A Littlewood—Paley inequality for analytic measures

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Let $\mathbf{T}$ be the circle group and $\mathbf{Z}$ the additive group of integers; designate by $M(\mathbf{T})$ the customary space of Borel measures on $\mathbf{T}$, and, given $\mu \in M(\mathbf{T})$ and $n \in \mathbf{Z}$, let

$$\hat{\mu}(n) = \int_{\mathbf{T}} e^{-in\theta}\, d\mu(\theta).$$

A measure $\mu \in M(\mathbf{T})$ is said to be of *analytic type* if $\hat{\mu}(n)=0$ for all $n<0$; as usual, $H^1(\mathbf{T})$ will denote the classical space of all measures of analytic type on $\mathbf{T}$. In 1933, Paley [4] published this remarkable inequality:

Theorem P. *There is a* $C>0$ *such that if* $\langle n_k \rangle_0^\infty \subset \mathbf{Z}^+$ *and* $n_{k+1}/n_k \geqq 2$ *for all* k *then*

$$\left\{\sum_{k=0}^{\infty} |\hat{\mu}(n_k)|^2\right\}^{1/2} \leqq C\,\|\mu\|$$

provided $\mu \in H^1(\mathbf{T})$.

For generalizations of Paley's Theorem see the work of J. Fournier [3] and the references cited therein.

In this paper we shall prove yet another generalization of Paley's inequality; before we describe our result we shall require some notation concerning quotient norms for $M(\mathbf{T})$:

For $\omega \in M(\mathbf{T})$ and $E \subset \mathbf{Z}$ put

$$\|\omega\|_E = \inf\{\|\nu\| : \hat{\nu} = \hat{\omega} \text{ on } E\};$$

here $\|\cdot\|$ is the usual total variation norm on $M(\mathbf{T})$. We say $\langle D_n \rangle_0^\infty$ is a sequence of *positive dyadic intervals* in $\mathbf{Z}$ if there exists a sequence $\langle n_k \rangle_0^\infty \subset \mathbf{Z}^+$, $n_{k+1}/n_k \geqq 2$ for all k and $D_k = [n_{2k}, n_{2k+1})$. If the sequence of positive dyadic intervals $\langle D_k \rangle_0^\infty$

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satisfies $n_{k+1}/n_k \leqq \lambda$ for some λ and all k we say $\langle D_k\rangle_0^\infty$ is a *standard sequence of positive dyadic intervals.*

Our generalization of Theorem P is the following Littlewood—Paley type inequality for the quotient norms of an analytic measure:

Theorem P′. *There is a* $C>0$ *such that for any sequence* $\langle D_n\rangle_0^\infty$ *of positive dyadic intervals*

$$\{\sum_0^\infty \|\mu\|_{D_k}^2\}^{1/2} \leqq C\|\mu\|$$

provided $\mu\in H^1(\mathbf{T})$.

The proof of Theorem P′ uses a variant on the construction of Cohen—Davenport [1, 2] as well as some ideas in [3] and [5]. Before beginning the proof of Theorem P′ we shall need the following lemma.

Lemma. *Let* $0\leqq a\leqq 1$, $t, z\in\mathbf{C}$ *and* $|t|\leqq 1$, $|z|\leqq 1$. *Then*

$$\left|\frac{at}{10}+\left(1-\frac{a^2}{5}\right)z-\frac{a\bar{t}}{10}z^2\right| \leqq 1.$$

Proof. Consider the function $F(z)=\frac{at}{10}\bar{z}+\left(1-\frac{a^2}{5}\right)-\frac{a\bar{t}}{10}z$; since $0\leqq a\leqq 1$ it is easy to check that $|F(z)|\leqq 1$ for $|z|\leqq 1$ and $|t|\leqq 1$.

Assume for the moment $|z|=1$; as a consequence of $|F(z)|\leqq 1$ we gather that

$$\left|\frac{at}{10}+\left(1-\frac{a^2}{5}\right)z-\frac{a\bar{t}}{10}z^2\right| \leqq 1.$$

Our result now follows from the maximum modulus principle for analytic functions.

We turn to the proof of Theorem P′: Let $\mu\in H^1(\mathbf{T})$ and let $\langle D_k\rangle_0^\infty$ be any sequence of positive dyadic intervals. Suppose $\langle a_n\rangle_0^\infty$ satisfies

$$\sum_0^\infty |a_n|^2 \leqq 1. \tag{1}$$

Let $\langle t_n\rangle_0^\infty$ be any sequence of trigonometric polynomials on $\mathbf{T}$ such that supp $\hat{t}_k\subset -D_k$ and $\|t_k\|_\infty\leqq 1$ for all k. We also arrange for $\int_T t_k(\theta)\,d\mu(\theta)\equiv\hat{\mu}(t_k)\geqq 0$ for all k. Put

$$F_0 = \frac{1}{10}|a_0|\,t_0$$

and define inductively for $n=1, 2, \ldots.$

$$F_n = \frac{1}{10}|a_n|\,t_n+\left(1-\frac{|a_n|^2}{5}\right)F_{n-1}-\frac{|a_n|\,\bar{t}_n}{10}F_{n-1}^2.$$

As a consequence of inequality (1), $|a_n|\leqq 1$, so we may infer from the Lemma

$$\|F_n\|_\infty \leqq 1 \quad \text{for all } n. \tag{2}$$

Well, on the one hand,

$$\left|\int_{\mathbf{T}} F_n(\theta)\,d\mu(\theta)\right| \leqq \|\mu\|, \tag{3}$$

because of inequality (2), while on the other hand,

$$10\int_{\mathbf{T}} F_n(\theta)\,d\mu(\theta) = |a_n|\,\hat{\mu}(t_n)+\left(1-\frac{|a_n|^2}{5}\right)|a_{n-1}|\,\hat{\mu}(t_{n-1}) \tag{4}$$

$$+\left(1-\frac{|a_n|^2}{5}\right)\left(1-\frac{|a_{n-1}|^2}{5}\right)|a_{n-2}|\,\hat{\mu}(t_{n-2})+\dots$$

$$+\left(1-\frac{|a_n|^2}{5}\right)\left(1-\frac{|a_{n-1}|^2}{5}\right)\dots\left(1-\frac{|a_1|^2}{5}\right)|a_0|\,\hat{\mu}(t_0),$$

because the sequence $\langle D_k\rangle_0^\infty$ is dyadic and $\mu\in H^1(\mathbf{T})$.

As a consequence of inequalities (3) and (4), we obtain

$$10\|\mu\| \geqq \left\{\prod_0^n\left(1-\frac{|a_k|^2}{5}\right)\right\}\left\{\sum_0^n |a_k|\,\hat{\mu}(t_k)\right\} \tag{5}$$

since $\hat{\mu}(t_k)\geqq 0$ for all k. It now follows from inequalities (1) and (5) that

$$\left\{\sum_0^\infty \|\mu\|_{D_k}^2\right\}^{1/2} \leqq C\|\mu\|$$

for some universal constant $C>0$. Our proof is complete.

Corollary. (Meyer [7].) *Let $\langle D_k^*\rangle_0^\infty$ be a sequence of standard symmetric dyadic intervals, i.e., $D_k^*=[n_{2k}, n_{2k+1})\cup(-n_{2k+1}, -n_{2k}]$ where $n_{k+1}/n_k\geqq 2$ for all k and $n_{k+1}/n_k\leqq\lambda$ for some λ. Then there exists a $C(\lambda)>0$ such that*

$$\left\{\sum_0^\infty \|\mu\|_{D_k^*}^2\right\}^{1/2} \leqq C(\lambda)\|\mu\|$$

provided $\mu\in H^1(\mathbf{T})$.

Comments. (a) Theorem P′ tells us that the quotient norms of an analytic measure vanish very quickly at "$+\infty$"; cf. [5].

(b) Versions of Theorem P′ hold for all compact abelian groups with ordered duals.

(c) For a different approach to Littlewood—Paley inequalities which generalize Theorem P see [6].

(d) The method of proof of Theorem P′ can easily be adapted to give this generalization of Paley's inequality: Suppose φ is a multiplicative linear functional with representing measure m; by $H^1(dm)$ we mean the closure in $L^1(dm)$ of A and by H_0^∞ the weak-*closure of A_0 in $L^\infty(dm)$.

There is a universal constant $C>0$ *such that if* $\{\mu_k\}_0^\infty$ *is any sequence of unimodular elements of* $H^\infty(dm)$ *such that* $(\bar{\mu}_{k-1})^2\mu_k \in H_0^\infty$ *for all k then, for any* $h\in H^1(dm)$,

$$\left\{\sum_0^\infty \left|\int h\bar{\mu}_k\, dm\right|^2\right\}^{1/2} \leqq C\|h\|_1.$$

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