BOUNDED APPROXIMATION BY POLYNOMIALS

BY

L. A. RUBEL and A. L. SHIELDS

Columbia University and the University of Illinois, University of Michigan(1)

1. Introduction

In this paper we present a complete solution to the following problem: if G is an arbitrary bounded open set in the complex plane, characterize those functions in G that can be obtained as the bounded pointwise limits of polynomials in G. Roughly speaking, the answer is that a function is such a limit if and only if it has a bounded analytic continuation throughout a certain bounded open set G^* that contains G. This set G^* is the inside of the "outer boundary" of G. More precisely, if G is a bounded open set and if H is the unbounded component of the complement of G^- (the closure of G), then G^* denotes the complement of H^-.

A sequence of polynomials $\{p_n\}$ is said to converge boundedly to a function f in an open set G if the polynomials are uniformly bounded in G, and if $p_n(z)$ converges to $f(z)$ at each point $z \in G$. It follows that f is bounded in G. Also, by the Stieltjes-Osgood theorem (see [8], Chapter II, § 7) the convergence is uniform on compact subsets of G and thus f is analytic in G.

MAIN THEOREM. *Let G be a bounded open set in the plane and let f be a bounded analytic function in G. If there is a function F, analytic in G^* and agreeing with f in G, with $|F(z)| \leqslant M$ in G^*, then there is a sequence of polynomials $\{p_n\}$ such that*

(i) $\lim p_n(z) = F(z) \qquad (z \in G^*)$,

(ii) $|p_n(z)| \leqslant M \qquad (z \in G^*; \quad n = 1, 2, \ldots)$.

Conversely, if there is a sequence of polynomials converging to f at each point of G, and uniformly bounded in G, then there is a bounded analytic function F in G^* that agrees with f in G.

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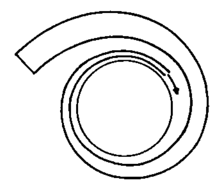

Fig. 1.

It is possible for a bounded function f to have a continuation F that is unbounded in G^*, even when both G and G^* are connected and simply connected (see § 4).

To illustrate the theorem, let G be the open set consisting of the open unit disc with a ribbon winding around it infinitely often (see Fig. 1). Thus G has two components G_1 and G_2. Let f_1 and f_2 be any two bounded analytic functions in G_1 and G_2, respectively. Then there is a single sequence of polynomials converging boundedly to f_1 in G_1 and to f_2 in G_2.

It is a classical result (see [5], Chapter I, § 1) that if G is the open unit disc, then every bounded analytic function in G can be boundedly approximated by polynomials. This result was extended to general Jordan domains by Carleman [3, pp. 3–5]. The next result, a corollary to the main theorem, is the most general result in this direction. The sufficiency, in the special case where G is connected, was proved by Farrell [4].

COROLLARY. *Let G be a simply connected (but not necessarily connected) bounded open set. The necessary and sufficient condition that every bounded analytic function in G be the bounded limit of a sequence of polynomials is that G and H have the same boundary.*

Bounded open sets G whose boundary coincides with the boundary of H will be called Carathéodory sets, following the terminology in the book of Markushevitch [6, Chapter V, 4.7]. E. Bishop [1] gives an equivalent definition and calls them "balanced" open sets. We shall see that Carathéodory sets are simply connected. See § 2.2 and § 2.8 for a more detailed discussion.

More generally, in case G is not simply connected, the necessary and sufficient condition that every bounded analytic function in G be boundedly approximable by a sequence of polynomials is that the "inner boundary" of G should form a set of removable singularities for bounded analytic functions. The inner boundary is the set of those boundary points of G that are not boundary points of H.

It is interesting to compare these results with those obtained for other methods of approximation. For example, Runge's theorem may be thought of as answering the following question: if G is an open set, characterize those analytic functions in G that can be approximated, uniformly on compact subsets of G, by polynomials. One version of Runge's theorem is the following.

RUNGE'S THEOREM. *Let G be an open set and let $G^\wedge$ be the complement of the unbounded component of the complement of G. Then a function f can be approximated, uniformly on compact subsets of G, by a sequence of polynomials if and only if f has an analytic continuation throughout $G^\wedge$.*

(In this statement, if G is not a bounded set, then the complements must be taken with respect to the extended plane.)

In the proof of our main theorem we shall use the following form of Runge's theorem (see [8], Chapter IV, § 1): if G is a simply connected open set (not necessarily connected), if f is holomorphic in G, and if K is a compact subset of G, then for each $\varepsilon > 0$ there is a polynomial p such that $|p - f| < \varepsilon$ on K. We do not use Mergelyan's theorem on uniform approximation by polynomials, but we state one version of it for comparison (see [7], Chapter I, § 4).

MERGELYAN'S THEOREM. *Let K be a compact set and let $K^\wedge$ be the complement of the unbounded component of the complement of K. Then a function f can be approximated uniformly on K by a sequence of polynomials if and only if f has an extension that is continuous on $K^\wedge$ and analytic in the interior of $K^\wedge$.*

A slight modification of our main theorem can be stated in a similar form.

THEOREM. *Let G be a bounded open set and let G^* be the complement of the closure of the unbounded component of the complement of the closure of G. Then a function f can be approximated pointwise on G by a uniformly bounded (on G) sequence of polynomials if and only if f has a bounded analytic continuation throughout G^*.*

One might think that our theorem describes the closure of the set of polynomials in some reasonable topology. There are locally convex topologies on $B_H(G)$ (the bounded holomorphic functions on G) in which a sequence is convergent if and only if it converges boundedly, and our theorem implies that in such a topology the closure of the set of polynomials coincides with the closure of $B_H(G^*:G)$ (those bounded functions f in G that have bounded analytic extensions throughout G^*). However, we do not know any simple concrete description of the closure of $B_H(G^*:G)$. In many cases $B_H(G^*:G)$ is already closed, but this is not always the case (see § 4); it seems to depend in part on questions about the harmonic measure of subsets of the boundary. In short, the best formulation of our theorem appears to be in terms of sequences rather than of topological closure.

The proof of the theorem is long. We sketch here an argument due to Farrell [4], for the case where G^* is connected. The proof for the general case requires additional techniques. $P(G)$ denotes the set of all those f in G that are bounded limits of polynomials in G. Thus $P(G) \subset B_H(G)$. We wish to show that $P(G) = B_H(G^*:G)$.

First, $P(G) \subset B_H(G^*:G)$, since if $p_n \to f$ then the p_n are uniformly bounded on the outer boundary of G, hence on the boundary of G^*, and consequently the p_n are uniformly bounded in G^*. Hence, some subsequence converges on G^* to a function which must be an extension of f. In the other direction, there exist simply connected regions G_n, with $G^* \subset G_n \subset G_n^- \subset G_{n-1}$ $(n=2,3,\ldots)$ that "squeeze down" onto G^* in the sense that G^* is the largest connected open set that contains G and is contained in all the G_n. The sets G_n can be constructed metrically, or they can also be obtained as the inside of a sequence of equipotential curves of the equilibrium potential on G. By a theorem of Carathéodory, if Φ_n is the normalized mapping function of G_n onto the unit disc, then $\Phi_n \to \Phi$, where Φ is the normalized mapping function of G^*. Given f in $B_H(G^*)$, the functions $f_n(z) = f(\Phi^{-1}(\Phi_n(z)))$ converge boundedly on G^* to f. By Runge's theorem, since G^{*-} is a compact subset of G_n, f_n can be approximated on G^* by a polynomial p_n with a uniform error at most $1/n$, and hence the p_n converge boundedly to f in G^*.

2. Topological preliminaries

Throughout this paper G will be a bounded open set in the plane and H will be the unbounded component of the complement of the closure of G. For any set A in the plane we use the following notations.

∂A = boundary of A.

A^- = closure of A.

DEFINITION 2.1. G^* is the complement of H^-.

Clearly $G \subset G^*$; we shall see that $(G^*)^* = G^*$.

DEFINITION 2.2. If $\partial G = \partial H$, then G is called a *Carathéodory set.* If in addition G is connected, then it is called a *Carathéodory domain.*

Corollary 2.8 says that G^* is always a Carathéodory set. It can be shown that G is a Carathéodory set if and only if G is the union of some of the components of G^*.

We state the following facts without proof.

2.3. If B is any plane set and C is its complement, then $\partial B = \partial C$.

2.4. If B is open and A is the union of some of the components of B, then $\partial A \subset \partial B$.

2.5. $\partial(A^-) \subset \partial A$ for all sets A.

LEMMA 2.6. $\partial G^* = \partial H^- = \partial H \subset \partial G$.

Proof. The first equality is 2.3. From 2.3–5 we have

$$\partial H^- \subset \partial H \subset \partial(\text{compl } G^-) = \partial G^- \subset \partial G.$$

Thus it only remains to prove that $\partial H \subset \partial(H^-)$.

Let J be the complement of G^-. Since $\partial J = J^- \cap G^-$, every neighborhood of every boundary point of J contains points of G.

H is a component of J and so $\partial H \subset \partial J$. Let $p \in \partial H$ and let U be any neighborhood of p. Then U contains points of G. Thus every neighborhood of p contains points of H^- and points of the complement of H^-, and so p is a boundary point of H^-. Q.E.D.

LEMMA 2.7. *H is the complement of $(G^*)^-$.*

Proof. Clearly $G^* \subset \text{compl } H$, but this is a closed set and so $(G^*)^- \subset \text{compl } H$.

In the other direction, if p is not in H, then either p is not in H^-, in which case $p \in G^*$, or $p \in \partial H$. In this case, from Lemma 2.6 we know that every neighborhood of p contains points of G^*, and thus p is in the closure of G^*. In short, compl $H \subset (G^*)^-$. Q.E.D.

COROLLARY 2.8. *G^* is a Carathéodory set and $(G^*)^* = G^*$.*

This is immediate from the two preceding lemmas and the definition.

2.9. Of the many equivalent definitions of simple connectedness we shall use the following: an open set, connected or not, will be said to be *simply connected* if its complement (on the Riemann sphere) is connected. A bounded disconnected open set is simply connected if each of its components is simply connected.

We require three more topological facts, the first two of which we state without proof.

2.10. If A and B are connected and if $A \cap B^- \neq \emptyset$, then $A \cup B$ is connected.

2.11. If $B_1 \subset B_2 \subset \ldots$ and if each B_n is connected, then the union of all the sets B_n is also connected.

2.12. If $A = A_0 \cup A_1 \cup \ldots$ where each set A_n is connected and $A_0 \cap A_n^- \neq \emptyset$ for all n, then A is connected.

Proof. Let $B_n = A_0 \cup A_1 \cup \ldots \cup A_n$. It follows inductively from 2.10 that each B_n is connected, and the result now follows from 2.11.

LEMMA 2.13. *Each component of G^* is simply connected.*

Proof. Let B be a component of G^*. We must show that the complement of B is connected. Indeed, the complement of B consists of the union of H^- and all the other components of G^*. By 2.11 it will be sufficient to show that the closure of each component of G^* meets H^-.

Let C be a component of G^*. By 2.3 and 2.4, $\partial C \subset \partial G^* = \partial H^- \subset H^-$, and the result follows.

Carathéodory kernels. We recall a special case of Carathéodory's theorem on the convergence of regions and mapping functions (see [2], Chapter 5, pp. 120–123).

DEFINITION 2.14. Let $\{D_n\}$ be a sequence of bounded, simply connected regions and let D be a region with $D \subset D_{n+1} \subset D_n$ (all n). Then $\ker[D_n : D]$ denotes the union of all the connected open sets that contain D and are contained in $\bigcap D_n$.

It is easy to see that $\ker[D_n : D]$ is connected and simply connected.

THEOREM 2.15 (Carathéodory). *Let $B = \ker[D_n : D]$ and let $z_0 \in D$. If Φ_n are the normalized mapping functions of the domains D_n onto the unit disc, then $\Phi_n \to \Phi$ uniformly on compact subsets of B, where Φ is the normalized mapping function of B onto the unit disc. Consequently,*

$$\Phi^{-1}(\Phi_n(z)) \to z$$

uniformly on compact subsets of B.

(By normalized mapping functions, we mean that z_0 is taken into zero, and the derivative at z_0 is positive.)

Our main topological result is the construction, in case G is a Carathéodory set, of suitable open sets G_n that "squeeze down" on G. The construction given here is similar to that given by Markushevitch for the case where G is a Carathéodory region (see [6], Chapter V, § 4.2, 4.7).

According to a private communication from Morton Brown, it is possible to choose the numbers ε_n of our construction in such a way that each G_n has only finitely many components, each of which is bounded by a Jordan curve. In the appendix, we give a different construction, using potential theory, in which the boundary of each component is actually an analytic Jordan curve.

THEOREM 2.16. *Let the bounded open set G be a Carathéodory set. Then there exists a sequence of bounded, simply connected open sets G_n with the following properties:*

(i) $G^- \subset G_n \subset G_n^- \subset G_{n-1}$ $(n=2,3,\ldots)$;

(ii) *if B is any component of G and if B_n is the component of G_n containing B, then*

$$B^- \subset B_n \subset B_n^- \subset B_{n-1} \quad (n=2,3,\ldots)$$

and
$$B = \ker[B_n : B].$$

The converse to this theorem is true but will not be proved here: if the sets G_n exist with the properties (i) and (ii), then G is a Carathéodory set.

Proof. Let $F_n = \{z \mid \operatorname{dist}(z, G) < \varepsilon_n\}$ $(n=1,2,\ldots)$, where ε_n decreases to 0. Then F_n is open, and

$$G^- \subset F_n \subset F_n^- \subset F_{n-1} \quad (n=2,3,\ldots).$$

2.17. Let J_n be the complement of F_n^-. Then $J_n \subset J_n^- \subset J_{n+1}$. Indeed, $F_{n+1}^- \subset F_n \subset F_n^-$ and so $J_n \subset \operatorname{compl} F_n \subset J_{n+1}$. But $\operatorname{compl} F_n$ is closed.

2.18. Let H_n be the unbounded component of J_n. Then H_n is open and connected and $H_n \subset H_n^- \subset H_{n+1} \subset H$.

Indeed, H_n^- is a connected subset of J_{n+1} and therefore lies in one component, which must be H_{n+1}.

2.19. Let G_n be the complement of H_n^-. Then G_n is open and

$$G^- \subset G_n \subset G_n^- \subset G_{n-1}$$

just as in 2.17. Also, each component of G_n is simply connected by Lemma 2.13.

Let B be a component of G and let B_n be the component of G_n that contains B.

2.20. $B^- \subset B_n \subset B_n^- \subset B_{n-1}$. (The proof is similar to 2.18.)

2.21. Let $C = \ker[B_n : B]$. Then $B = C$.

Indeed, clearly $B \subset C$. If $B \neq C$, then, since C is connected, there is a point $p \in \partial B \cap C$. But G is a Carathéodory set and thus by 2.4 $\partial B \subset \partial G \subset \partial H$. Since C is open, some neighborhood of p lies in C. But every neighborhood of p contains points of H, and so there is a point $q \in C \cap H$. Let Γ be an arc in H joining q to the point $z = \infty$, and let d be the distance from Γ to G^-. When $\varepsilon_n < d$ holds, this arc is disjoint from F_n^- and therefore lies in H_n. But this is impossible, for if $q \in H_n$ then q cannot lie in B_n and hence cannot be in C. This completes the proof of the theorem.

DEFINITION 2.22. Let G be a Carathéodory set and let B be a component of G. By the *cluster at B*, denoted by $K(B)$, we mean the union of all those components of G that are contained in the component of G^- that contains B.

2.23. With the hypotheses and notations of Theorem 2.16 and Definition 2.22 we have

$$K(B) \subset B_n \quad (n=1,2,\ldots).$$

Here B_n is the component of G_n containing B.

Indeed, $G^- \subset G_n$ and hence each component of G^- is contained in some component of G_n.

3. The main theorem

We come now to the proof of the main theorem stated in the introduction. The second half of the theorem is easy to prove.

LEMMA 3.1. *Let f be analytic in G. If there is a sequence of polynomials $\{p_n\}$, uniformly bounded in G and converging to f at each point of G, then there is a bounded analytic function F in G^*, agreeing with f in G.*

Proof. Since $|p_n| \leqslant M$ in G we have $|p_n| \leqslant M$ on ∂G; hence by Lemma 2.6 and the maximum modulus theorem, this holds throughout G^*. Therefore some subsequence of $\{p_n\}$ converges throughout G^* to an analytic function F, which furnishes the desired extension of f. Q.E.D.

The other half of the theorem states that every bounded analytic function in G^* is the bounded limit of polynomials (with the same bound). Since G^* is a Carathéodory set (Corollary 2.8) it will be sufficient to prove the following result.

THEOREM 3.2. *Let the bounded open set G be a Carathéodory set. Let f be analytic in G with $|f| \leqslant 1$ there. Then there is a sequence of polynomials, uniformly bounded by 1 in G, converging to f at each point of G.*

The proof will require a series of lemmas; we use the notations of § 2.

LEMMA 3.3. *Let E be a finite subset of G and let B be a component of G. Assume that $f(z)=1$ in all the other components of G. Let $\varepsilon>0$ be given. Then there is a polynomial $p(z)$ such that*

(i) $|f(z)-p(z)|<\varepsilon \quad (z\in E)$,

(ii) $|p(z)|\leqslant 1 \qquad (z\in G)$.

The proof of this lemma will be given later; we first show how the lemma can be used to prove the theorem.

Let $C_1, C_2, \ldots$ be an enumeration of all the components of G. If f is any bounded analytic function in G ($|f|\leqslant 1$), then f_n will be defined by

$$f_n(z) = f(z) \quad (z \in C_k;\ k = 1, 2, \ldots, n),$$
$$f_n(z) = 1 \quad (z \in C_k;\ k > n).$$

Let $z_1, z_2, \ldots$ be a countable dense subset of G and let $E_n = \{z_1, \ldots, z_n\}$.

LEMMA 3.4. *Let n be a given positive integer and let $\varepsilon > 0$ be given. Then there is a polynomial $p(z)$ such that*

(i) $|f_n(z) - p(z)| < \varepsilon \quad (z \in E_n)$,
(ii) $|p(z)| \leqslant 1 \quad (z \in G)$.

Proof. Let $g_i(z) = f(z)$ in C_i and let $g_i(z) = 1$ in all the other components of G. By the previous lemma, for each $i \leqslant n$ there is a polynomial p_i such that

$$|g_i(z) - p_i(z)| < \varepsilon/n \quad (z \in E_n),$$
$$|p_i(z)| \leqslant 1 \quad (z \in G).$$

Let $p = p_1 p_2 \ldots p_n$. Clearly p is a polynomial bounded by 1, and on E_n we have

$$|f_n - p| \leqslant |g_1 \ldots g_n - p_1 g_2 \ldots g_n| + |p_1 g_2 \ldots g_n - p_1 p_2 \ldots g_n| + \ldots + |p_1 \ldots p_{n-1} g_n - p| < \varepsilon/n + \ldots + \varepsilon/n = \varepsilon.$$

3.5. The theorem follows easily from this lemma. Indeed, choose $\varepsilon_n \to 0$. By the previous lemma there is a sequence of polynomials $\{p_n\}$, uniformly bounded by 1 in G, with $|f_n - p_n| < \varepsilon_n$ on E_n. But then p_n converges to f on the dense subset $\{z_n\}$ and hence at all points of G by the Stieltjes-Osgood theorem ([8], Chapter II, § 7).

Thus it remains to prove Lemma 3.3. To do this it will be sufficient to prove the following lemma.

LEMMA 3.6. *Let E be a finite subset of G, let $\varepsilon > 0$ be given, let B be a component of G and assume that $f = 1$ in all the other components. Then there is a simply connected open set Q containing G^-, and an analytic function g, $|g| \leqslant 1$ in Q, such that $|f - g| < \varepsilon$ on E.*

Indeed, suppose that this lemma has been established. Since G^- is a compact subset of the simply connected set Q, we may apply Runge's theorem to obtain a polynomial p such that $|p| \leqslant 1$ and $|p - g| < \varepsilon$ on G^-, and hence $|p - f| < 2\varepsilon$ on E.

The proof of Lemma 3.6 will be given in several steps.

3.7. From Theorem 2.16 we have a sequence of sets $\{G_n\}$ "squeezing down" on G. Let us fix an integer n for the moment. We shall see later that for a suitable choice of n, the set G_n will serve as the set Q called for in Lemma 3.6.

Let B_n be the component of G_n containing B. We wish to define a function g in all of G_n as called for in the lemma. We begin by putting $g = 1$ in all the components of G_n other than B_n. In Section 3.14 we shall show that if n is large enough, then g can be defined in B_n so as to satisfy the conditions of the lemma.

3.8. Let z_0 be a point in B, and let Φ_n and Φ be the normalized mapping functions from B_n and B, respectively, onto the unit disc D ($|w| < 1$). As $n \to \infty$ we have, by Theorems 2.15 and 2.16,

$$\Phi_n(z) \to \Phi(z) \quad (z \in B).$$

3.9. As in 2.22, the cluster at B, denoted by $K(B)$, means the union of all those components of G that are contained in the component of G^- that contains B. Let $B, A_1, A_2, \ldots$ be an enumeration of these components. Since the mapping functions $\{\Phi_n\}$ are uniformly bounded on $K(B)$, some subsequence, which we continue to denote by $\{\Phi_n\}$, will converge on all of $K(B)$ to a function $s(z)$. We already know that $s(z) = \Phi(z)$ on B.

3.10. In each of the components $A_1, A_2, \ldots$ $s(z)$ is a constant of modulus 1:

$$s(z) \equiv \zeta_i \quad (z \in A_i), \ |\zeta_i| = 1.$$

By the maximum modulus theorem, it is sufficient to show that in each component A_i there is at least one point where $|s| = 1$. Actually we shall show that $|s(z)| \equiv 1$ in each A_i. Indeed, assume that at some point z' in one of the components we had $\Phi_n(z') \to w'$ and $|w'| < 1$. By 3.8, there is a point z'' in B at which $\Phi_n(z'') \to w'$. But by Hurwitz' theorem, this would mean that for all sufficiently large n the function Φ_n would take the value w' in a neighborhood of z' and also in a neighborhood of z'', contradicting the fact that Φ_n is one-to-one in B_n.

Next, we require some results on functions analytic in the unit disc.

LEMMA 3.11. *Given $\varepsilon > 0$, $\varrho < 1$, and a finite set of points $\{\zeta_i\}$, $|\zeta_i| = 1$, there is a function $h(w)$, continuous in $|w| \leqslant 1$ and analytic in the open disc, such that $|h(w)| \leqslant 1$ ($|w| \leqslant 1$) and*

$$|h(w) - 1| < \varepsilon \quad (|w| \leqslant \varrho),$$
$$h(\zeta_i) = 0 \qquad (\textit{all } i).$$

Proof. It is sufficient to prove this result in case there is just a single point ζ_1 on the boundary; the general case follows by forming a product. We take $\zeta_1 = -1$.

Let $g(w) = (1 + w)/2$, and let

$$h_r(w) = \frac{w + r}{1 + rw} \quad (0 < r < 1).$$

Thus $h_r(-1) = -1$, and for r sufficiently close to 1 we have $h_r(w)$ close to 1 uniformly on $|w| \leq \varrho$. Fix r sufficiently close to 1 and for this r let $h(w) = g(h_r(w))$. Since g is uniformly continuous and $g(1) = 1$ and $g(-1) = 0$, this h will be the required function.

LEMMA 3.12. *Given a finite set of points ζ_i, $|\zeta_i| = 1$ $(i = 1, \ldots, m)$, and disjoint neighborhoods U_i of these points, and $\varepsilon > 0$, there are neighborhoods V_i of the points ζ_i with $V_i \subset U_i$, and there is a polynomial p such that*

(a) $|p(w)| \leq 1 \qquad (|w| \leq 1)$,
(b) $|p(w)| < \varepsilon \qquad (|w| \leq 1, w \notin U_1 \cup U_2 \cup \ldots \cup U_m)$
(c) $|p(w) - 1| < \varepsilon \quad (|w| \leq 1, w \in V_1 \cup V_2 \cup \ldots \cup V_m)$.

Proof. It will be sufficient to take the points one at a time: for each i we find a polynomial $p_i(z)$ that is bounded by one, takes the value 1 at ζ_i and is small everywhere in the unit disc outside of U_i. The function p is then obtained by adding the p_i and dividing by a suitable constant to achieve the estimate (a).

If k is sufficiently large, then the functions

$$p_i(w) = \frac{(\zeta_i + w)^k}{2} \quad (i = 1, \ldots, m)$$

will satisfy our requirements. Q.E.D.

LEMMA 3.13. *Given a finite subset E' of the open disc $|w| < 1$, and a finite number of boundary points ζ_i, and a function $F(w)$ analytic in $|w| < 1$ with $|F| \leq 1$, and given $\varepsilon > 0$, then there is a function $q(w)$ analytic in the open disc with $|q| \leq 1$, and there are neighborhoods V_i of the points ζ_i such that*

(i) $|q - F| < \varepsilon$ *on* E',
(ii) $|q(w) - 1| < \varepsilon$ *for* $w \in V_1 \cup \ldots \cup V_m$.

Proof. (We wish to thank Norman Hamilton for suggesting this line of proof.) Choose $\varrho < 1$ so that $E' \subset (|w| \leq \varrho)$. Let h be the function of Lemma 3.11, using a smaller ε, and let U_i be neighborhoods of the points ζ_i in which h is small. Now let p be the function of Lemma 3.12, using a smaller ε.

Put $q_1 = hF + p$. This function is close to F on $(|w| \leq \varrho)$, and is close to 1 near the points ζ_i. It will be bounded in the whole disc by a number only slightly larger than 1. Dividing by this bound we obtain the desired q. Q.E.D.

3.14. We are now able to complete the proof of Lemma 3.6. We use the notations and results of the preceding sections. In Section 3.7 we showed that the chief problem was to define the function g in the component B_n.

The finite set E meets only a finite number of the components A_i in the cluster $K(B)$: say that it meets $A_1, \ldots, A_m$. Let $\zeta_1, \ldots, \zeta_m$ be the points on the boundary of the unit disc that were described in 3.10, and let $E' = \Phi(E \cap B)$. Then E' is a finite set of points in the open unit disc.

Let $F(w) = f(\Phi^{-1}(w))$. Then F is analytic and bounded by 1 in the disc. We may apply Lemma 3.13 to obtain a function $q(w)$ that is close to F on E' and close to 1 near the points ζ_i $(i = 1, \ldots, m)$.

Fix n sufficiently large and let

$$g(z) = q(\Phi_n(z)) \quad (z \in B_n).$$

Since Φ_n is close to Φ on compact subsets of B, and q is close to F on E', g will be close to f on $E \cap B$.

Also, Φ_n is close to ζ_i on compact subsets of A_i $(i = 1, \ldots, m)$, and q is close to 1 in a neighborhood of ζ_i. Thus g is close to 1 on $E \cap A_i$. This completes the proof of Lemma 3.6; therefore Theorem 3.2 and the main theorem are established.

3.15. We turn now to the proof of the corollary stated in the introduction. We have already shown in Theorem 3.2 that if G is a Carathéodory set, then every bounded analytic function in G is the bounded limit of a sequence of polynomials.

In the other direction, let G be a bounded simply connected open set that is not a Carathéodory set, and let z_0 be a boundary point of G that is not a boundary point of H. We claim that z_0 is in G^*. Indeed, if not, then $z_0 \in H^-$. But $H^- = H \cup \partial H$. By assumption, z_0 is not in ∂H, and so $z_0 \in H$. But this is impossible since $z_0 \in G^-$, the complement of H.

Since G is simply connected, there is a single-valued function f such that $f^2 = z - z_0$ in G. This function is not the bounded limit of polynomials in G. Indeed, if it were, then by the easy half of the main theorem, f could be extended to be analytic in G^*; in particular, f would be analytic at z_0, which is impossible.

4. Further remarks

The correct formulation of the main theorem is in terms of sequences, and not of topological closure. The next result supports this assertion.

THEOREM 4.1. *There exists a bounded, connected, simply connected open set G and functions $f, f_1, f_2, \ldots$ analytic and uniformly bounded in G such that*

(i) *each f_n is the bounded limit of a sequence of polynomials,*
(ii) *f is the bounded limit of the sequence f_n,*
(iii) *f is not the bounded limit of any sequence of polynomials.*

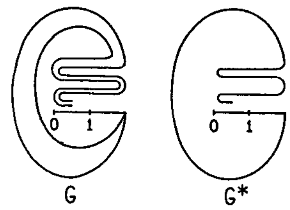

Fig. 2.

Proof. The region G and the associated G^* are shown in Fig. 2.

Let
$$f(z) = \frac{1}{1+\sqrt{z}} \quad (z \in G),$$
where the square root is defined in the plane slit along the positive real axis, with the branch chosen so that $(-1)^{1/2} = i$. Then f is bounded in G, since in G the point $z=1$ can only be approached from above, so that as z approaches 1 within G, $f(z) \to 1/2$. Also, f has an analytic continuation throughout G^*, but the continuation is unbounded since now the point $z=1$ can be approached from below. Since G^* is connected, the continuation of f is unique, and so f does not have a bounded continuation. By the main theorem, f is not the bounded limit of any sequence of polynomials in G.

Let z_n be a sequence of points in the upper half plane that approach 1. Let
$$f_n(z) = \frac{1}{z_n + \sqrt{z}},$$
where the same branch of the square root is chosen as before. These functions f_n are uniformly bounded in G, and each has a bounded continuation throughout G^* (the continuations are not uniformly bounded in G^*). Indeed, $z^{\frac{1}{2}}$ lies in the upper half plane for all z in G^*, whereas $-z_n$ is in the lower half plane. By the main theorem, each f_n is the bounded limit of a sequence of polynomials. Finally, the f_n converge boundedly to f in G. Q.E.D.

Instead of bounded approximation by polynomials, one could consider bounded approximation by rational functions with assigned poles. This seems to lead to new difficulties.

Consider, for example, the case of two poles, at 0 and ∞, and let G be an open set such that neither 0 nor ∞ is in G^-. If $R(z)$ is a rational function with poles at 0 and ∞, then $R(z) = p(z) + q(1/z)$ where p and q are polynomials. By analogy with the polynomial case, one might conjecture that a bounded f in G is the bounded limit of a sequence of such rational functions if and only if f has a bounded continuation throughout a larger open set G^*, where G^* is now defined as follows. Let J be the complement of G^-. Let $H = H_1 \cup H_2$

where H_1, H_2 are the components of J containing 0 and ∞, respectively. Then G^* is the complement of H^-.

The simplest example is the annulus: $G=\{z: r_1<|z|<r_2\}$. This presents no difficulty, since if f is bounded and analytic in G then, by the Laurent expansion, $f=f_1+f_2$ where f_1 is bounded and analytic for $|z|<r_2$, and f_2 is bounded and analytic for $|z|>r_1$.

This method does not work in general. For example, let G be a lune: $G=G_1\cap G_2$, where G_1 is the disc $|z|<1$ and G_2 is the set $|z-\frac{1}{4}|>\frac{3}{4}$. What we show is that the function $f=\log z$ (defined in the plane slit along the positive real axis, with $\log(-1)=i\pi$) admits no decomposition $f=f_1+f_2$, with f_1 bounded in G_1 and f_2 bounded in G_2.

Assume such a decomposition exists. Then f_1 has an analytic continuation (which we still call f_1) throughout the complement of the ray $L=[1,\infty)$, and f_1 is bounded except near $z=\infty$. Indeed, f_1 is already bounded and analytic in $|z|<1$, by assumption. Also $f_1=f-f_2$ in G, and both f and f_2 are analytic in G_2-L, and bounded except near $z=\infty$. Note that as z crosses L from below, f_1 jumps by $2\pi i$.

Now let $g(z)=f_1(z)-\log(z-1)$, where the logarithm is a translation of the one previously chosen and thus is defined on the complement of L. Also, g is continuous across L, except possibly at $z=1$, since both f_1 and the logarithm jump by $2\pi i$ as z crosses L from below. Thus g is analytic except possibly at $z=1$. Near $z=1$ we have $|g|\leqslant c+|\log(z-1)|$, and thus g has a removable singularity at $z=1$. But $f_1=g+\log(z-1)$ and hence f_1 is unbounded as $z\to 1$ in the complement of L, which is impossible.

5. Appendix

We present here a different proof of Theorem 2.16 based on potential theory. It has the advantage that the sets G constructed here have extremely smooth and simple boundaries.

We first state a number of results from classical potential theory that will be needed; definitions and proofs may be found in [9, Chapter III]. Our exposition is self-contained, in the sense that we use only the properties and results stated below. As before, we assume that G is a bounded open set and that H is the unbounded component of the complement of G^-.

THEOREM 5.1. *There exists a positive Borel measure μ, of total mass 1, supported in ∂H, and a positive number V such that if*

$$P(z)=-\int\log|z-w|\,d\mu(w),$$

then

(a) *$P(z) \leqslant V$ everywhere;*

(b) *$P(z) = V$ throughout G^*;*

(c) *P is harmonic in H and $P(z) < V$ there;*

(d) *P is superharmonic everywhere; in particular, P is lower semicontinuous;*

(e) *P is bounded on each compact set;*

(f) $\lim P(z) = -\infty \quad (z \to \infty)$*;*

(g) *if $p \in \partial H$ is a regular point for the Dirichlet problem, then* $\lim P(z) = V \quad (z \to p)$*;*

(h) *P is continuous except on the irregular points of ∂H;*

(i) *the set I of irregular points of ∂H is an F_σ set of capacity zero.*

If G has only a finite number of components then ∂H will be a finite union of continua and by known results there are no irregular points. But if G is, for example, a union of disjoint discs with centers at $1/n$ and radii r_n $(n = 1, 2, \ldots)$, then by Wiener's criterion [9, Theorem III.62], if r_n approaches zero rapidly, then the point $z = 0$ will be an irregular point. We handle the irregular points by introducing an Evans' function $E(z)$ for I. The next result is a simple extension of [9, Theorem III.27].

THEOREM 5.2. *Given a bounded F_σ set I of capacity zero, there exists a positive Borel measure m, supported in I^-, with total mass 1, such that if*

$$E(z) = -\int \log |z - w| \, dm(w),$$

then

(a) *$E(z)$ is finite and harmonic for $z \notin I^-$, and $E(z) = +\infty$ for $z \in I^-$;*

(b) *$E(z)$ is superharmonic, and in particular is lower semicontinuous;*

(c) *$E(z)$ is everywhere continuous in the extended sense; in particular,* $\lim E(z) = +\infty$ *as z approaches any point in I^-;*

(d) $\lim E(z) = -\infty \quad (z \to \infty)$.

The remaining results that we need from potential theory are the maximum and strong maximum principles.

THEOREM 5.3. *If F is a superharmonic function in an open set W, if F is bounded below in W, if D is a bounded open set whose closure is contained in W, and if $F \geqslant t$ on ∂D, then $F \geqslant t$ throughout D.*

In addition, if D is connected and if $F(z_0) = t$ for some point z_0 in D, then $F(z) \equiv t$ throughout D.

We are now ready to prove Theorem 2.16. Recall the hypotheses: G is a bounded open set and $\partial G = \partial H$. The theorem asserts the existence of open sets G_n containing G and having certain properties. Roughly speaking, the sets G_n are the insides of the level lines of the potential function $P(z)$ of Theorem 5.1. However, these level lines may touch ∂H at an irregular point, and to avoid this we use the Evans' function of Theorem 5.2.

5.4. Without loss of generality, we may assume that G^- is contained in the disc $|z| < \frac{1}{2}$. A simple estimate shows that $E(z) > 0$ in this disc.

5.5. For $0 < t < V$ let

$$F(z,t) = P(z) + (V-t)\,E(z),$$

$$\Gamma_t = \{z \mid F(z,t) = t\}.$$

Γ_t is a closed set since $P(z)$ is continuous away from I, $E(z)$ is continuous, and $E(z) \to \infty$ in I^-.

5.6. There is a number t_1 such that for $t \geqslant t_1$ the sets Γ_t are all contained in the disc $|z| < \frac{1}{2}$.

Indeed, by 5.1 (c) and 5.2 (a), there is an $\varepsilon > 0$ and an $M > 1$ such that if $|z| \geqslant \frac{1}{2}$ then $P(z) < V - \varepsilon$ and $E(z) \leqslant M$. For such z and all t we have

$$F(z,t) < V - \varepsilon + (V-t)\,M.$$

Let $t_1 = V - \varepsilon/2M$. Then for $t \geqslant t_1$ we have

$$F(z,t) < V - \varepsilon/2 < V - \varepsilon/2M \leqslant t$$

and so z is not in Γ_t.

5.7. Choose a sequence $t_1 < t_2 < \ldots \to V$, where t_1 is the number from 5.6. Let $\Gamma_j = \Gamma_{t_j}$. Then Γ_i and Γ_j are disjoint for $i \neq j$.

Indeed, if $z \in \Gamma_i \cap \Gamma_j$ then $(t_i - t_j)\,E(z) = t_j - t_i$, which is impossible since $E(z) > 0$ by 5.4.

DEFINITION 5.8. For each positive integer n let G_n be the union of all the bounded components of the complement of Γ_n.

We shall show that these sets satisfy all the conditions of Theorem 2.16. We require two preliminary results.

5.9. $F(z,t_n) > t_n$ in G_n and $F(z,t_n) < t_n$ in the unbounded component of the complement of Γ_n.

Proof. The first statement follows from 5.3. The second inequality is true for large z since the potential functions $P(z)$ and $E(z)$ tend to $-\infty$. If there were a point in the unbounded component at which $F > t_n$, then by connectedness, there would be a point where $F = t_n$, which is impossible.

LEMMA 5.10. *The sets G_n are simply connected.*

Proof. Fix n and let C be a component of G_n. We must show that C is simply connected. We shall consider the set C^* (the complement of the closure of the unbounded component of the complement of the closure of C). By Lemma 2.13 each component of C^* is simply connected, so it will be sufficient to show that C is one of the components of C^*.

Since $C \subset C^*$, C will be contained in one of the components, A, of C^*. We must show that C coincides with A. If not, then since A is connected, some boundary point p of C lies in A.

From 2.3 and 2.4 we have

$$\partial C \subset \partial(\operatorname{compl} \Gamma_n) = \partial \Gamma_n.$$

Hence $F(z,t_n)=t_n$ on ∂C. In particular, $F(p,t_n)=t_n$. But by 5.9, $F>t_n$ throughout A, which is a contradiction.

5.11. $G \subset G_n$ for all n.

Indeed, by 5.1(b) $P(z)=V$ in G and hence $F(z,t_n) \geq V > t_n$ for all n and all z in G. Thus G and Γ_n are disjoint, and by 5.9, G must be contained in G_n.

5.12. $G_n^- \subset G_{n-1}$ $(n \geq 2)$.

Proof. $F(z,t)$ decreases monotonically as t increases; hence (by 5.9)

$$F(z,t_{n-1}) > F(z,t_n) \geq t_n > t_{n-1} \quad (z \in G_n^-).$$

Thus by 5.9, $G_n^- \subset G_{n-1}$.

5.13. Let B be a component of G, and let B_n be the component of G_n that contains B. Then $B = \ker[B_n : B]$.

Proof. Let $Q = \ker[B_n : B]$. If B were a proper subset of Q, then Q would contain a boundary point p of B since Q is connected. Since

$$p \in \partial B \subset \partial G = \partial H,$$

every neighborhood of p contains points of H. In particular, Q must contain a point q of H.

But by 5.1(c), $P(q) < V$ in H and hence, for t_m sufficiently close to V,

$$F(q,t_m) < t_m.$$

Thus by 5.9, q is not contained in G_m, contradicting the fact that Q is contained in all the sets G_m. Q.E.D.

This completes the proof of Theorem 2.16. It can be shown that the boundary of each component of each G_n is an analytic Jordan curve since it is locally a level line of a harmonic function of non-vanishing gradient. Finally, each G_n has only finitely many components.

Otherwise, there would be a point p, any neighborhood of which intersects infinitely many components of G_n. It follows that $p \in \Gamma_n$. But by Theorem 5.3, each component of G_n intersects the support of $d\mu$, so that p is in the support of $d\mu$ and hence $p \in \partial H$. But $\partial H \cap \Gamma_n$ is empty, and we are done.

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