TEICHMÜLLER SPACES OF GROUPS OF THE SECOND KIND

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I. Introduction

1. Let U be the upper half plane. A normalized Fuchsian group G is a discontinuous group of conformal self-mappings of U with limit points at 0, 1, and ∞. All Fuchsian groups in this paper are normalized. G is of the first (second) kind if its limit set is dense (nowhere dense) on the real axis.

Let f be a normalized quasiconformal self-mapping of U. (Throughout this paper, a normalized mapping is one that leaves 0, 1, and ∞ fixed.) f is compatible with the group G if $f \circ A \circ f^{-1}$ is conformal for all A in G. The set of mappings compatible with G is denoted by $\Sigma(G)$.

Each f in $\Sigma(G)$ induces an isomorphism of G onto $f \circ G \circ f^{-1}$. The mappings f and g induce the same isomorphism if $f \circ A \circ f^{-1} = g \circ A \circ g^{-1}$ for all A in G. This is an equivalence relation on $\Sigma(G)$. The set of equivalence classes is denoted by $S(G)$.

It is easy to see that f and g are equivalent if and only if $f = g$ on the limit set of G. Hence, for groups of the first kind, $S(G)$ equals the space $T(G)$ defined in III. If G is of the second kind, however, $T(G)$ and $S(G)$ are unequal. Thus, $T(G)$ and $S(G)$ are different generalizations of the notion of Teichmüller space to groups of the second kind. Following the terminology of Bers in [4], we shall call $T(G)$ the Teichmüller space of G. Our purpose here is to study the space $S(G)$.

Bers [4] has recently proved that $T(G)$ always carries a complex analytic structure. By contrast, if G is of the second kind, the natural structure on $S(G)$ is real analytic. Indeed, the region of discontinuity D of G is symmetric about the real axis. If one represents

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D/G in the form U/H for some group H of the first kind, the symmetry of D induces a symmetry in H. Roughly speaking, one expects $S(G)$ to correspond to a symmetric part of $S(H)=T(H)$.

This idea is developed here in a precise way. In the following section we record the necessary facts about the universal Teichmüller space T. In II and III the symmetric parts of T and $T(G)$ are defined and shown to have real analytic structures. In IV the space $S(G)$ of a group of the second kind is mapped on the symmetric part of a suitable $T(H)$. This mapping induces a natural real analytic structure on $S(G)$.

2. Let T be the set of mappings $h: R \to R$ which are boundary values of normalized quasiconformal self-mappings of U. T is the universal Teichmüller space of Bers.

There is a natural map of the open unit ball M in $L_\infty(U)$ onto T. For each μ in M let f^μ be the unique normalized self-mapping of U which satisfies the Beltrami equation

$$f_{\bar{z}} = \mu f_z. \tag{1}$$

We map M onto T by sending μ to the boundary mapping of f^μ. T is given the quotient topology induced by the L_∞ topology on M. The right translations, of the form $h \to h \circ h_0$, are homeomorphisms of T.

We next associate to each μ in M a function ϕ^μ holomorphic in the lower half plane U^*. For each μ, let w^μ be the unique normalized quasiconformal mapping of the plane onto itself which is conformal in U^* and satisfies (1) in U. ϕ^μ is the Schwarzian derivative $\{w^\mu, z\}$ of w^μ in U^*. By Nehari [6], ϕ^μ belongs to the complex Banach space B of holomorphic functions ψ on U^* which satisfy

$$\|\psi\| = \sup |(z-z^*)^2 \psi(z)| < \infty.$$

It is easy to see that $\phi^\mu = \phi^\nu$ if and only if $f^\mu = f^\nu$ on R. A much deeper theorem of Bers [4] states that the mapping $\mu \to \phi^\mu$ is open and continuous. T may therefore be mapped homeomorphically on the image of M in B. We shall identify T with its image under this mapping. (Ahlfors [1] gave the first proof that T is an open subset of B. Formula (13) of [1] implies that the map $\mu \to \phi^\mu$ is open.)

II. The symmetric parts of T and B

3. The symmetric part of B is the real Banach space B' consisting of the ψ in B which are real on the y-axis. Let J be the reflection in the y-axis; that is, $Jz = -z^*$. Then $\psi \in B'$ if and only if $\psi \in B$ and $\psi(Jz) = \psi(z)^*$. By elementary properties of the Schwarzian derivative, $\{w, z\} \in B'$ if and only if $w \circ J \circ w^{-1}$ is the conjugate of a linear transformation in $w(U^*)$.

The symmetric part of T, denoted by T', is the set of boundary mappings h in T which are odd functions of x. We shall identify T' with its image in B.

THEOREM 1. $T' = T \cap B'$.

Proof. First, suppose $\phi^\mu \in T \cap B'$. Let $w = w^\mu$, $D = w(U)$, and $D^* = w(U^*)$. Let Q be the anticonformal involution of the plane which agrees with $w \circ J \circ w^{-1}$ in D^*. Evidently $Q \circ w = w \circ J$ on the real axis.

We must prove that $f = f^\mu$ is an odd function of x. Since $g = w \circ f^{-1}$ maps U conformally on D, the function $g^{-1} \circ Q \circ g$ is an anticonformal involution of U. Since g and Q leave 0 and ∞ fixed, $g^{-1} \circ Q \circ g = J$ in the closure of U. Therefore, f commutes with J on the real axis, as required.

Conversely, suppose $\phi^\nu \in T'$. Then f^ν is an odd function on the real axis. According to Ahlfors and Beurling ([2], formula (14)), there is a quasiconformal mapping f^μ of U on itself which agrees with f^ν on R and commutes with J in U.

Let $w = w^\mu$. Since $g = w \circ (f^\mu)^{-1}$ is conformal in U, $w \circ J \circ w^{-1} = g \circ J \circ g^{-1}$ is anticonformal in $w(U)$. Therefore $w \circ J \circ w^{-1}$ is anticonformal in the entire plane, and its conjugate is a linear transformation. Hence $\{w, z\} \in B'$, and the theorem is proved.

4. Let M' be the set of μ in M such that f^μ commutes with J. It is easy to prove:

THEOREM 2. *The image of M' under the map $\mu \to \phi^\mu$ is T'. Moreover, $\mu \in M'$ if and only if $\mu \in M$ and*

$$\mu(Jz) = \mu(z)^*. \tag{2}$$

Proof. We observed in the proof of Theorem 1 that each ϕ in T' has the form ϕ^μ where f^μ commutes with J. This proves the first part of the theorem. As for the rest, it is clear that each μ in M' satisfies (2). Conversely, if μ in M satisfies (2), then $f^\mu \circ J \circ (f^\mu)^{-1}$ is an anticonformal involution of U leaving 0 and ∞ fixed. Therefore f^μ commutes with J and $\mu \in M'$. This completes the proof.

III. $T(G)$ and its symmetric part

5. Let G be a Fuchsian group. We denote by $M(G)$ the set of μ in M such that f^μ is compatible with G. $\mu \in M(G)$ if and only if $\mu \in M$ and

$$\mu(Az) = \mu(z) A'(z)/A'(z)^* \text{ for all } A \text{ in } G. \tag{3}$$

The Teichmüller space $T(G)$ is the image of $M(G)$ under the natural map $\mu \to \phi^\mu$ of M onto T.

$B(G)$, the space of quadratic differentials, is the set of ψ in B such that $(\psi \circ A)(A')^2 = \psi$

for all A in G. If f^μ is compatible with G, then $w^\mu \circ A \circ (w^\mu)^{-1}$ is a linear transformation for every A in G. Therefore $T(G)$ is a subset of $B(G)$.

Let $M'(G) = M(G) \cap M'$. The symmetric part of $T(G)$, denoted by $T'(G)$, is the image of $M'(G)$ in T under the natural map. Since the image of M' is T', $T'(G)$ is contained in the real Banach space $B'(G) = B' \cap B(G)$. Our purpose is to prove:

THEOREM 3. *$T'(G)$ is an open subset of $B'(G)$.*

6. Let $\Delta(G) = B'(G) \cap T = B(G) \cap T'$. The function $\phi(z) = \{w, z\}$ in T' belongs to $\Delta(G)$ if and only if for each A in G, the restriction of $w \circ A \circ w^{-1}$ to $w(U^*)$ is a linear transformation.

LEMMA 1. *$\Delta(G)$ is open in $B'(G)$. $T'(G) \subset \Delta(G)$. If $\phi \in B'(G)$ and $\|\phi\| < 2$, then $\phi \in T'(G)$.*

Proof. Since T is open in B, $\Delta(G)$ is open in $B'(G)$. It is obvious that $T'(G) \subset \Delta(G)$. Finally, it is well-known ([1], pp. 297–299) that every ϕ in B with $\|\phi\| < 2$ has the form ϕ^μ for

$$\mu(z) = \tfrac{1}{2}(z - z^*)^2 \phi(z^*).$$

By (2) and (3), if $\phi \in B'(G)$, then $\mu \in M'(G)$ and $\phi \in T'(G)$ as required. This proves the lemma.

Now let ν be an arbitrary member of $M'(G)$ and let $\alpha : T \to T$ be the right translation of T which carries ϕ^ν to zero. We recall from I that α is a homeomorphism. Since ϕ^ν belongs to T', α maps T' on itself.

Let $G_1 = f^\nu \circ G \circ (f^\nu)^{-1}$. Since f^ν is compatible with G, G_1 is a Fuchsian group.

LEMMA 2. *$\alpha(T'(G)) = T'(G_1)$.*

Proof. For each ϕ^μ in T, $\alpha(\phi^\mu) = \phi^\lambda$ where λ is such that $f^\mu = f^\lambda \circ f^\nu$. Obviously, f^μ commutes with J if and only if f^λ does. Moreover, f^μ is compatible with G if and only if f^λ is compatible with G_1. This completes the proof.

LEMMA 3. *$\alpha(\Delta(G)) = \Delta(G_1)$.*

Proof. It is enough to show that α maps $\Delta(G)$ into $\Delta(G_1)$, for by the same token α^{-1} maps $\Delta(G_1)$ into $\Delta(G)$.

Let f be the quasiconformal extension of f^ν to the whole plane by $f(z^*) = f(z)^*$. Let ϕ^μ belong to $\Delta(G)$ and let $\phi^\lambda = \alpha(\phi^\mu)$ with λ as in Lemma 2. Since ϕ^λ belongs to T', it suffices to find for each $A_1 = f \circ A \circ f^{-1}$ in G_1 a linear transformation A^λ which agrees with $w^\lambda \circ A_1 \circ (w^\lambda)^{-1}$ in $w^\lambda(U^*)$.

Let A^μ be the linear transformation that agrees with $w^\mu \circ A \circ (w^\mu)^{-1}$ in $w^\mu(U^*)$. Let $g = w^\lambda \circ f \circ (w^\mu)^{-1}$. g is quasiconformal and maps $w^\mu(U)$ conformally on $w^\lambda(U)$. We define $A^\lambda = g \circ A^\mu \circ g^{-1}$. Then A^λ agrees with $w^\lambda \circ A_1 \circ (w^\lambda)^{-1}$ in $w^\lambda(U^*)$. Moreover, A^λ is conformal in $w^\lambda(U)$. Since a quasiconformal map which is conformal almost everywhere is conformal, A^λ is everywhere conformal, and the lemma is proved.

7. We can now prove the theorem. By Lemma 3, α maps $\Delta(G)$ homeomorphically on $\Delta(G_1)$. Let N be the set of ϕ in $B'(G_1)$ with $\|\phi\| < 2$. By Lemmas 1 and 2, $\alpha^{-1}(N)$ is contained in $T'(G)$. But N is open in $\Delta(G_1)$, so $\alpha^{-1}(N)$ is open in $\Delta(G)$ and hence in $B'(G)$. Therefore $T'(G)$ contains a neighborhood of ϕ^ν. Since ν was arbitrary in $M'(G)$, this completes the proof.

COROLLARY. *$T'(G)$ inherits a real analytic structure from $B'(G)$. The mapping α of $T'(G)$ on $T'(G_1)$ is real analytic.*

In fact, α is a holomorphic mapping of T on itself (see [4]).

IV. The real analytic structure of $S(G)$

8. Let G be a Fuchsian group of the second kind with the region of discontinuity D. We choose a holomorphic function $\varrho : U \to D$ which represents U as a regular covering surface of D and satisfies $\varrho(Jz) = \varrho(z)^*$. (By [3], p. 99, there must be an involution Q of U such that $\varrho(Qz) = \varrho(z)^*$. Replacing ϱ by $\varrho \circ A$ if necessary, we can put the real fixed points of Q at 0 and ∞, so that $Q = J$.)

Let H be the group of linear transformations $A : U \to U$ such that $\varrho \circ A = C \circ \varrho$ for some C in G. Let H_0 be the group of A such that $\varrho \circ A = \varrho$. Both H and H_0 are Fuchsian groups of the first kind. By [3], p. 99, for each C in G there exists A in H such that $\varrho \circ A = C \circ \varrho$.

The existence of a real analytic structure on $S(G)$ is a consequence of:

THEOREM 4. *The mapping ϱ induces a bijection ϱ_* between $S(G)$ and $T'(H)$.*

The proof is again preceded by several lemmas.

9. Let each μ in $M(G)$ be extended to D so that $\mu(z^*) = \mu(z)^*$. The function $\varrho \cdot \mu$ in M is defined by

$$(\varrho \cdot \mu)(z) = \mu(\varrho(z))\varrho'(z)^*/\varrho'(z). \tag{4}$$

We record the obvious

LEMMA 4. *The map $\mu \to \varrho \cdot \mu$ is a bijection from $M(G)$ to $M'(H)$.*

If f is compatible with G, we extend it to D by $f(z^*) = f(z)^*$. We denote by $\varrho^\#(f)$ the normalized self-mapping of U such that $f \circ \varrho \circ \varrho^\#(f)^{-1}$ is holomorphic. Evidently $\varrho^\#(f^\mu) = f^{\varrho \cdot \mu}$. Therefore, $\varrho^\#(f)$ is compatible with H and commutes with the mapping J. The map $\varrho^\#$ is injective.

LEMMA 5. *If $\varrho^\#(f)$ commutes with H, then f commutes with G, and $f \circ \varrho = \varrho \circ \varrho^\#(f)$.*

Proof. Let $g: D \to D$ be defined by $g \circ \varrho = \varrho \circ \varrho^{\#}(f)$. g is quasiconformal, and $g(z^*) = g(z)^*$. Moreover, g commutes with G. Therefore, g is a normalized self-mapping of U, and $g = f$. This completes the proof.

LEMMA 6. *If f commutes with G, then $\varrho^{\#}(f)$ commutes with H, and $f \circ \varrho = \varrho \circ \varrho^{\#}(f)$.*

Remark. For a geometric interpretation of Lemma 6 when G contains no elliptic transformations, see [5], Theorems 1 and 2.

Proof. Since f leaves every limit point of G fixed, it maps each component of $D \cap R$ onto itself. Hence, for each z in D the line segment joining z to $f(z)$ is in D, and f is homotopic to the identity.

By a familiar theorem ([3], p. 99), there exists $g: U \to U$ such that $f \circ \varrho = \varrho \circ g$ and g commutes with H_0. Since H_0 is of the first kind, g leaves every real x fixed. Therefore, g commutes with H, g is normalized, and $g = \varrho^{\#}(f)$. This completes the proof.

LEMMA 7. *f and g are equivalent if and only if $\varrho^{\#}(f)$ and $\varrho^{\#}(g)$ are equivalent.*

Proof. We recall from the introduction that f and g are equivalent if and only if $h = f^{-1} \circ g$ commutes with G. If h commutes with G, then $h \circ \varrho = \varrho \circ \varrho^{\#}(h)$. Therefore,

$$\varrho^{\#}(g) = \varrho^{\#}(f \circ h) = \varrho^{\#}(f) \circ \varrho^{\#}(h).$$

Since $\varrho^{\#}(h)$ commutes with H, $\varrho^{\#}(f)$ and $\varrho^{\#}(g)$ are equivalent. The converse is proved similarly.

10. The proof of Theorem 4 is immediate. Let ϱ_* map the equivalence class of f on the equivalence class of $\varrho^{\#}(f)$. By Lemma 7, ϱ_* is a one-to-one mapping into $T(H)$. By Lemma 4, the image is $T'(H)$, and the theorem is proved.

11. The real analytic structure of $T'(H)$ induces via ϱ_* a real analytic structure on $S(G)$ which we call the natural structure. We must show that this structure does not depend on the function ϱ.

We may replace ϱ by $\sigma = \varrho \circ A$, where A is a linear transformation of U onto itself such that $\varrho(AJz) = \varrho(Az)^*$. The map σ_* has the form $\theta \circ \varrho_*$, where

$$\theta(\phi^{\varrho \cdot \mu}) = \phi^{\sigma \cdot \mu}.$$

By (4),

$$(\sigma \cdot \mu)(z) = (\varrho \cdot \mu)(Az) A'(z)^* / A'(z).$$

Therefore, $\theta(\phi) = (\phi \circ A)(A')^2$, and θ is a norm-preserving automorphism of B. We conclude that σ_* induces the natural structure on $S(G)$.

12. Finally, suppose f_1 is compatible with G. The group $G_1 = f_1 \circ G \circ f_1^{-1}$ is a Fuchsian group of the second kind discontinuous on $D_1 = f_1(D)$. If f is compatible with G, then $f \circ f_1^{-1}$

is compatible with G_1. The mapping $f \to f \circ f_1^{-1}$ induces a natural map of $S(G)$ onto $S(G_1)$. It is important to prove that the natural map is analytic.

Let $\varrho^{\#}(f_1) = f^{\mu}$, and let $\sigma = f_1 \circ \varrho \circ (f^{\mu})^{-1}$. Then σ represents U as a regular covering surface of D_1, and $\sigma(Jz) = \sigma(z)^*$. Hence, σ induces a mapping σ_* from $S(G_1)$ into T'. Let α be the right translation of T' which carries ϕ^{μ} to zero. The natural map of $S(G)$ on $S(G_1)$ is given by $\sigma_*^{-1} \circ \alpha \circ \varrho_*$. Since α is real analytic, we have proved:

THEOREM 5. *$S(G)$ has a natural analytic structure such that the map $\varrho_* : S(G) \to T'(H)$ is analytic. This structure depends only on G. If $G_1 = f_1 \circ G \circ f_1^{-1}$, where f_1 is compatible with G, then the natural map from $S(G)$ to $S(G_1)$ is analytic.*

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