-1}(\delta)=\{f\in A: f(x)=0\}$.*

Δ-filters are very closely related to z-filters. The proofs given by Gillman and Jerison [7] for the corresponding results for z-filters can be easily modified to prove these results.

We will call an ideal I of A *fixed* or *free* according as $Z(I)$ is fixed or free. It is clear that all fixed prime ideals of A are maximal. An ideal I of A will be called a *Δ-ideal* if $I=Z^{-1}Z(I)$. Let P be a prime Δ-ideal. If P is fixed then it is maximal. Assume that P is free; then $\delta=Z(P)$ enjoys the following property: given $D_i\in\Delta$ such that $D_0\cup D_1\in\delta$, then D_0 or $D_1\in\delta$ (i.e., δ is a *prime Δ-filter*). Let D_0 be a discrete subset of δ and let $\delta_0=\{D\cap D_0: D\in\delta\}$. Then δ_0 is an ultrafilter on D_0. Conversely, given an ultrafilter δ_0 on a non-empty discrete subset D_0 of X, then $\delta=\{D\in\Delta: D\cap D_0\in\delta_0\}$ is a Δ-ultrafilter. Thus P is a maximal ideal. We see therefore that the only prime Δ-ideals of A are the maximal ideals and that the study of prime Δ-filters is not going to help in the study of non-maximal prime ideals in A. Let us, however, record the following useful fact discussed above.

PROPOSITION 1.8. *If δ is a Δ-ultrafilter and D_0 is a discrete subset of δ then $\delta_0=\delta\cap D_0\equiv\{D\cap D_0: D\in\delta\}$ is an ultrafilter on D_0, fixed or free according as δ is fixed or free. Conversely, given a non-empty discrete subset D_0 of X and an ultrafilter δ_0 on it, let $\delta=\operatorname{ext}\delta_0=\{D\in\Delta: D\cap D_0\in\delta_0\}$. δ is a Δ-ultrafilter, fixed or free according as δ_0 is fixed or free. Finally, $\delta=\operatorname{ext}(\delta\cap D_0)$ and $\delta_0=(\operatorname{ext}\delta_0)\cap D_0$.*

We now will investigate the quotient fields of A. Let M be a maximal ideal of A. Assume, first, that M is fixed and let $\bigcap_{D\in Z(M)} D=x$. In this case let $M=M_x$. Then it is clear that two elements $f, g\in A$ are congruent modulo M if and only if $f(x)=g(x)$. Thus, the subfield C of constant functions maps onto A/M. Assume now that M is free, let $\delta = Z(M)$, let $K = A/M$ and let λ be the canonical homomorphism of A onto K. Let C be identified with $\lambda(C)$. Clearly $\lambda(f) = 0$ if and only if $f \mid D = 0$ for some $D \in \delta$; thus we have the following proposition.

PROPOSITION 1.9. *K is canonically isomorphic to* inj $\lim_{D\in\delta} A \mid D$, *where* $A \mid D = \{f \mid D: f \in A\}$.

Proof. The kernel of the canonical homomorphism of A onto this injective limit is exactly $\{f \in A: Z(f) \in \delta\}$: i.e., M, proving the proposition.

By Theorem 1.3, if D_0 is a discrete subset of X, $A \mid D_0$ is merely C^{D_0}, the set of *all* mappings of D_0 into C. Thus we have the following corollary.

COROLLARY 1.10. *K is isomorphic to* inj $\lim_{D\in\gamma} C^N \mid D$, *where N is the set of natural numbers and γ is a free ultrafilter on N; thus K is an algebraically closed proper extension of C, the image of the constant functions.*

Proof. The algebraic closure of K can be shown by choosing a monic polynomial with coefficients in K choosing a monic polynomial with coefficients in C^N whose coefficients map to the corresponding coefficients of the original polynomial, and for each $n \in N$ choose a root of the corresponding polynomial with coefficients in C. Then the element in C^N having this value at n goes to a root of the original polynomial. That K is a proper extension of C follows from the existence of unbounded elements in C^N.

Remark. A more elegant proof can be given by observing that K is an ultrapower of an algebraically closed field, and thus is algebraically closed. See Kochen [15] for details.

Of greater importance to us, in this paper, is the fact that K has a natural valuation whose residue class field is the complexes. Since δ is a Δ-ultrafilter, given $f \in A$, $\lim_{D\in\delta} f(D)$ always exists in the Riemann sphere Σ; let $f(M)$ be this limit. It can easily be shown that δ has a unique limit p in βX, the Stone-Čech compactification of X. Every $f \in A$ admits a continuous extension f^* from βX into Σ. $f(M)$ is merely $f^*(p)$. Given f, $g \in A$ that are congruent modulo M, then $f = g + m$, $m \in M$. Let $D_0 = Z(m)$; then $f \mid D_0 = g \mid D_0$. Hence $f(M) = g(M)$, and we see that the mapping $f \to f(M)$ induces a corresponding mapping p of K onto Σ. Using results proved in [3] we have the following.

THEOREM 1.11. *Let M be a maximal free ideal of A, $\delta = Z(M)$, and let λ be the canonical homomorphism of A onto $A/M \equiv K$. Given $a \in K$, choose $f \in A$ such that $\lambda(f) = a$. Define $p(a)$ to be* $\lim_{D\in\delta} f(D)$. *p, independent of the choice of f, is a place of K over C onto C whose value group is a divisible group that is an η_1-set of power $2^{\aleph_0}$, and whose valuation is 1-maximal.*

In proving the theorem, first observe that, according to Proposition 1.9, K is canonically isomorphic to inj $\lim_{D\in\delta_0} A\,|\,D$, where $\delta_0 = \delta \cap D_0$ and D_0 is a discrete subset of δ; thus K is canonically isomorphic to a residue class field modulo a maximal free ideal of the ring of complex-valued continuous functions on D_0. Applying [3], the theorem follows.

Background. Before going on to § 2, let us recall some of the definitions that occur in this theorem. In saying that p is a place of K over C onto C we mean (see, e.g., Zariski and Samuel [22]) it is a place whose valuation ring contains C such that C maps onto its residue class field. Since K is algebraically closed, a value group G associated with p (see, e.g., [22]) must be divisible. That G is an *η_1-set*, an idea due to Hausdorff [8], means that given any two countable subsets G_i of G, that may be empty, such that $G_0 < G_1$, then there exists $g \in G$ such that $G_0 < g < G_1$. Let V be the valuation of K associated with p whose range is $G \cup \{\infty\}$ (see, e.g., [22]). A sequence $(a_n)_{n\in N}$ in K is called *pseudo-convergent*, if given $n < m < k$ then $V(a_m - a_n) < V(a_k - a_m)$ (see, e.g., Schilling [19, pp. 39–43]). To show that $(a_n)_{n\in N}$ is pseudo-convergent it is necessary and sufficient to show that $V(a_{n+1} - a_n) = g_n$ is a strictly increasing sequence in G. Assume that $(a_n)_{n\in N}$ is pseudo-convergent. An element a in K is called a *pseudo-limit* of $(a_n)_{n\in N}$ if $V(a - a_n) = g_n$ for all n. V is called 1-*maximal* if every countable pseudo-convergent sequence $(a_n)_{n\in N}$ in K has a pseudo-limit in K.

Historical note. Helmer's study [9] of the ideal structure of A, in case $X = C$, seems to have been the first strictly algebraic study of this ring; Helmer's lemma (Lemma 1.1) and its ideal theoretic consequences occur in that paper. In [10] Henriksen adapts many of the ideas of Hewitt [12] to the study of A, in case $X = C$; in particular the correspondence between maximal ideals and Δ-ultrafilters is there in essence. Henriksen introduces algebraic zero sets, in which the multiplicity of the zero is noted, rather than zero sets; these he later used to great effect to study prime ideals [11]. Kakutani [13] is responsible for the correspondece between maximal ideals in A and Δ-ultrafilters, as it appears here (Theorem 1.7). Henriksen [10] also showed that A/M is algebraically closed. In [18] Royden suggests the generalization of Henriksen's results using Florack's generalization [6] of the Weierstrass and Mittag–Leffler theorems (Theorem 1.3). The ideas of Δ-ideal and Prime Δ-filter appear, in modified form, in Gillman and Jerison [7]. The valuation theory of these residue class fields is due to the author [3]. Schilling has, in an unpublished manuscript, obtained Helmer's lemma, in this setting, in his study of the closed fractionary ideals of A, extending his results on the subject [20] to the general case. I am indebted to Professor Schilling for making these unpublished results available to me.

2. Quotient rings and valuations

Let M be a maximal ideal of A and let $A_M = \{a/b\colon a \in A \text{ and } b \in A - M\}$; then A_M is a local ring whose maximal ideal M' is $\{a/b\colon a \in M \text{ and } b \in A - M\}$. In case $M = M_x$, for some $x \in X$, then A_M is clearly O_x, a valuation ring of F. With the aid of the following lemma, A_M will be seen to be a valuation ring of F in case M is free.

PROPOSITION 2.1. *Let M be a maximal ideal in A and let $\delta = Z(M)$. Then $A_M = \{f \in F$: there exists $D \in \delta$ such that f has no poles on $D\}$ and $M' = \{f \in F$: there exists $D \in \delta$ such that $f(D) = 0\}$.*

Proof. Let $f \in A_M$; then there exist $a \in A$ and $b \in A - M$ such that $f = a/b$. Since $b \notin M$, $Z(b) \notin \delta$; thus there exists $D \in \delta$ such that $Z(b) \cap D = \emptyset$. Hence f has no poles on D. Let $f \in M'$; then we may require that $a \in M$. Thus $Z(a)$ and $Z(a) \cap D = D' \in \delta$. On D', f is zero.

Let $f \in F$. By Theorem 1.3, there exist a, $b \in A$ such that $f = a/b$ and $Z(a) \cap Z(b) = \emptyset$. Assume there exists $D \in \delta$ such that f has no poles on D. Then $Z(b) \cap D = \emptyset$ and $Z(b) \notin \delta$: i.e., $b \notin M$, showing that $f \in A_M$. Assume now that $f(D) = 0$; then $D \cap Z(a)$, hence $Z(a) \in \delta$ and $a \in M$, showing that $f \in M'$, proving the proposition.

THEOREM 2.2. *A_M is a valuation ring of F.*

Proof. Let $f \in F - A_M$ and let P be the set of poles of f. By Proposition 2.1, $P \cap D \neq \emptyset$ for all $D \in \delta$; thus $P \in \delta$. Since $Z(1/f) = P$, we may apply Proposition 2.1, and conclude that $1/f \in M'$, proving the theorem.

The rest of this section will be devoted to considering the value group and valuation associated with A_M in case M is free.

For $f \in F^*$ let $d(f)(x) = V_x(f)$ for all $x \in X$; thus $d(f) \in J^X$, J denoting the ring of integers. $d(f)$ is called the *divisor of* f. Let $d(0) = \infty$ and let $\infty > u$ for all $u \in J^X$. Clearly J^X is a lattice-ordered group. For $u \in J^X$, the *support of* u is $\{x \in X: u(x) \neq 0\}$.

PROPOSITION 2.3. *d is a homomorphism of F^* into J^X whose range is $\{u \in J^X$: the support of u is a discrete subset of $X\}$. Given f, $g \in F$ then $d(f \pm g) \geqslant d(f) \wedge d(g)$.*

Clearly δ is a directed set; thus using the restriction mappings of $J^X | D (= \{u | D: u \in J^X\})$ onto $J^X | D'$ if $D' \subset D$ we can consider the following injective limit, inj $\lim_{D \in \delta} J^X | D = H$. Let τ be the canonical homomorphism of J^X onto H. Let H inherit the order of J^X: i.e., let u, $v \in J^X$ and let $\tau(u) \leqslant \tau(v)$ ($\tau(u) < \tau(v)$) if there exists $D \in \delta$ such that $u | D \leqslant v | D$ ($u | D < v | D$). Clearly τ is order-preserving.

PROPOSITION 2.4. *τ maps $d(F^*)$ onto H, and H is a totally ordered group.*

Proof. By Proposition 2.3, and Theorem 1.3, if $D_0 \in \delta - X$ then $d(F^*) | D_0 = J^{D_0}$; thus τ maps $d(F^*)$ onto H. Clearly H is a partially ordered group. Let $u \in J^{D_0}$, let $D_1 = \{x \in D_0$: $a(x) \geqslant 0\}$, and let $D_2 = \{x \in D_0$: $u(x) < 0\}$. Clearly $D_0 = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$. Since δ is a Δ-ultrafilter, $\delta_0 = \delta \cap D_0$ is an ultrafilter on D_0 (Proposition 1.8). Thus either D_1 or $D_2 \in \delta_0$; accordingly either $\tau(u) \geqslant 0$ or $\tau(u) < 0$, proving the proposition.

Let $\tau(\infty)=\infty$, ∞ being greater than all $h \in H$. Let $W=\tau d$. Thus W maps F onto $H \cup \{\infty\}$.

THEOREM 2.5. *W is a valuation of F associated with A_M.*

Proof. By Propositions 2.3, and 2.4, W is a valuation of F over C; thus it suffices to show that $W(f) \geqslant 0$ if and only if $f \in A_M$. By definition the following statements are equivalent: $W(f) \geqslant 0$, $\tau(d(f)) \geqslant 0$, there exists $D \in \delta$ such that $d(f) \mid D \geqslant 0$. By Proposition 2.1, the last statement is equivalent to the statement that $f \in A_M$, proving the theorem.

We will now investigate the group H. Clearly H may be regarded as inj $\lim_{D \in \gamma} J^N \mid D$, where γ is a free ultrafilter on N, the set of positive integers. Let σ be the canonical homomorphism of J^N onto H. That the divisibility of elements in H by positive integers is rather complex can be seen from the following examples: let $a(m)=m!$, $b(m)=2^m$, and let $c(m)$ be the mth prime number. Then $\sigma(a)$ is divisible in H by all $n \in N$, $\sigma(b)$ is divisible by all powers of two and no other integers, and $\sigma(c)$ has no divisors other than 1.

Background. By a *near η_1-set* is meant a non-empty totally ordered set T such that given any two non-empty countable subsets T_i of T such that $T_0 < T_1$, then there exists $t \in T$ such that $T_0 \leqslant t \leqslant T_1$. Clearly an η_1-set is a near η_1-set. The converse is not true, since the set of real numbers is a near η_1-set but is not an η_1-set. Let G be a totally ordered Abelian group. A subgroup G' of G is called *convex* if given $g' \in G'$ and $g \in G$ such that $|g| \leqslant |g'|$ (where $|g| = \max\ g, -g$), then $g \in G'$. The convex subgroups of G form a complete totally ordered set under inclusion. For $g \in G$ let $v(g)$ be the smallest convex subgroup of G that contains g. Note: $0 \leqslant g \leqslant h$ implies $v(g) \leqslant v(h)$, $v(g)=\{0\}$ if and only if $g=0$, $v(g \pm h) \leqslant \max v(g),\ v(h)$, and if $v(g) \neq v(h)$, $v(g+h)=\max v(g),\ v(h)$; thus v has many of the properties of a valuation. v is called the *natural valuation on* G and $S=v(G^*)$ is called the *value set of* G. Let $s \in S$ and let

$$G(s)=\{g \in G\colon v(g) \leqslant s\}/\{g \in G\colon v(g) < s\}.$$

$G(s)$ is an Archimedean totally ordered group under the order induced on it by G; thus $G(s)$ is isomorphic to a subgroup of the reals. $G(s)$ is referred to as a *factor* of G. (See [2] for references.)

THEOREM 2.6. *H is a near η_1-set whose value set S has no countable cofinal subset and has a least element s_0.*

Proof. Let $a(m)=1$ for all $m \in N$, and let v be the natural valuation on H. Then $s_0 = v(\sigma(a))$ is the least element of S. Clearly $H(=\text{inj} \lim_{D \in \gamma} J^N \mid D)$ is a cofinal subgroup of $G = \text{inj} \lim_{D \in \gamma} R^N \mid D$, R denoting the reals. Hewitt [12] has shown that G, a totally ordered group, has no countable cofinal subsets, proving that H, and thus S, have no countable cofinal subsets. It remains to show that H is a near η_1-set. Let (h_n) and (k_n) be countable subsets of H such that $h_n \leqslant_{n+1} < k_{n+1} \leqslant k_n$. By Lemma 13.5 [7], there exist pre-images h'_n and k'_n in J^N of h_n and k_n respectively such that $h'_n \leqslant h'_{n+1} < k'_{n+1} \leqslant k'_n$ for all $n \in N$. Let $b(m) = h'_m(m)$ for all $m \in N$. Then $b \in J^N$. Let $D_n = \{m \in N: m \geqslant n\}$. Since γ is a free ultrafilter on N, $D_n \in \gamma$. Let $m \in D_n$: i.e., let $m \geqslant n$. Then $h'_n(m) \leqslant h'_m(m) = b(m) < k'_m(m) \leqslant k'_n(m)$; hence $h'_n \mid D_n \leqslant b \mid D_n \leqslant k'_n \mid D_n$. Thus $h_n \leqslant \sigma(b) \leqslant k_n$ for all $n \in N$, proving the theorem.

Let us now apply the results obtained in [3] on near η_1-sets.

COROLLARY 2.7. *$S - \{s_0\}$ is an η_1-set and the natural valuation on H is 1-maximal.*

Applying classical valuation theory we get the following.

COROLLARY 2.8. *The set of all prime ideals of A that are contained in M is in one-to-one order reversing correspondence with the lower sets of S. Further M' is a principal ideal in A_M.*

Proof. It is well known (see, e.g., [21, p. 228]) that the mapping $P \to PA_M$ is a one-to-one order preserving mapping between the prime ideals of A contained in M and the prime ideals of A_M. Since A_M is a valuation ring, its prime ideals are totally ordered under inclusion. Let P' be a prime ideal in A_M and let $H_{P'} = \{h \in H: |h| < V(y) \text{ for all } y \in P'\}$. The mapping $P' \to H_{P'}$ is well known [22] to be a one-to-one order reversing mapping of the prime ideals of A_M onto the convex subgroups of H. Finally, it is well known that the natural valuation v of H induces a one-to-one order preserving mapping of the convex subgroups of H onto the lower sets of S. Since H has a least positive element, M' is a principal ideal of A_M; proving the corollary.

Since H is a near η_1-set we may apply results obtained in [3] and conclude the following: the factors of H are either discrete or real. In the the following, more will be proved.

THEOREM 2.9. *All factors of H are real except the factor associated with s_0, the least element of S, which is discrete.*

Proof. Let $a(m) = 1$ for all $m \in N$. Then, since $\sigma(a)$, the smallest positive element of H, generates $H(s_0)$, this group is discrete. Let $s \in S$, $s > s_0$. A non-zero element in $H(s)$ is the image of an element $b \in J^N$ such that $v(\sigma(b)) = s$. Let $D_j = \{m \in N: b(m) = j \pmod 2\}$, $j = 0, 1$. Clearly $D_0 \cup D_1 = N$ and $D_0 \cap D_1 = \phi$; thus either D_0 or $D_1 \in \gamma$. If $D_0 \in \gamma$ then $\sigma(b)$ is divisible by 2 in H and thus its image will be divisible by 2 in $H(s)$. If $D_1 \in \gamma$ then $\sigma(b-a)$ is divisible by 2 in H. Since $v(\sigma(b)) = s > s_0 = v(\sigma(a))$, $\sigma(b-a)$ and $\sigma(b)$ have the same image in $H(s)$, showing that every element in $H(s)$ is divisible by 2 in $H(s)$. Since H is a near η_1-set, its factors are either real or discrete [3]; thus $H(s)$ is real, proving the theorem.

COROLLARY 2.10. *Let P' be a non-zero, non-maximal prime ideal in A_M. P' is not a principal ideal; thus P' is not finitely generated. There exist such P' that are countably generated and such P' that admit only an uncountable set of generators; in particular, this is so if P' is the largest non-maximal prime ideal in A_M.*

Proof. Since P' is a non-zero, non-maximal prime ideal in A_M, $H_{P'}$ (see the proof of Corollary 2.8 for the definition) is a proper, non-zero convex subgroup H. $S-v(H_{P'})$ may have a least element s_1. Since $H_{P'}$ is non-zero, $s_1 > s_0$; thus, by Theorem 2.9, $H(s_1)$ is isomorphic to the reals. Hence $W(P')$ has no least element but does have a countable coinitial subset, showing that P' is not principal but is countably generated (see, e.g., Schilling [19, p. 10]). P' will also be countably generated if $S-v(H_{P'})$ has a countable coinitial subset. It can also occur that $S-v(H_{P'})$ has no countable coinitial subset, since $S-\{s_0\}$ is an η_1-set. In this case P' is only uncountably generated, proving the corollary.

Using this corollary we can get lower bounds for the number of generators needed for the corresponding prime ideals in A, observing that $W(A^*)=H(\geqslant 0)$. However, from Helmer's Lemma we know that no free ideal in A is finitely generated.

We conclude this section by proving a result that indicates the amount of interplay existing between F and H, namely the following.

THEOREM 2.11. *W is 1-maximal.*

Proof. Let $(f_n)_{n\in N}$ be a countable pseudo-convergent sequence in F. Let $D_1 \in \delta - \{X\}$ and let x be a one-to-one mapping of N onto D_1; thus $D_1=(x(j))_{j\in N}$. Since $\delta \cap D_1$ is a free ultrafilter on D_1, $\{x(j): j\in N \text{ and } j\geqslant n+1\}\in\delta$. Assume that D_n has been chosen in δ such that

(1) $d(f_{n+1}-f_n)\,|\,D_n > \ldots > d(f_2-f_1)\,|\,D_n$ and
(2) $1\leqslant j<n$ implies $x_j \notin D_n$.

Since (f_n) is pseudo-convergent, $W(f_{n+1}-f_n)=h_n$ is strictly increasing; thus, there exists $D\in\delta$ such that $d(f_{n+2}-f_{n+1})\,|\,D > d(f_{n+1}-f_n)\,|\,D$. Let $D_{n+1}=D\cap D_n\cap\{x(j): j\in N$ and $j\geqslant n+1\}$. Clearly $D_{n+1}\in\delta$, and D_{n+1} satisfies conditions (1) and (2). Thus, (D_n) is defined, each element having properties (1) and (2).

Let $j\in N$. By (2), $x(j)\in D_n$ implies $j\geqslant n$. Let $p(j)$ be the largest integer such that $x(j)\in D_{p(j)}$. Clearly $j\geqslant p(j)\geqslant n$. Let $k(j)=d(f_{p(j)+1}-f_{p(j)})\,x(j)$. Note: $k(j)\geqslant V_{x(j)}(f_{p(j)})$. By Corollary 1.4, there exists $f\in F$ such that $V_{x(j)}(f-f_{p(j)})\geqslant k(j)$ for all $j\in N$.

Let $n\in N$ and let $x\in D_n$. Since $D_n\subset D_1$, there exists a unique $j\in N$ such that $x(j)=x$. By (2) $j\geqslant n$. Let $p=p(j)$; then $x\in D_p$ and $p\geqslant n$. Then $V_x(f-f_p)\geqslant k(j)=V_x(f_{p+1}-f_p)$. If $p=n$, then $V_x(f-f_n)\geqslant V_x(f_{n+1}-f_n)$. Assume $p>n$. Then

$$V_x(f-f_n)=V_x(f-f_p+f_p-f_n)\geqslant \min V_x(f-f_p),\ V_x(f_p-f_n).$$

Since $x\in D_p$ we can apply (1) and conclude that

$$V_x(f_{p+1}-f_p)>V_x(f_p-f_{p-1})>\ldots>V_x(f_{n+1}-f_n).$$

Let $n<j\leqslant p$. We wish to show that $V_x(f_j-f_n)=V_x(f_{n+1}-f_n)$. Clearly it is true if $j=n+1$. Assume it is true for $n<j<p$.

$$V_x(f_{j+1}-f_n)=V_x(f_{j+1}-f_j+f_j-f_n)=V_x(f_{n+1}-f_n),$$

showing that $V_x(f_p-f_n)=V_x(f_{n+1}-f_n)$ and hence that

$$V_x(f-f_n)=\min V_x(f_{p+1}-f_p),\ V_x(f_{n+1}-f_n)=V_x(f_{n+1}-f_n).$$

Hence $d(f-f_n)\,|\,D_n=d(f_{n+1}-f_n)\,|\,D_n$. Thus $W(f-f_n)=W(f_{n+1}-f_n)$, showing that f is a pseudo-limit of $(f_n)_{n\in N}$, proving the theorem.

Historical note. Henriksen [11] analyzed the prime ideals of A, in case $X=C$, and found that the prime ideals of A contained in M are totally ordered under inclusion; his results on the order type of this set have been sharpened slightly in Corollary 2.7 and 2.8. Banaschewski [4] employed the divisor mapping d on A, in case $X=C$, to take ideals in A to "ideals" in $d(A)$. He also employed injective limits along δ to analyze the "ideals" in $d(A)$ that come from ideals in A that contain M. Kochen [15] has analyzed the order type of H, using the continuum hypothesis, finding it to be $(\omega^*+\omega)\eta_1$, when η_1 is the order type of an η_1-set of power $\aleph_1$. Without the continuum hypothesis, an analogous result holds, letting η_1 be merely the order type of an η_1-set. Theorem 2.11 and its proof are closely related to Lemma 6 [11], in which it is shown that if P is the largest non-maximal prime ideal of A contained in M, then A/P, a valuation ring, is complete.

Henriksen [11] also shows that if P is a non-maximal prime ideal of A, then A/P is a valuation ring. These results hold in the general case. Let F_P be the quotient field of A/P; then the value group of F_P, under the valuation W_P associated with A/P, is $H_P=\{h\in H: |h|<W(p)$ for all $p\in P\}$. Further W_p is 1-maximal, giving alternate proofs to a number of Henriksen's results [11, § 4].

3. Composite places

Let M be a maximal free ideal of A. Let λ' be the unique extension to A_M of λ, the canonical homomorphism of A onto $A/M=K$. Let r extend λ' taking $f\in F-A_M$ to ∞; thus r is a place of F over C onto K associated with A_M. Let p be the place of K over C onto C defined in Theorem 1.11. Extend p to $K\cup\{\infty\}$ by letting $p(\infty)=\infty$ and let $pr=s(=s_M)$. Then s is a place of F over C onto C determined by M. Let O be its valuation ring and P its maximal ideal.

PROPOSITION 3.1. *Given* $f \in F$, $s(f) = \lim_{D\in\delta} f(D)$.

Proof. Let $f \in A_M$. There exists $g \in A$ such that $\lambda'(f) = \lambda(g)$. Clearly $s(f) = s(g)$, which by Theorem 1.11, is $\lim_{D\in\delta} g(D)$. Since $\lambda'(f) = \lambda(g)$, $f - g \in M'$. By Proposition 2.1, there exists $D_0 \in \delta$ such that $(f-g)(D_0) = 0$, and hence $\lim_{D\in\delta} g(D) = \lim_{D\in\delta} f(D)$. Let $f \in F - A_M$. By Proposition 2.1, f has poles on D for all $D \in \delta$, showing that $\lim_{D\in\delta} f(D) = \infty$, proving the proposition.

Since s is a composite place, we have the following.

COROLLARY 3.2. $M' \subset P \subset O \subset A_M$.

(This may also be seen from Proposition 3.1, and Proposition 2.1.)

Applying the classical theory of composite places and valuations (see, e.g., [22]) we have the following. Let Y be a valuation of F associated with O, and let Ω be its value group.

THEOREM 3.3. *Let* $G = Y(A_M - M')$. *Then* Y *and* λ' *induce a valuation* V *of* K *associated with* p *which has* G *as its value group. Let* Ψ *be the canonical homomorphism of* Ω *onto* $\Omega/G = H$ *and let* $W = \Psi Y$. *Then* W *is a valuation of* F *associated with* r.

In § 1 and § 2 the structure of G and of H was described. Combining these results we have the following.

THEOREM 3.4. Ω *is an* η_1*-set whose factors are real, save one, which is discrete.*

Proof. By Theorem 1.11 and [1], all of the factors of G are real. By Theorem 2.9, all but one of the factors of H are real, that one being discrete; thus the statement concerning the factors of Ω follows. By Theorem 1.11, and [1], G is 1-maximal. By Theorem 2.6, and [3], H is 1-maximal; thus Ω is 1-maximal. By Theorem 1.11, the value set P of G is an η_1-set. By Corollary 2.7, the value set S of H has a least element s_0 and $S - \{s_0\}$ is an η_1-set; thus the value set of Ω, which is similar to $P + S$ is an η_1-set. Applying [1], we see that Ω is an η_1-set, proving the theorem.

Using the classical analysis of ideals in a valuation ring [22], as was done in the proof of Corollary 2.8, we get the following.

COROLLARY 3.5. *The prime ideals in* O *are in one-to-one order reversing correspondence with the lower sets of an* η_1*-set. Further,* O *is not countably generated.*

We are able to conclude the following.

COROLLARY 3.6. *The transcendence degree of* F *over* C *is* $2^{\aleph}$

Proof. Since the cardinal number of F is $2^{\aleph}$, its transcendence degree over C cannot exceed $2^{\aleph}$. Ω can be imbedded, in an essentially unique way, in a divisible totally ordered group Ω' such that Ω' is the divisible subgroup of Ω generated by Ω; further, the value set of Ω is mapped onto the valued set of Ω' by a one-to-one mapping. By the *rational rank* of Ω is meant the dimension of Ω' over the rationals. It is well known (see, e.g., [22]) that the transcendence degree of F over C is at least the rational rank of Ω. Clearly the dimension of Ω' over the rationals is at least the cardinality of Ω', which is the cardinality of Ω, which by Theorem 3.4 is an η_1-set. Hausdorff [8] showed that the cardinality of such sets is at least $2^{\aleph}$, proving the corollary.

We can apply the same argument to show that the trascendence degree of $K(=A/M)$ over C is $2^{\aleph}$ in case M is a maximal free ideal.

We have seen in § 1 and § 2 that V and W are 1-maximal. These results will now be combined to form the following.

THEOREM 3.7. *Y is 1-maximal.*

Proof. Let $(f_n)_{n\in N}$ be a countable pseudo-convergent sequence, under Y, in F and let $\omega_n = Y(f_{n+1}-f_n)$; then by the definition of pseudo-convergence, (ω_n) is a strictly increasing sequence in Ω. Let $h_n = \Psi(\omega_n)$, Ψ being the canonical homomorphism of Ω onto $H=\Omega/G$. Since Ψ is order-preserving, (h_n) is an increasing sequence in H. Either,

(1) $(h_n)_{n\in N}$ has a greatest element h, or

(2) no such element exists.

Assume that (2) holds. Let j be a strictly increasing function of N into N such that $(h_{j(n)})$ is a strictly increasing sequence in H; then, by definition, $(f_{j(n)})$ is pseudo-convergent under W. By Theorem 2.11, W is 1-maximal. Thus there exists $f\in F$ such that $W(f-f_{j(n)}) = h_{j(n)}$, for all $n\in N$. Clearly $Y(f-f_{j(n)})=\omega_{j(n)}+\gamma_{j(n)}$, where $\gamma_{j(n)}\in G$. Since $h_{j(n)}<h_{j(n+1)}$ and $\gamma_{j(n)}\in G$, $\omega_{j(n)}<\omega_{j(n+1)}+\gamma_{j(n+1)}$. $Y(f-f_n)=Y(f-f_{j(n+1)}+f_{j(n+1)}-f_n)=\min\ \omega_{j(n+1)}+\gamma_{j(n+1)}, \omega_n=\omega_n$, proving that f is a pseudolimit of (f_n) under Y.

Assume now that (1) holds. By dropping a finite number of terms from (f_n) and re-indexing, we may assume that $h_n=h$ for all n. Clearly (f_n) is still pseudo-convergent under Y. Let $b_n=f_{n+1}-f_1$ for all $n\in N$. Note: (b_n) is pseudo-convergent under Y and $W(b_n) = h = W(b_{n+1}-b_n)$ for all n. Let $d_n=b_n b_1^{-1}$ for all n. Then (d_n) is pseudo-convergent under Y. To show that (f_n) has a pseudo-limit under Y in F, it suffices to show that (d_n) has a pseudo-limit under Y in F. Since $W(d_n)=0$, $d_n\in A_M-M'$. Let $e_n=\lambda'(d_n)$ for all n. Since $W(d_{n+1}-d_n)=0$, $Y(d_{n+1}-d_n)=g_n\in G$. By the definition of V, $V(e_{n+1}-e_n)=g_n$; thus (e_n) is a pseudo-convergent sequence in K under V. By Theorem 1.11, V is 1-maximal. Thus there exists $e\in K$ such that $V(e-e_n)=g_n$ for all n. Let $d\in A_M$ such that $\lambda'(d)=e$. Then $Y(d-d_n)=g_n$ for all n; hence d is a pseudo-limit of (d_n) under Y, proving the theorem.

4. Place spaces

Let S be the set of all places of F over C onto C: i.e., all places of F that contain C in their valuation rings and map C onto their residue class fields. For $s \in S$ and $f \in F$ let $f(s) = s(f)$ and we thus regard f as a mapping of S into the Riemann sphere Σ. Let S be given the weakest topology making the mapping $s \to f(s)$ continuous, for all $f \in F$. Using an argument given by Chevelley [5, Chapt. VII, § 1] we obtain the following.

THEOREM 4.1. *S is a compact Hausdorff space.*

Let $x \in X$ and let s_x be the place of F over C onto C obtained by "evaluating f at x". Let $j(x) = s_x$; then j is a homeomorphism of X into S. It will frequently be convenient to identify X and $j(X)$. Let T, the closure of $j(X)$ in S, be called the *set of topological places* of F. Clearly we have the following.

COROLLARY 4.2. *T is a compact Hausdorff space in which X is everywhere dense. Every $f \in F$ extends to a continuous mapping of T into Σ. These extended mappings separate points of T; further, T has the weakest topology making all these functions continuous.*

Let βX denote the Stone-Čech compactification of X, (see [7] for details). βX has the following characteristic properties:

(a) βX is a compact Hausdorff space that contains X as an everywhere dense subset, and

(b) every continuous mapping from X into a compact set Y has a continuous extension to βX into Y.

Let Λ be the set of all closed subsets of X. Since X is a metrizable space, Λ is also the set of zero sets of continuous real-valued functions on X. Following Gillman and Jerison [7], a filter in Λ will be called a z-filter on X. It has been shown [7] that the points of βX are in one-to-one correspondence with the z-ultrafilters on X. The correspondence is the following: every z-ultrafilter on X has a unique limit $p \in \beta X$. Conversely, given $p \in \beta X$, let $\zeta = \{U \in \Lambda : p \in \mathrm{cl}_{\beta X}\, U\}$.

THEOREM 4.3. *j has a unique continuous extension k that maps βX onto T. Each $f \in F$ has a unique continuous extension f^* that maps βX into Σ. If given $p \in \beta X$, let $s = k(p)$, then $f^*(p) = f(s)$. Finally $f^*(p) = \lim_{U \in \zeta} f(U)$.*

This follows from the characteristic properties of βX. (See [7] for details.)

As remarked above, each point in βX is the limit of a unique z-ultrafilter on X. A z-ultrafilter on X will be called *discrete* if it contains a discrete subset of X; let δX be the set of all points in βX that are the limits of discrete z-ultrafilters on X, or equivalently, let δX be the set of all points in βX that are adherence points of discrete subsets of X. Clearly $X \subset \delta X$, and by [7], the cardinal number of δX and βX is $2^{2^{\aleph_0}}$. Clearly the restriction of a discrete z-ultrafilter on X to Δ gives rise to a Δ-ultrafilter; conversely a Δ-ultra filter engenders a discrete z-ultrafilter on X. We have seen in Theorem 1.7 that there is a one-to-one correspondence between the Δ-ultrafilters and the maximal ideals of A. Let $S_A = \{s_M: M$ is a maximal ideal in $A\}$; thus there is a natural one-to-one correspondence between δX and S_A, an observation made by Kakutani [13] for schlicht plane domains X. In the next theorem we will see that this correspondence is $k \mid \delta X$.

THEOREM 4.4. *k is a one-to-one mapping of δX onto S_A.*

Proof. Let $p \in \delta X$ and let ζ be the z-ultrafilter on X that converges to p. By definition, ζ is a discrete z-ultrafilter. Let $\delta = \zeta \cap \Delta$ and let $M = Z^{-1}(\delta)$; then M is the maximal ideal of A associated with p by the correspondence discussed above. By Theorem 4.3, $O_{k(p)} = \{f \in F: \lim_{U \in \zeta} f(U) \in C\}$ which is also $\{f \in F: \lim_{D \in \delta} f(D) \in C\}$. But by Theorem 3.1, this is exactly the valuation ring of s_M, proving that $k(p) = s_M$. As this correspondence is one-to-one, the theorem is proved.

It is easily seen (cf. [7, 4F]) that $\beta X \neq \delta X$, showing that δX is not compact.

Using the next result, together with Corollary 3.6, we can see how very arbitrary places of F over C onto C can be.

LEMMA 4.5. *Let $(x_i)_{i \in I}$ be a transcendence base of F over C, let G be a divisible totally ordered (Abelian) group and let $(g_i)_{i \in I}$ be a set of positive elements of G. There exists a place s of F over C onto C whose valuation V takes x_i to g_i and whose value group is contained in the smallest divisible subgroup of G containing $(g_i)_{i \in I}$.*

Proof. Let $V_0(c) = 0$ for all $c \in C^*$ and let $V_0(x_i) = g_i$, for all $i \in I$. Then V_0 extends, by linearity over the integers, to the monomials of $C[x_i]_{i \in I}$. $f \in C[x_i]_{i \in I}$ can be uniquely expressed as a sum $\sum_{i=1}^{n} c_i m_i$, $c_i \in C^*$ the m_i's distinct monomials in $C[x_i]_{i \in I}$. Let $V_0(f) = \min(V_0(m_i))_{i=1,\ldots,m}$. For $f, g \in C[x_i]_{i \in I}$, $g \neq 0$, let $V_0(f/g) = V_0(f) - V_0(g)$; thus V_0 is a valuation of $C(x_i)_{i \in I}$ over C. Since $g_i > 0$ for all i, the place s_0, of $C(x_i)_{i \in I}$ associated with V_0, maps x_i to zero, showing that it maps an element in $C[x_i]_{i \in I}$ to its constant term: i.e., V_0 has C as its residue class field. By the place extension theorem (see, e.g., [16, p. 8]), s_0 extends to a place s of F over C onto C. Since F is an algebraic extension of $C(x_i)_{i \in I}$, the value group of s_0 is contained in the smallest divisible subgroup of G containing $(g_i)_{i \in I}$ (see [22, § 11] for details), proving the lemma.

COROLLARY 4.6. *There exists a place $s \in S - X$ with an Archimedean value group.*

Proof. Let $(g_i)_{i \in I}$ be a set of positive real numbers that does not generate a discrete subgroup and let I be of power $2^{\aleph_0}$. By Lemma 4.5, there exists $s \in S$ such that $s(x_i) = g_i$.

Since the group generated by $(g_i)_{i\in I}$ is non-discrete, the value group of s is not the integers, thus $s\notin X$, proving the corollary.

As a result of the following lemmas we will show that $T \neq S$.

LEMMA 4.7. *Given $s\in T-X$ there exists $f\in A$ such that $f(s)=\infty$.*

Proof. Narasimhan [17] has shown that X has a closed, nonsingular imbedding into C^3; let X be so imbedded. Let $p\in k^{-1}(s)$ and let ζ be the z-ultrafilter on X that converges to p. Clearly there exists $U_0\in\zeta$ such that $(0,0,0)\notin U_0$; thus on, U_0 $r(z_1,z_2,z_3)=(z_1,z_2,z_3)/|(z_1,z_2,z_3)|$, the modulus denoting the distance to the origin, is a continuous mapping into $S^5=\{(z_1,z_2,z_3)\colon |(z_1,z_2,z_3)|=1\}$. Let $\zeta_0=\zeta\cap U_0$ and let p_0 be the limit of ζ_0 in βU_0. Clearly r extends to r^*, a mapping of βU_0 into S^5. Let $\alpha=r^*(p_0)$. Clearly α is independent of the choice of U_0. Let π_α be the orthogonal projection of C^3 onto the plane $C\alpha$, and let $f=\pi_\alpha|X$. Clearly $f\in A$. By the choice of α, $f^*(p)=\infty$, proving the lemma.

LEMMA 4.8. *Given $s\in T-X$ and $f\in A$ such that $V(f)<0$, where V is a valuation of F associated with s, then there exists $h\in A$ such that $V(h)<mV(f)$ for all $m\in N$.*

Proof. Since $V(f)<0$, $f(s)=\infty$. By Theorem 4.3, $f^*(p)=f(s)$, and there exists $U_0\in\zeta$, the z-ultrafilter on X that converges to p, such that $0\notin f(U_0)$. Hence $f/|f|$ is a continuous mapping of U_0 into $S^1=\{\alpha\in C\colon|\alpha|=1\}$. Since S^1 is compact, $f/|f|$ extends to a continuous mapping $(f/|f|)^*$ of βU_0 into S^1. Let $\zeta_0=\zeta\cap U_0$ and let p_0 be the limit of ζ_0 in βU_0. Clearly $\alpha=(f/|f|)^*(p_0)$ is independent of the choice of U_0. Let α be denoted by $(f/|f|)^*(p)$, let $\bar\alpha$ be the conjugate of α in C, and let $g=\bar\alpha f$. Then $(g/|g|)^*(p)=1$, showing that there exists $U_1\in\zeta_0$ such that for $x\in U_1$, the angle between the vectors $g(x)$ and 1 is between $-\pi/4$ and $\pi/4$. Let m be a positive integer and let $\varepsilon>0$. There exists $n>0$ such that $t>n$ implies $(2t)^m/e^t<\varepsilon$. Since $g^*(p)=\infty$, there exists $U_2\in\zeta$ such that $|g(U_2)|>n\sqrt{2}$. Let $U=U_0\cap U_1\cap U_2$. Clearly $U\in\zeta$. Let $x\in U$. Then the real part, $a(x)$, of $g(x)$ is greater than n. Let $h=e^g$. Since $g\in A$, $h\in A$. Further $|(g(x))^m/h(x)|\leqslant(2(a(x)))^m/e^{a(x)}<\varepsilon$; thus $\lim_{U\in\zeta}(g^m/h)(U)=0$ and $g^m/h\in P_s$, the valuation ideal of s. Then $0<V(g^m/h)=mV(g)-V(h)$ for all $m\in N$, showing that $V(h)<mV(g)$ for all $m\in N$. Clearly $V(f)=V(g)$, proving the theorem.

As a consequence of Lemmas 4.10 and 4.11 we have the following.

COROLLARY 4.9. *Let $s\in T-X$ and let G be a value group of s. The value set of G has an infinite ascending sequence in it.*

Combining this result with Corollary 4.6 gives us the following.

THEOREM 4.10. $S\neq T$.

Thus we have shown that $X<S_A\subset T<S$.

Open questions

The following questions raised by these researches seem, at this writing, to be open.

1. Are δX and S_A homeomorphic?

2. Is $S_A \neq T$?

3. Are T and βX homeomorphic? To show they are, it suffices to show that the elements of A or of F, all of which extend to βX, separate the points of βX.

4. Given $s \in T - S_A$, if such elements exist, what is the value group of s like? We know only that its value set has in it an infinite ascending sequence.

5. Given $s \in T - S_A$, is the valuation associated with s 1-maximal?

6. Is there $s \in S - T$ whose value group is the integers? If not, then X can be extracted from S, and thus can be reconstructed from F.

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