On polyharmonic continuation by reflection formulas

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1. Introduction and summary*

Let Ω be an open connected set in $\mathbf{R}^n$, which is contained in the half space $\mathbf{R}^n_+ = \{x\colon\ x_1 \geqq 0\}$, and let an open connected subset ω of the boundary of Ω be situated in the hyperplane $x_1=0$. Then $\Omega\cup\omega$ is open in $\mathbf{R}^n_+$. A p-harmonic function in Ω is a $2p$ times differentiable solution of the equation

$$\Delta^p u = 0, \quad u\in C^{2p}(\Omega), \tag{1.1}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator. We denote the set of all such functions by $H^p(\Omega)$. It will be seen that if $u\in H^p(\Omega)$, u is in fact analytic in Ω. We shall consider functions $u\in H^p(\Omega)$ which also satisfy a set of p boundary conditions

$$\lim_{x_1\to+0} q_i(D_1)u(x_1, x') = 0,\ (0, x')\in\omega, \quad i = 1, \dots, p, \tag{1.2}$$

where $q_i(D_1)$ are linearly independent polynomials in $D_1 = \frac{\partial}{\partial x_1}$, with constant coefficients and x' denotes $(x_2, \dots, x_n)$. In (1.2) we do not suppose x' to be fixed as $x_1\to+0$. We shall also use the notation $q_i(D_1)u(x)=o(1)$ as $x_1\to+0$. It will be shown that these functions can be continued as polyharmonic functions across ω into the half space $\mathbf{R}^n_- = \{x\colon\ x_1<0\}$. Very general theorems of this type have been given by Hörmander in [7], where he considers solutions of general elliptic and hypoelliptic differential equations with constant coefficients.

* The main part of this work constituted a PH.D. thesis accepted at Stockholm University 1973.

His results, however, do not tell anything about the extent of the domain into which the continuation is possible. An example of the type of theorems we are aiming at, is the Schwarz' reflection principle, stating that a harmonic function defined in Ω, satisfying the single condition

$$\lim_{x_1 \to +0} u(x) = 0, \quad x \in \Omega$$

can be analytically continued by the formula

$$u(-x_1, x') = -u(x_1, x') \quad (x_1, x') \in \Omega$$

into the whole of the domain $\Omega_1 = \Omega \cup \omega \cup \underline{\Omega}$, where $\underline{\Omega}$ is obtained by reflecting Ω geometrically in the plane $x_1 = 0$, (fig. 1).

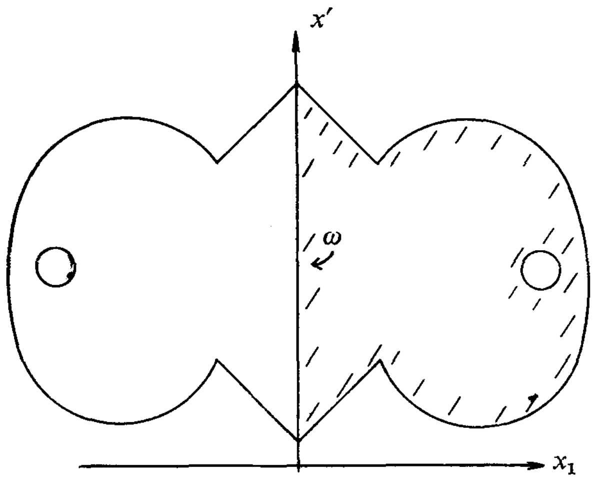

Fig. 1

It was proved by Almansi [1] that, under certain conditions on Ω, a function $u \in H^p(\Omega)$ can be represented by p functions v_j, $j=0, 1, ..., p-1$, each of which is simply harmonic in Ω, in the following way (Theorem 2.2)

$$u(x) = \sum_{j=0}^{p-1} \frac{x_1^j}{j!} v_j(x). \tag{1.3}$$

In a closed sphere $S_R(x_0) \subset \Omega$ with center x_0 and radius R, we can also represent u as (Theorem 2.4)

$$u(x) = \sum_{j=0}^{p-1} r^{2j} w_j(x) \quad r \leqq R \tag{1.4}$$

where w_j, $j=0, 1, \ldots, p-1$ also are harmonic and r denotes the distance from x to x_0. Formulas (1.3) and (1.4) are usually called Almansi representations.

The representation (1.4) can be used to prove a mean value theorem for polyharmonic functions. Let $M_R(f, x_0)$ denote the arithmetical mean value of f over the boundary of the sphere $S_R(x_0)$.

Then we get (Theorem 2.4)

$$M_r(u, x_0) = \sum_{j=0}^{p-1} A_{n,j} r^{2j} \Delta^j u(x_0) \quad r \leqq R,$$

where $A_{n,j}$ are constants which depend only on j and the dimension n. We use this to show that if u, given by (1.3), has the property

$$u(x_1, x') = o(1) \quad \text{as} \quad x_1 \to +0 \quad x' \in \omega$$

then

$$v_0(x_1, x') = o(1) \quad \text{as} \quad x_1 \to +0 \quad x' \in \omega \quad \text{(Theorem 2.5)}$$

This was proved in the biharmonic case by Duffin [5].

With help of these theorems, we shall prove in section 3 (Theorem 3.1), that a p-harmonic function satisfying the p conditions (1.2) can be analytically continued into the domain Ω_2, defined as follows. Let $\Omega' \subset \Omega$ have the property

$$(x_1, x_2, \ldots, x_n) \in \Omega' \Rightarrow (t_1, x_2, \ldots, x_n) \in \Omega \quad \text{all } t_1 \text{ such that } \quad 0 < t_1 \leqq x_1. \tag{1.5}$$

Then $\Omega_2 = \Omega \cup \omega \cup \underline{\Omega}'$, where $\underline{\Omega}'$ is obtained by reflecting Ω' geometrically in $x_1=0$, (fig. 2).

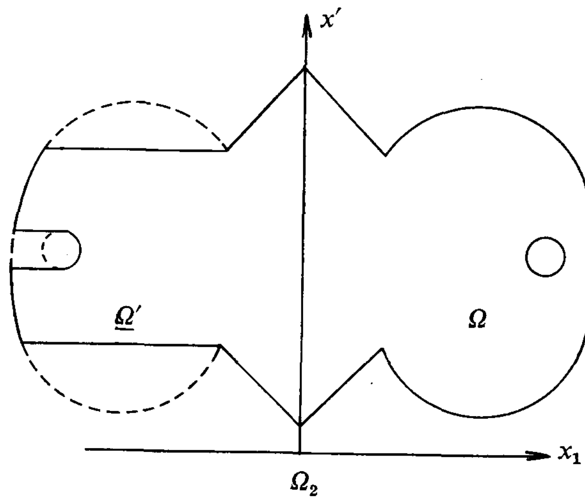

Fig. 2

Such theorems have already been given when the boundary conditions (1.2) are those of Dirichlet:

$$q_\nu(D_1) \equiv D_1^{\nu-1}, \quad \nu = 1, \dots, p,$$

namely for $p=2$ by Poritsky [13] and Duffin [5], and for general p by Huber [6]. Huber proved that the continuation in this case is given by the formula

$$u(-x_1, x') = (-1)^p \sum_{i=0}^{p-1} (-1)^i (i!)^{-2} x_1^{p+i} \Delta^i \left(\frac{u(x_1, x')}{x_1^{p-i}} \right). \tag{1.6}$$

Here the continuation is possible into Ω_1.

Some authors have also studied the problem of the continuation of the solutions of other partial differential equations than (1.1), when the boundary conditions are those of Dirichlet. See Canavan [2], John [9] and Lewy [11].

In the sequel $Q(D)$ (and $Q(x, D)$) denotes a differential polynomial in all $D_\iota = \dfrac{\partial}{\partial x_\iota}$, $\iota = 1, 2, \dots, n$. We shall also use the notation $Q(D')$ for a differential polynomial in the "boundary" differentials D_ι, $\iota = 2, 3, \dots, n$, only.

In section 4—7, we study such boundary conditions (1.2), for which a reflection formula of the form

$$u(-x_1, x') = Q(x, D)\, u(x_1, x'), \quad x \in \Omega \tag{1.7}$$

exists, where Q is some differential operator, and we also study the form of the corresponding operator $Q(x, D)$. In doing so, we assume that the p polynomials q_ν are homogeneous, that is of the simple form

$$q_i(D_1)\, u(0, x') \equiv D_1^{\nu_i} u(0, x') = 0 \quad 0 \leqq \nu_1 < \nu_2 < \dots < \nu_p. \tag{1.8}$$

First, in section 4, we prove two auxiliary theorems which are, however, interesting in themselves. Let $H^p(\mathbf{R}^n)$ denote the set of all functions which are p-harmonic in the whole of $\mathbf{R}^n$. Let $Q(D)$ be a differential operator with constant coefficients and such that for a certain point $y = (y_1, y')$,

$$[Q(D)\, u(x)]_{x=y} = 0$$

for all $u \in H^p(\mathbf{R}^n)$ satisfying (1.8). If $y_1 \neq 0$, then $Q(D)$ contains the factor Δ^p, that is:

$$Q(D) = P(D) \Delta^p, \quad \text{(Theorem 4.1).}$$

If $y_1 = 0$, then

$$Q(D) = \sum_{i=1}^{p} P_{\nu_i}(D') D_1^{\nu_i} + P(D) \Delta^p,$$

where the $P_{\nu_i}(D')$ are operators in the boundary differentials only, (Theorem 4.2 and Corollary 4.1). Both theorems are stated for more general boundary conditions than (1.8).

In section 5 we first prove that if there is a differential operator $Q(D)$ with constant coefficients and such that (1.7) is valid for a single point $x=y$ with $y_1 \neq 0$ and all u satisfying (1.8) which are p-harmonic in R^n, then there is an operator $Q_1(x, D)$ of the form

$$Q_1(x, D) \equiv \sum_\alpha \sum_{\beta<p} A_{\alpha,\beta} x_1^{\alpha+2\beta} D_1^\alpha \Delta^\beta \tag{1.9}$$

such that (1.7) is valid with $Q=Q_1$ and for every x (Lemma 5.1). Since

$$\big(Q(D)-Q_1(x, D)\big)u(y) = 0$$

for all $u \in H^p(R^n)$ it follows that $Q(D)=Q_1(y, D)+p(y, D)\Delta^p$. Hence Q and Q_1 differ only by terms containing the factor Δ^p. After proving that Q (or equivalently Q_1) must map the set H^p of all p-harmonic functions in itself we prove that the set of operators of the form (1.9) mapping H^p into itself is p-dimensional. Because of this, we can prove (Lemma 5.3), that such a Q_1 can be brought into the form

$$Q_1(x, D)\, u \equiv \sum_{i=0}^{p-1} B_i x_1^{p+i} \Delta^i \left(\frac{u}{x_1^{p-i}}\right) \tag{1.10}$$

Since $Q_1(x, D)$ in (1.10) is invariant if we replace x_1 by $-x_1$ we get using (1.7) twice that

$$Q_1^2 u - u = 0 \tag{1.11}$$

for all $u \in H^p$ satisfying (1.8). Having proved in section 4 that (1.11) must in fact be valid for all $u \in H^p$, if it is valid for all $u \in H^p$ satisfying (1.8), we prove (Theorem 5.2), that precisely 2^p of the operators $Q_1(x, D)$ of the form (1.10) have the property (1.11), and denote this last set of operators by T^p.

Hitherto we have supposed that a set of boundary conditions (1.8) is given, and have obtained necessary conditions on the operator Q_1 in order that Q_1 be a reflection operator in the sense that (1.7) holds for all u satisfying (1.8). These conditions may be summarized: Q_1 must belong to T^p.

In order to obtain sufficient conditions, we reverse the reasoning in the following way. Let Ω be as defined above and let Q be an arbitrary operator such that $Qu \in H^p(\Omega)$ if $u \in H^p(\Omega)$. For a given $u \in H^p(\Omega)$, define a continuation of u into $\underline{\Omega}$ by means of (1.7). A necessary and sufficient condition for u thus defined in $\Omega \cup \underline{\Omega}$ to be p-harmonic in $\Omega_1 = \Omega \cup \omega \cup \underline{\Omega}$ is that u is continuous over ω together with its $2p-1$ first derivatives. This furnishes $2p$ boundary conditions on the function u to be continued by Q. Analytically expressed, these conditions are

$$q_j(D)\, u(0, x') = 0, \quad j = 0, 1, \ldots, 2p-1, \tag{1.12}$$

where $q_j(D)$ is defined by

$$q_j(D)u(0, x') = \lim_{x_1\to 0}[D_1^j u(x_1, x')-(-1)^j D_1^j Qu(x_1, x')].$$

We shall call them the boundary conditions corresponding to Q.

For some operators Q, the corresponding boundary conditions may be reduced in number. We define two sets of operators, S^p and M^p. $Q\in T^p$ is said to belong to S^p if out of the $2p$ corresponding boundary conditions (1.12) we can find p conditions such that if they are satisfied by u, then the remaining p conditions are automatically satisfied.

$Q\in T^p$ is said to be in M^p, if there is a set of p boundary conditions of the special type (1.8), such that if they are satisfied, then the $2p$ boundary conditions corresponding to Q are also satisfied.

The first main result of section 7 is that M^p contains $p+1$ elements. All operators Q in M^p and their corresponding boundary conditions are listed, (Theorem 7.1). The second main result is that S^p and T^p are equal, (Theorem 7.3). The vital idea in the proofs of sections 6 and 7 is to proceed by induction in the order of harmonicity. To each operator $Q\in T^p$ written in the form (1.10) we define an operator Q^* by (Definition 6.3).

$$Q^*(x, D)u = -\sum_{i=0}^{p-2} B_i x_1^{p-1+i}\Delta^i\left(\frac{u}{x_1^{p-1-i}}\right).$$

It is shown that $Q^*\in T^{p-1}$. We then define the boundary conditions $q_*^j(D)$ corresponding to Q^* by

$$q_j^*(D)v = \lim_{x_1\to 0}[D_1^j v(x_1, x')-(-1)^j Q^* v(x_1, x')].$$

It turns out that, apart from a term containing Δ^{p-1},

$$q_j'(D) = j\cdot q_{j-1}^*(D), \quad j = 1, \ldots, 2p-1,$$

where $q_j'(\xi)$ denotes $\partial q_j/\partial\xi_1$ (Lemma 6.8). This observation is the main point in the induction step.

In the final section 8 we deal with sets of boundary conditions (1.8) for which there do not exist reflection operators Q of purely differential type. A p-harmonic function satisfying such boundary conditions is continued by an operator containing integrations, and the continuation is therefore only proved up to Ω_2 (fig. 2).

Acknowledgement

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2. Integration and representation of polyharmonic functions

Let Ω be an open set defined as in the introduction and let u be harmonic in Ω. Since a harmonic function u in Ω is analytic, see Courant—Hilbert [3] p. 269, it is clear that any derivative of u is harmonic in the whole of Ω. We shall now study the existence of a harmonic primitive function to u. Such a function does not necessarily exist in the whole of Ω for all Ω, and we must therefore impose some restriction on Ω. With future application in mind, we choose to assume that u is defined in an open set $\Omega_\varepsilon \subset \Omega$ with the property (see fig. 3)

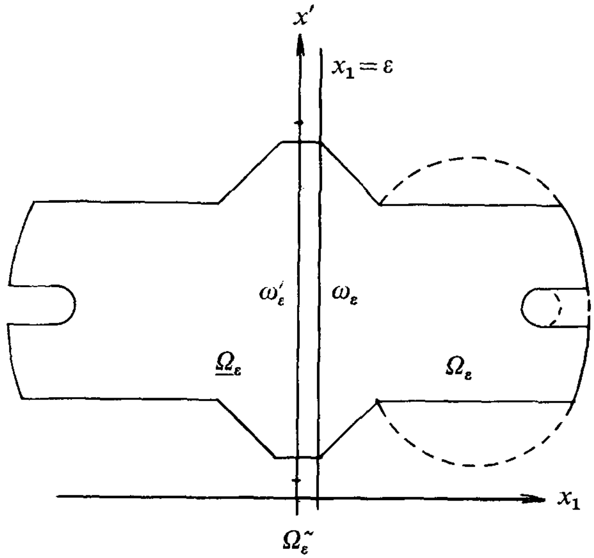

Fig. 3

$$(x_1, x_2, \ldots, x_n) \in \Omega_\varepsilon \Rightarrow (t_1, x_2, \ldots, x_n) \in \Omega \quad \text{all } t_1, \quad 0 < t_1 \leqq T_1 \tag{2.1}$$

where ε is a positive number sufficiently small so that Ω_ε is not void, and $T_1 = \max(\varepsilon, x_1)$. Later, in section 3, we shall let $\varepsilon \to 0$. It is seen from (1.5) that

$\lim_{\varepsilon\to 0}\Omega_\varepsilon=\Omega'$. The intersection of Ω_ε and the hyperplane $x_1=\varepsilon$ is called ω_ε. Since $\Omega_\varepsilon\subset\Omega$, it is clear that any u harmonic in Ω is also harmonic in Ω_ε.

Theorem 2.1. *Let u be a harmonic function in Ω_ε, where Ω_ε has the properties* (2.1). *Then there is a harmonic function U in Ω_ε such that*

$$D_1 U = u. \tag{2.2}$$

See Duffin [5].

Proof. We shall prove that there is a function $g(x')$, $x'=(x_2, \ldots, x_n)$ in the hyperplane $x_1=\varepsilon$ such that

$$U(x_1, x') = \int_\varepsilon^{x_1} u(t, x')\,dt+g(x'), \quad x = (x_1, x')\in\Omega_\varepsilon, \tag{2.3}$$

has the required properties. That (2.2) is fulfilled is obvious. Furthermore,

$$\Delta U(x_1, x') = D_1 u(x_1, x')+\int_\varepsilon^{x_1} \Delta' u(t, x')\,dt+\Delta' g(x'), \tag{2.4}$$

where Δ' is the Laplace operator in the boundary variables x'. Since u is harmonic, we have $\Delta' u=-D_1^2 u$. Hence (2.4) becomes if we write $D_1 u=u_1$

$$\Delta U(x_1, x') = u_1(\varepsilon, x')+\Delta' g(x').$$

Since $u_1(\varepsilon, x')$ is in $C^\infty(\omega_\varepsilon)$, there is a solution g in $C^2(\omega_\varepsilon)$ of the Poisson equation $\Delta' g(x')=-u_1(\varepsilon, x')$. See Courant—Hilbert [3] p. 246. This completes the proof.

Remark. In special cases the result may be valid under much weaker hypothesis concerning Ω_ε. When $n=2$ in particular, the result holds for any simply connected Ω_ε. To see this, we note that there is an analytic function $f(z)$, $z=x_1+ix_2$ in Ω_ε such that $u(x_1, x_2)=\operatorname{Re} f(z)$. Since there exists a primitive analytic function $F(z)$ to $f(z)$ in the whole of Ω_ε for any simply connected Ω_ε, it follows that $U(x_1, x_2)=\operatorname{Re} F(z)$ satisfies (2.2) in the whole of such a Ω_ε.

That the restriction (2.1) is rather natural when $n>2$, is seen by considering the harmonic function $u(x)=|x-y_0|^{2-n}$. For $n=3$, $u(x)$ has the only singularity $x=y_0$. Hence $\Omega=\mathbf{R}^3\setminus y_0$. Choose y_0 to be origo. Then a primitive function U will be

$$U(x) = -\log\left|x_1-\sqrt{x_1^2+x_2^2+x_3^2}\right|+g(x_2, x_3).$$

If we choose $g(x_2, x_3)\equiv 0$, then $U(x)$ will be defined for $x_1<0$, all x_2 and x_3. On the other hand $U(x)$ will not exist on the half line $x_2=x_3=0$, $x_1\geqq 0$. See also Diaz and Ludford [4].

This remark also applies to the next theorem.

Theorem 2.2. *(Almansi Representation Theorem). Let Ω_ε be the same set as in Theorem* 2.1. *Then every p-harmonic function u in Ω_ε can be written*

$$u = \sum_{j=0}^{p-1} \frac{x_1^j}{j!} v_j, \quad x \in \Omega_\varepsilon, \tag{2.5}$$

where each v_j is harmonic in Ω_ε. Conversely, every such function is p-harmonic, and

$$\Delta^{p-1} u = 2^{p-1} D_1^{p-1} v_{p-1}. \tag{2.6}$$

Proof. We shall make a proof by induction over p. Assume the theorem to be true when p is replaced by $1, 2, \ldots, p-1$. We start by proving that u defined by (2.5) is p-harmonic.

Since each v_j is analytic in Ω_ε, it is clear that u is also analytic in Ω_ε, and hence differentiable. Direct computation gives the formula

$$\Delta(f\cdot g) = f\Delta g + 2\langle \operatorname{grad} f, \operatorname{grad} g\rangle + g\Delta f, \tag{2.7}$$

where

$$\langle \operatorname{grad} f, \operatorname{grad} g\rangle = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial x_i}.$$

It follows from (2.7), if u is defined by (2.5) with v_j harmonic, that

$$\Delta u = \sum_{j=2}^{p-1} \frac{x_1^{j-2}}{(j-2)!} v_j + 2\sum_{j=1}^{p-1} \frac{x_1^{j-1}}{(j-1)!} D_1 v_j. \tag{2.8}$$

By the induction hypothesis the first sum in (2.8) is $(p-2)$-harmonic, the second $(p-1)$-harmonic. Hence $\Delta^{p-1}\Delta u = 0$, which proves that u is p-harmonic.

Now apply Δ^{p-2} to the $(p-1)$-harmonic function Δu in (2.8). Since we suppose (2.6) to be true when $p-1$ is replaced by $p-2$, we obtain

$$\Delta^{p-1} u = \Delta^{p-2}\Delta u = \Delta^{p-2} 2\sum_{j=1}^{p-1} \frac{x_1^{j-1}}{(j-1)!} D_1 v_j = 2^{p-1} D_1^{p-1} v_{p-1},$$

i.e. (2.6) holds also for $p-1$.

Now assume that u is a p-harmonic function. Repeated use of Theorem 2.1 shows that we can find a harmonic function v_{p-1} in Ω_ε, such that

$$2^{p-1} D_1^{p-1} v_{p-1} = \Delta^{p-1} u,$$

for $\Delta^{p-1} u$ is harmonic. Then the difference

$$u_1 = u - \frac{x_1^{p-1}}{(p-1)!} v_{p-1}$$

is $(p-1)$-harmonic, since by (2.6)

$$\Delta^{p-1}u_1 = \Delta^{p-1}u - 2^{p-1}D_1^{p-1}v_{p-1} = 0.$$

Since the theorem is trivial when $p=1$, the proof is complete.

Corollary 2.1. *Let Ω_ε be defined as before and let $u\in H^p(\Omega_\varepsilon)$. Then there is one function $u_1\in H^k(\Omega_\varepsilon)$, $0\leqq k\leqq p$, and one function $u_2\in H^{p-k}(\Omega_\varepsilon)$ such that*

$$u = u_1 + \frac{x_1^k}{k!}u_2 \quad x\in\Omega_\varepsilon.$$

Conversely, every such function is p-harmonic.

Proof. Dropping, as we may, the numerical factors $k!$ above and $j!$ in (2.5), which are there for computational purpose only, we obtain from Theorem 2.2.

$$u = \sum_{j=0}^{p-1} x_1^j v_j = \sum_{j=0}^{k-1} x_1^j v_j + x_1^k \sum_{j=k}^{p-1} x_1^{j-k} v_j = u_1 + x_1^k u_2$$

where, by the same theorem $u_1\in H^k(\Omega_\varepsilon)$ and $u_2\in H^{p-k}(\Omega_\varepsilon)$. The second statement is proved in the same way.

Because of the Almansi representation, the analogue of Theorem 2.1 can now be proved for polyharmonic functions.

Theorem 2.3. *Let $u\in H^p(\Omega_\varepsilon)$, Ω_ε being the same set as in Theorem 2.1. Then there is a function $U\in H^p(\Omega_\varepsilon)$ such that*

$$D_1 U = u$$

Proof. We proceed by induction. The case $p=1$ was proved in Theorem 2.1. Assume the result to be known when p is replaced by $p-1$. Corollary 2.1 shows with $k=1$ that

$$u = v_0 + x_1 u_1, \tag{2.9}$$

where v_0 is harmonic and u_1 is $(p-1)$-harmonic. By assumption there is a harmonic function V_0 and a $(p-1)$-harmonic function U_1 in Ω_ε such that $D_1 V_0 = v_0$ and $D_1 U_1 = u_1$. There is also a $(p-1)$-harmonic function U_2 in Ω_ε such that $D_1 U_2 = U_1$. Then by Corollary 2.1 the function

$$U = V_0 + x_1 U_1 - U_2$$

is p-harmonic. Since u satisfies (2.9), U satisfies the requirements of the theorem.

We shall now give another representation theorem which is also a mean value theorem for polyharmonic functions. Let Ω be as before and $x_0\in\Omega$. For a point x, let $\bar{r}$ denote the vector from x_0 to x and let $r\equiv|\bar{r}|$. Also, let $S_R(x_0)$ or simply S_R denote the solid, closed n-sphere with center in x_0 and radius R. R is chosen so that $S_R\subset\Omega$. Let $M_R(u, x_0)$ denote the mean value of u over the boundary ∂S_R

of S_R. It is well known that if v is harmonic in Ω, then

$$M_R(v, x_0) = v(x_0).$$

Theorem 2.4. *Let S_R be as above, and let $u \in H^p(\Omega)$. Then there are p functions w_j, $j=0, 1, \ldots, p-1$, each harmonic in S_R such that for $r \leqq R$ we have*

$$u(x) = \sum_{j=0}^{p-1} r^{2j} w_j(x). \tag{2.10}$$

Conversely, every such function is p-harmonic in S_R, and finally we have the following mean value relation for $u(x)$:

$$M_r(u, x_0) = \sum_{j=0}^{p-1} r^{2j} w_j(x_0) = \sum_{j=0}^{p-1} \frac{(n-2)!!}{2j!!(2j+n-2)!!} r^{2j} \Delta^j u(x_0) \tag{2.11}$$

where n, as before, is the dimensional number.

Proof. The proof is again carried out by induction over p. The result is trivial for $p=1$. We begin by showing that every function given by (2.10) is p-harmonic. Formula (2.7) shows that for w harmonic

$$\Delta r^{2j} w = w\, \Delta r^{2j} + 4j r^{2j-2} \langle \bar{r}, \operatorname{grad} w \rangle = w\, \Delta r^{2j} + 4j r^{2j-2} r \frac{\partial w}{\partial r}.$$

Now, if f is a function of r alone, we have in R^n

$$\Delta f = \frac{d^2 f}{dr^2} + \frac{(n-1)}{r} \frac{df}{dr},$$

so that

$$\Delta r^{2j} w = 2j(2j+n-2) r^{2j-2} w + 4j r^{2j-1} \frac{\partial w}{\partial r}. \tag{2.12}$$

Since $\Delta r \dfrac{\partial w}{\partial r} = 0$ for w harmonic, it follows from the induction hypothesis that $\Delta r^{2j} w$ is j-harmonic for $j \leqq p-1$, and hence $r^{2j} w$ is $j+1$-harmonic. Since each term in (2.10) is of this type, u is p-harmonic.

We shall now prove that for $u \in H^p(\Omega)$ we can find p harmonic functions w_j, such that (2.10) is true in S_R. The induction step will be to show that we can find w_{p-1} such that $u - r^{2p-2} w_{p-1}$ is $p-1$-harmonic. Assume by induction that we can express Δu by $p-1$ harmonic functions $\bar{w}_j$, $0 \leqq j \leqq p-2$ by means of

$$\Delta u = \sum_{j=0}^{p-2} r^{2j} \bar{w}_j.$$

Let (r, θ) denote the "spherical" coordinates in R^n, and define a function w by

$$\frac{1}{2p-2} w(r, \theta) = \frac{1}{r^\alpha} \int_0^r t^{\alpha-1} \bar{w}_{p-2}(t, \theta)\, dt \quad r \leqq R,$$

where $\alpha = \dfrac{1}{2}(2p+n-4) \geqq n/2 \geqq 1$, since $p > 1$. A direct computation shows that

w solves the equation

$$(2p-2)\left(r\frac{\partial w}{\partial r}+\alpha w\right) = \bar{w}_{p-2} \tag{2.13}$$

and

$$\frac{1}{2p-2}\Delta w = \frac{1}{r^{\alpha+2}}\int_0^r t^{\alpha+1}\Delta\bar{w}_{p-2}\,dt = 0$$

so that w is harmonic.

In view of (2.12) and (2.13) we get

$$\Delta(u-r^{2p-2}w) = \Delta u - r^{2p-4}(2p-2)\left[(2p+n-4)w+2r\frac{\partial w}{\partial r}\right] =$$

$$= \Delta u - r^{2p-4}w_{p-2} = \sum_{j=0}^{p-3} r^{2j}\bar{w}_j.$$

This shows that $\Delta(u-r^{2p-2}w)$ is $p-2$-harmonic, and that $(u-r^{2p-2}w)$ is $p-1$-harmonic. Hence we can find harmonic functions w_j, $0\leqq j\leqq p-1$, with $w_{p-1}=w$ such that (2.10) holds.

To see the mean value relation, we first note that

$$M_r(u, x_0) = M_r\left(\sum_{j=0}^{p-1} r^{2j}w_j, x_0\right) = \sum_{j=0}^{p-1} r^{2j}M_r(w_j, x_0) =$$

$$= \sum_{j=0}^{p-1} r^{2j}w_j(x_0) \quad \text{for} \quad r\leqq R. \tag{2.14}$$

Since all w_j are harmonic in S_R which is compact, they are continuous and uniformly bounded there together with their derivatives of order $\leqq 2p$. Hence

$$(\Delta^k u)(x_0) = \lim_{r\to 0}\Delta^k\left(\sum_{j=0}^{p-1} r^{2j}w_j\right) = \frac{(2k)!!(2k+n-2)!!}{(n-2)!!}w_k(x_0),$$

since all other terms disappear as $r\to 0$. The coefficient in this formula comes from repeated application of (2.12). Together with (2.14) this gives the desired mean value relation.

In the next theorem the notation $x'\in\omega'_\varepsilon$ means that the point $(\varepsilon, x')\in\omega_\varepsilon$ (fig. 3)

Theorem 2.5. *Let v_j, $0\leqq j\leqq p-1$, be harmonic in Ω_ε, and let*

$$u(x) = \sum_{j=0}^{p-1} x_1^j v_j(x) = o(1) \quad \text{as} \quad x_1\to +0,\ x'\in\omega'_\varepsilon.$$

Then $v_0(x)=o(1)$ as $x_1\to+0$, $x'\in\omega'_\varepsilon$.

Proof. The theorem is trivial for u simply harmonic. Suppose by induction that the theorem is proved for $p-1$-harmonic functions. It is then enough to show that $x_1^{p-1}v_{p-1}(x)=o(1)$.

The condition on u means that for every $\varepsilon_1>0$, there is a $\delta>0$, such that

$$|u(x_1, x')| \leqq \varepsilon_1 \quad \text{for} \quad 0 < x_1 \leqq 3\delta, \quad x' \in \omega'_\varepsilon, \tag{2.15}$$

and this holds uniformly in every compact of ω'_ε.

Take a x'_0 such that the sphere $S_\delta(x_0)$, where $x_0=(2\delta, x'_0)$, lies entirely in a compact of Ω_ε. The mean value relation (2.11) used for p different values of $r\leqq\delta$, e.g. $r_i=i\delta/p$, $i=1, 2, \ldots, p$, gives a system of p equations in the p unknowns $\Delta^j u(x_0)$, $j=0, 1, \ldots, p-1$, namely

$$\sum_{j=0}^{p-1} A_{n,j} r_i^{2j} \Delta^j u(x_0) = M_{r_i}(u, x_0), \quad i = 1, 2, \ldots, p,$$

where $A_{n,j}=\dfrac{(n-2)!!}{(2j)!!(2j+n-2)!!}$.

The determinant, Det, of this system is of the form

$$\text{Det} = K \cdot \delta^h \det |i^{2j}|, \quad i = 1, 2, \ldots, p, \ j = 0, 1, \ldots, p-1, \ K \neq 0,$$

and $h=\sum_{j=0}^{p-1} 2j$. Hence $\text{Det}=K_1\delta^h$, where $K_1\neq 0$. We solve this system by means of Cramer's rule, and obtain for $\Delta^{p-1}u(x_0)$

$$\Delta^{p-1}u(x_0) = \text{Det}_1/\text{Det}.$$

Det_1 is obtained by substituting $M_{r_i}(u, x_0)$ for $A_{n,p-1}(i\delta/p)^{2p-2}\Delta^{p-1}u(x_0)$ $i=1, \ldots, p$, in the last column of Det. Now expand Det_1 by means of this last column, and note that, because of (2.15), the mean value of u over the sphere $S_{r_i}(x_0)$ also satisfies $|M_{r_i}(u, x_0)|<\varepsilon_1$, $i=1, 2, \ldots, p-1$. This gives the following estimate for Det_1:

$$|\text{Det}_1| \leqq \varepsilon_1 \sum_{i=1}^{p} |K_i \delta^{h'}|$$

where $h'=h-(2p-2)$.

This implies that

$$|\Delta^{p-1}u(x_0)| \leqq K_2\varepsilon_1\delta^{-(2p-2)} = o(\delta^{2-2p}) \quad \text{as} \quad \delta \to +0,$$

since K_2 is independent of ε_1 and δ. Since by Theorem 2.2

$$\Delta^{p-1}u(x) = 2^{p-1}D_1^{p-1}v_{p-1}(x),$$

it is clear that $v_{p-1}(x)=o(x_1^{1-p})$ and hence

$$x_1^{p-1}v_{p-1} = o(1) \quad \text{as} \quad x_1 \to +0.$$

This completes the proof.

Remark. Let $v_0=1$, and $v_1=-1/r$ in R^3. Then $u=v_0+x_1v_1=0$ on the line $x_2=x_3=0$, but $v_0 \nrightarrow 0$ as $x_1\to 0$ on this line. This shows that it is essential that ω'_ε does not degenerate in Theorem 2.5.

We shall now give some examples of polyharmonic functions. Let

$$\xi = (\xi_1, \xi_2, \dots, \xi_n) = (\xi_1, \xi')$$

be a fixed vector in an n-dimensional complex space, and such that the scalar product

$$\langle \xi, \xi \rangle = \sum_{i=1}^n \xi_i^2 = 0,$$

while

$$\langle \xi', \xi' \rangle = \sum_{i=2}^n \xi_i^2 \neq 0.$$

Denote $\xi^* = (-\xi_1, \xi_2, \dots, \xi_n)$. Then $\langle \xi^*, \xi^* \rangle = 0$. For later use (section 4), we also introduce a complex parameter τ. As in the introduction, $x = (x_1, x_2, \dots, x_n)$ is a vector in R^n.

Define

$$v(x) = e^{\tau\langle \xi, x \rangle}, \tag{2.16}$$

then

$$\Delta v = \Delta e^{\tau\langle \xi, x\rangle} = e^{\tau\langle \xi, x\rangle} \cdot \tau^2 \cdot \sum_{i=1}^n \xi_i^2 = 0.$$

Hence $v \in H^1(R^n)$. We get the same result if ξ is replaced by ξ^*. Hence, by Theorem 2.2, $u(x)$ defined by

$$u(x) = \sum_{j=0}^{p-1} A_j \frac{x_1^j}{j!} e^{\tau\langle \xi, x\rangle} + \sum_{j=0}^{p-1} B_j \frac{x_1^j}{j!} e^{\tau\langle \xi^*, x\rangle} \tag{2.17}$$

where A_j and B_j are arbitrary constants, belongs to $H^p(R^n)$, and hence to $H^p(\Omega)$ for any open $\Omega \in R^n$.

The functions (2.17) are called "exponential solutions", and since any statement about the set $H^p(\Omega)$ must take such exponential solutions into account, they furnish necessary conditions on such statements, and it will be seen that very often these conditions are also sufficient. This is very natural in view of the fact that the set of exponential solutions is dense in $H^p(R^n)$. See Hörmander [8] p. 76 ff.

Another example of functions in $H^p(R^n)$ is given by $u = x_1^k$, $0 \leqq k < 2p$. These functions will be used since their and their derivatives behaviour on the boundary $x_1 = 0$ is easily determined. A straightforward application of the proof of the Almansi representation gives for the biharmonic function $u = 2x_1^3$ the following as a possible representation among others.

$$u = 2x_1^3 = v_0 + x_1 v_1 = (-x_1^3 + 3x_1 x_i^2 + Ax_1) + x_1(3x_1^2 - 3x_i^2 - A)$$

where $i \neq 1$ and A is arbitrary. It is seen that the Almansi representation is by no means unique. It is easy to find still more representations.

Remark. As a final remark to this section, we observe that the function

$$u = x_1^{p-1} v$$

where v is defined by (2.16) is an example of a function in H^p such that

$$D_1^j u \notin H^{p-1} \quad j = 1, 2, \ldots$$

and

$$x_1^l u \in H^{p+l} \text{ but } \notin H^{p+l-1} \quad l = 1, 2, \ldots .$$

This observation will be helpful in section 5.

3. General transversal boundary conditions

We consider p-harmonic functions $u(x_1, x')$ defined in an open set Ω of the type considered in the introduction, satisfying in the limit on ω the p boundary conditions

$$\lim_{x_1 \to +0} q_i(D_1) u(x_1, x') = 0, \; x \in \Omega, \quad i = 1, \ldots, p \tag{3.1}$$

where the q_i are linearly independent differential polynomials in D_1. Let Ω_ε be the set of all points in Ω with the property (2.1). Theorem 2.2 shows that u has an Almansi representation (2.5) in Ω_ε with v_j, $j=0, \ldots, p-1$, harmonic in Ω_ε. Let $\underline{\Omega}_\varepsilon$ be the reflection of Ω_ε in $x_1=0$, i.e. $\underline{\Omega}_\varepsilon$ is the set of all $x=(x_1, x')$ such that $(-x_1, x') \in \Omega_\varepsilon$. Also let ω'_ε be the projection of ω_ε on the hyperplane $x_1=0$. Then $\Omega_\varepsilon^{\sim}$ defined by $\Omega_\varepsilon^{\sim} = \Omega_\varepsilon \cup \omega'_\varepsilon \cup \underline{\Omega}_\varepsilon$ is an open set. (See fig. 3.) By the definition of Ω' and Ω_2 in the introduction, $\lim_{\varepsilon \to 0} (\Omega \cup \omega'_\varepsilon \cup \Omega_\varepsilon) = \Omega \cup \omega \cup \underline{\Omega}' = \Omega_2$.

Theorem 3.1. *Let Ω and ω satisfy the conditions of the introduction. Every p-harmonic function u in Ω satisfying* (3.1) *can be extended to a p-harmonic function in Ω_2.*

The method of proof will be to use the Almansi representation of u to construct harmonic functions w_i, $i=1, \ldots, p$, in Ω_ε in such a way that the boundary conditions (3.1) imply that $\lim_{x_1 \to 0} w_i(x_1, x')=0$, $i=1, \ldots, p$. Hence each of the $w(x_1, x')$ can be continued into $\Omega_\varepsilon^{\sim}$ by means of Schwarz reflection principle. From the continuation of the w_i to $\Omega_\varepsilon^{\sim}$, we then conclude that u can be continued into $\Omega_\varepsilon^{\sim}$.

Proof. We know already that u has an Almansi representation

$$u = \sum_{j=0}^{p-1} \frac{x_1^j}{j!} v_j \tag{3.2}$$

in Ω_ε with v_j harmonic in Ω_ε. Using this and Leibniz' formula for the k:th derivative of a product we obtain

$$q_i(D_1) u = \sum_{j=0}^{p-1} \sum_{k=0}^{j} \frac{x_1^{j-k}}{(j-k)!} \cdot \frac{1}{k!} q_i^{(k)}(D_1) v_j \tag{3.3}$$

where $q_i^{(k)}(\tau)$ denotes $d^k q_i(\tau)/d\tau^k$. When $x_1 \to +0$ in (3.3), (3.1) and Theorem 2.5 imply that the harmonic functions w_i, defined by

$$w_i = \sum_{j=0}^{p-1} \frac{1}{j!} q_i^{(j)}(D_1) v_j \quad i = 1, \ldots, p$$

satisfy

$$\lim_{x_1 \to +0} w_i(x_1, x') = 0, \ x \in \Omega_\varepsilon, \quad i = 1, \ldots, p.$$

Hence the w_i can be extended to harmonic functions $w_i^{\sim}$ in $\Omega_\varepsilon^{\sim}$ by means of Schwarz' reflection principle, that is, we set

$$w_i^{\sim}(x_1, x') = \begin{cases} w_i(x_1, x'), & (x_1, x') \in \Omega_\varepsilon \\ 0, & (x_1, x') \in \Omega_\varepsilon' \\ -w_i(-x_1, x'), & (x_1, x') \in \underline{\Omega}_\varepsilon \end{cases}$$

We shall now define the extensions $v_j^{\sim}$ of the functions v_j by solving the system of ordinary differential equations

$$\sum_{j=0}^{p-1} \frac{1}{j!} q_i^{(j)}(D_1) v_j^{\sim} = w_i^{\sim}, \quad i = 1, \ldots, p \ x \in \Omega_\varepsilon' \tag{3.4}$$

with the conditions $v_j^{\sim} = v_j$ when $x_1 > 0$. This is a system of p ordinary differential equations in x_1 of the p functions $v_j^{\sim}$, $j = 0, \ldots, p-1$ and containing the parameters $x_2, \ldots, x_n$. The characteristic determinant of the system is the so-called Wronski determinant

$$W(\tau) = \det |D^j q_i(\tau)|, \quad i = 1, \ldots, p, \quad j = 0, \ldots, p-1.$$

Since the boundary conditions are linearly independent, $W(\tau)$ does not vanish identically. Hence we can solve the system (3.4), and since the $w_i^{\sim}$ are infinitely differentiable, each $v_j^{\sim}$ is also an infinitely differentiable function of $(x_1, \ldots, x_n)$. Applying the Laplace operator Δ to (3.4), we get the system

$$\sum_{j=0}^{p-1} \frac{1}{j!} q_i^{(j)}(D_1) \Delta v_j^{\sim} = 0, \quad i = 1, \ldots, p,$$

for $w_i^{\sim}$ is harmonic in $\Omega_\varepsilon^{\sim}$.

By solving this system for $\Delta v_j^{\sim}$, $j = 0, \ldots, p-1$, and observing that $\Delta v_j^{\sim} = 0$ in Ω_ε we infer that $\Delta v_j^{\sim}$ is identically 0, that is, $v_j^{\sim}$ is harmonic in $\Omega_\varepsilon^{\sim}$. If we set

$$u^{\sim} = \sum_{j=0}^{p-1} \frac{x_1^j}{j!} v_j^{\sim} \tag{3.5}$$

we have proved that $u^{\sim}$ is p-harmonic in $\Omega_\varepsilon^{\sim}$ and $u^{\sim} \equiv u$ in Ω_ε. Hence $u^{\sim}$ is a p-harmonic extension of u into $\Omega_\varepsilon^{\sim}$. It is trivial that $u^{\sim}$ is extendable to $\Omega \cup \omega_\varepsilon' \cup \underline{\Omega}_\varepsilon$, although the Almansi representation (3.5) of $u^{\sim}$ has only been proved to be valid in $\Omega_\varepsilon^{\sim}$. Finally, letting $\varepsilon\to+0$, we have proved that $u^{\sim}$ is extendable to $\Omega_2=\lim_{\varepsilon\to 0}(\Omega\cup\omega'_\varepsilon\cup\Omega_\varepsilon)$.

Example 3.1. Let u be harmonic in Ω, and satisfy the boundary condition $\lim_{x_1\to+0}(D_1u+ku)=0$ on ω, where k is a constant. Then the continuation of u to Ω_2 is given by

$$u(-x_1, x') = u(x_1, x')+2ke^{kx_1}\int_0^{x_1} e^{-kt}u(t, x')\,dt, \quad x\in\Omega_\varepsilon. \tag{3.6}$$

Remark. Let Ω and ω be the same sets as in the introduction. Let $q_i(D)$ be differential polynomials in all D_ι, $\iota=1, \dots, n$. A set of boundary conditions

$$q_i(D)u = 0, \quad x\in\omega, \quad i = 1, \dots, p, \tag{3.7}$$

is called elliptic (see Hörmander [7]) with respect to Δ^p if every p-harmonic function $u\in C^k(\Omega\cup\omega)$, where k is the maximum of $2p$ and the degrees of q_i, satisfying the conditions (3.7) can be continued across ω into some domain, independent of u. By Theorem 3.3 in Hörmander [7], the condition for ellipticity is that the Wronski determinant W^0 of the principal parts q_i^0 of the q_i has no zero $\neq 0$. Denote the degree of q_i by v_i, and note that we may assume that all q_i have different degrees. Then $q_i^0(\tau)=\tau^{v_i}$, and with the notation $R=\sum_{i=0}^p v_i-1-2-\dots-p-1$, we get

$$W^0(\tau) = \tau^R \prod_{i<k}(v_i-v_k)$$

which has no zero $\neq 0$. Hence the conditions (3.1) are elliptic with respect to Δ^p.

However, the theorem of Hörmander does not tell anything about the extent of the continuation, whereas our Theorem 3.1 extends u to a function in Ω_2.

4. Two auxiliary theorems on polyharmonic functions

Consider a biharmonic function u satisfying on $x_1=0$ the boundary conditions

$$\Delta u(0, x') = D_1\Delta u(0, x') = 0, \tag{4.1}$$

which are not elliptic in the sense of the remark at the end of section 3. These conditions imply, in view of the uniqueness of the Cauchy problem, that the harmonic function $v=\Delta u$ is identically zero. Hence all solutions u of $\Delta^2u=0$, satisfying (4.1), are also solutions of the "simpler" equation $\Delta u=0$. The object of the following theorem 4.1 is to prove that such a case cannot happen for the boundary conditions (1.2).

Theorem 4.1. *Let $Q(D)$ be a polynomial differential operator and let $y=(y_1, y')$ be a fixed point with $y_1 \neq 0$. Assume that*

$$Q(D)u(y) = 0 \tag{4.2}$$

for every $u \in H^p(R^n)$ satisfying the p linearly independent boundary conditions

$$q_i(D_1)u = 0, \quad x_1 = 0, \quad i = 1, \ldots, p. \tag{4.3}$$

Then $Q(D)$ contains Δ^p as a factor.

Proof. We shall prove the theorem by imposing (4.2) and (4.3) to the p-harmonic exponential functions (2.17)

$$u = \sum_{j=0}^{p-1} A_j \frac{x_1^j}{j!} e^{\tau\langle \xi, x\rangle} + \sum_{j=0}^{p-1} B_j \frac{x_1^j}{j!} e^{\tau\langle \xi^*, x\rangle} \tag{4.4}$$

where τ is a complex parameter and, as before, A_j and B_j are constants. Also, as is (2.17), ξ is a fixed complex n-vector such that $\langle \xi, \xi\rangle = \langle \xi^*, \xi^*\rangle = 0$ and $\xi_1 \neq 0$. Because of Leibniz' rule (3.3) for the derivative of a product, the boundary conditions (4.3) for $x_1=0$ applied to (4.4) give a system of p linear equations in the $2p$ constants A_j and B_j.

$$\sum_{j=0}^{p-1} \frac{1}{j!} A_j q_i^{(j)}(\tau\xi_1) + \sum_{j=0}^{p-1} \frac{1}{j!} B_j q_i^{(j)}(-\tau\xi_1) = 0, \quad i = 1, \ldots, p. \tag{4.5}$$

Here $q_i^{(j)}(\eta) = \dfrac{d^j}{d\eta^j} q_i(\eta)$. From section 3 we know that the determinant $W(-\tau\xi_1)$ consisting of the coefficients of the B_j is $\not\equiv 0$. Hence we can solve the equations with respect to the B_j for large τ and obtain by Cramer's rule

$$B_j = \sum_{k=0}^{p-1} C_{jk}(\tau\xi_1) A_k, \quad j = 0, 1, \ldots, p-1, \tag{4.6}$$

where the C_{jk} are rational functions in $\tau\xi_1$, with the denominator $W(-\tau\xi_1)$.

Applying the operator Q to one of the terms in the first sum in (4.4), we get by means of Leibniz' formula

$$Q(D)\left(\frac{x_1^j}{j!} e^{\tau\langle \xi, x\rangle}\right) = \sum_{l=0}^{j} \frac{x_1^{j-l}}{(j-l)!} \frac{1}{l!} e^{\tau\langle \xi, x\rangle} Q^{(l)}(\tau\xi), \quad j = 0, 1, \ldots, p-1$$

where

$$Q^{(l)}(\eta) = \frac{\partial^l}{\partial \eta_1^l} Q(\eta).$$

Now we eliminate in (4.4) the constants B_j by means of (4.6). Then we get from (4.2) the somewhat cumbersome expression

$$Q(D)u(y) = \left[\sum_{j=0}^{p-1} A_j \sum_{l=0}^{j} \frac{y_1^{j-l}}{(j-l)!\,l!} Q^{(l)}(\tau\xi)\right] e^{\tau\langle\xi, y\rangle}$$

$$+\left[\sum_{j=0}^{p-1} \left\{\sum_{k=0}^{p-1} C_{jk}(\tau\xi_1)A_k\right\} \sum_{l=0}^{j} \frac{y_1^{j-l}}{(j-l)!\,l!} Q^{(l)}(\tau\xi^*)\right] e^{\tau\langle\xi^*, y\rangle} = 0 \tag{4.7}$$

and the important thing is that (4.7) is linear and homogeneous in the A_j which still are arbitrary. Thus, the coefficient of each A_j must be zero. The coefficient of A_j in (4.7) is, after division of $e^{\tau\langle\xi, y\rangle}$

$$\left[\sum_{l=0}^{j} \frac{y_1^{j-l}}{(j-l)!\,l!} Q^{(l)}(\tau\xi)\right] e^{2\tau\xi_1 y_1} + \left[\sum_{k=0}^{p-1} C_{jk}(\tau\xi_1) \sum_{l=0}^{j} \frac{y_1^{k-l}}{(k-l)!\,l!} Q^{(l)}(\tau\xi^*)\right] \equiv 0 \tag{4.8}$$

for all τ such that the denominator of C_{jk} does not vanish.

This can be written

$$K_1(\tau)e^{2\tau\xi_1 y_1} + K_2(\tau) \equiv 0$$

where $K_1(\tau)$ and $K_2(\tau)$ are rational functions. Since $y_1 \neq 0$, $e^{2\tau\xi_1, y_1}$ is a transcendent function of τ, it follows that both K_1 and K_2 must be identically zero. Hence, we get from (4.8)

$$K_1(\tau)_{\tau=1} = \sum_{l=0}^{j} \frac{y_1^{(j-l)}}{(j-l)!\,l!} Q^{(l)}(\xi) \equiv 0, \quad j = 0, 1, \ldots, p-1,$$

which proves that

$$Q^{(l)}(\xi) = 0, \quad l = 0, \ldots, p-1 \tag{4.9}$$

if

$$\langle\xi, \xi\rangle = \sum_{i=1}^{n} \xi_i^2 = 0.$$

Now $\langle\xi, \xi\rangle^p$ is a polynomial in ξ_1 of degree $2p$. Hence, because of the division algorithm, we can write

$$Q(\xi) = \langle\xi, \xi\rangle^p Q'(\xi) + R(\xi)$$

where the degree of $R(\xi)$ as a polynomial in ξ_1 is less that $2p$, and the coefficients are polynomials in $\xi' = \xi_2, \ldots, \xi_n$. (4.9) shows that for any fixed ξ' with $\langle\xi', \xi'\rangle = t^2 \neq 0$, the equation $Q(\xi)=0$ has the zeros $\xi_1 = \pm it$, each of multiplicity p and so has $\langle\xi, \xi\rangle^p Q'(\xi)$. Hence $R(\xi)$ must also have the same zeros and of the same multiplicity, but since it is of degree less than $2p$ is must be identically zero and $Q(\xi) = \langle\xi, \xi\rangle^p Q'(\xi)$; that is, $Q(D)u = Q'(D)\Delta^p u = 0$ for all $u \in H^p$, which was to be proved.

Remark. The vital point in the proof is that we can eliminate the B_j in (4.5). This will be possible as long as the determinant $W(-\tau\xi_1) \not\equiv 0$. According to the remark at the end of section three, this is true for an elliptic set of boundary conditions, since for such a set $W^0(-\tau, \xi)$ has no zero $\neq 0$. Hence the theorem remains true for all elliptic boundary conditions.

In the next theorem we shall for later purposes study slightly more general boundary conditions than in Theorem 4.1. These may contain differentials also in the boundary variables. Let $q_i(D)$, $i=1, \ldots, p$ be p differential polynomials with constant coefficients, each of degree $r_i<2p$ (counted as a polynomial in all differentials D_ι). Suppose that $r_i>r_j$ for $i>j$ and that the coefficient of each $D_1^{r_i}\neq 0$. We may suppose the coefficient to be 1 and write

$$q_i(D) = \sum_{l=0}^{r_i} R_{il}(D')D_1^l, \quad i = 1, \ldots, p \tag{4.10}$$

where R_{il} are polynomials in the boundary differentials only, and $R_{ir_i}(D')\equiv 1$.

Theorem 4.2. *Let $q_i(D)$, $i=1, \ldots, p$ be p differential polynomials as described above. Let y be a fixed point with $y_1=0$, and $V(D)$ a differential polynomial such that for all $u\in H^p(R^n)$ satisfying the p boundary conditions*

$$q_i(D)u(0, x') = 0, \quad i = 1, \ldots, p \tag{4.11}$$

we have

$$V(D)u(y) = 0. \tag{4.12}$$

Then $V(D)$ can be written in the form

$$V(D) = \sum_{i=1}^{p} P_{r_i}(D')q_i(D)+P_{2p}(D)\Delta^p, \tag{4.13}$$

where $P_{r_i}(D')$ are polynomials in the boundary D' only, and $P_{2p}(D)$ can be any operator.

Proof. Consider a given polynomial $V(D)$, and suppose that it cannot be brought into the form (4.13). The proof will then consist of an explicit construction of a function $u\in H^p(R^n)$ which satisfies (4.11) but not (4.12).

Because of the division algorithm, any differential polynomial can be written in the form

$$V(D) = \sum_{k=0}^{2p-1} P_k(D')D_1^k+P_{2p}(D)\Delta^p. \tag{4.14}$$

We shall now separate $V(D)$ in two parts. One part which is of the form (4.13), and one which is not. Since in each $q_i(D)$ in (4.10) the coefficient of $D_1^{r_i}=1$, and since $2p>r_i>r_j$ for $i>j$, we can extract first $P_{r_p}(D')q_p(D)$ from the sum (4.14), and then the following q_{r_i}, $i=p-1, \ldots, 1$ in strictly descending order, and finally obtain

$$V(D) = \sum_{i=1}^{p} P_{r_i}(D')q_i(D)+\sum_{k=0}^{\prime 2p-1} P_k(D')D_1^k+P_{2p}\Delta^p,$$

where $\sum'$ in the second sum indicates that $k=r_i$, $i=p, \ldots, 1$ are not included in the summation.

We may assume that there is at least one $k=k'$ in the second sum such that $P_{k'}(D')\not\equiv 0$, since otherwise $V(D)$ is of the form (4.13) and the theorem is proved. As before, take a fixed $\xi=(\xi_1, \xi')$ such that $\langle\xi, \xi\rangle=0$, $\langle\xi', \xi'\rangle\neq 0$ and $P_k'(\xi')\neq 0$.

Let ξ^* be $(-\xi_1, \xi')$. Consider the p-harmonic functions (2.17) with $\tau=1$.

$$u = \sum_{j=0}^{p-1} A_j \frac{x_1^j}{j!} e^{\langle\xi,x\rangle} + \sum_{j=0}^{p-1} B_j \frac{x_1^j}{j!} e^{\langle\xi^*,x\rangle}.$$

For $x_1=0$ we have

$$D_1^k u = L_k(A_j, B_j) e^{\langle\xi',x'\rangle}, \quad k = 0, \ldots, 2p-1,$$

where L_k is a linear expression in the coefficients A_j and B_j. It is well known from the Cauchy problem that $D_1^k u(0, x')=0$, $k=0, \ldots, 2p-1$, implies $u(x)\equiv 0$. This shows that the $2p$ equations

$$L_k(A_j, B_j) = 0, \quad k = 0, \ldots, 2p-1$$

have only the trivial solution $A_j=B_j=0, j=1, \ldots, p-1$. Hence it follows from the theory of linear equations that there exists a unique solution of the following system of linear equations in the $2p$ "unknown" A_j and B_j:

$$\begin{cases} L_k' = 1 \\ L_k = 0, \quad 0 \leqq k < 2p, \quad k \neq k' \quad \text{and} \quad k \neq r_i, \quad i = 1, \ldots, p \\ L_{r_i} = \sum_{l=0}^{r_i-1} R_{il}(\xi') L_l, \quad i = 1, \ldots, p. \end{cases} \tag{4.15}$$

This system is constructed recursively from $k=0$ to $k=2p-1$. The reason for this is that the last set of the equations (4.15), (which comes from (4.10)), contains $L_1(A_j, B_j)$ in the right hand side also. But since the system is built up recursively and since the summation in the right hand side of L_{r_i} in (4.15) is brought only to $l=r_i-1$, we can express these L_l, $l\leqq r_i-1$, by means of ξ' only and not A_j or B_j.

The exponential p-harmonic function u whose coefficients A_j and B_j satisfy (4.15), has the following property for $x_1=0$.

$$\begin{cases} D_1^{k'} u = e^{\langle x',\xi'\rangle} \\ D_1^k u = 0, \quad 0 \leqq k < 2p, \ k \neq k', \ k \neq r_i, \quad i = 1, \ldots, p \\ q_i(D) u = 0, \quad i = 1, \ldots, p. \end{cases}$$

Thus u satisfies the conditions of the theorem, but

$$V(D)u(y) = P_{k'}(D') D_1^{k'} u(y) = e^{\langle y',\xi'\rangle} P_{k'}(\xi') \cdot L_{k'} \neq 0.$$

This proves the theorem.

Remark. It is clear that any $u\in H^p$ which satisfies (4.11) also satisfies (4.12) if $V(D)$ is defined by (4.13).

It is also clear from the proof of the theorem, that the number of boundary conditions (4.11) is irrelevant as long as it is less than $2p$. We shall however only be dealing with p conditions.

For the special case of Theorem 4.2 that the boundary conditions (4.10) are $q_i(D)=D_1^{r_i}$, we state:

Corollary 4.1. *Let y be a fixed point with $y_1=0$, and $V(D)$ a differential polynomial such that for all $u\in H^p(R^n)$ satisfying the p boundary conditions*

$$D_1^{r_i}u(0, x') = 0, \quad 0 \leqq r_1 < r_2 < \ldots < r_p < 2p$$

we have

$$V(D)\,u(y) = 0.$$

Then $V(D)$ can be written in the form

$$V(D) = \sum_{i=1}^{p} P_{r_i}(D')D_1^{r_i} + P_{2p}(D)\Delta^p,$$

where $P_{r_i}(D')$ are polynomials in the boundary D' only, and $P_{2p}(D)$ can be any operator.

5. Necessary conditions on reflection formulas of differential type

As was seen in the proof of Theorem 3.1, the continuation of a p-harmonic function in Ω across ω is effected by solving a system of differential equations (3.4). Hence we can expect that a continuation formula generally contains integrations as e.g. in formula (3.6). The example (1.6) shows, however, that sometimes a continuation formula, involving differentiations only, can be given. In such cases the restrictions on Ω given in (2.1) are superfluous, so that u can be continued into the whole of the domain Ω_1 defined in the introduction. We shall in this section determine necessary conditions for a differential operator Q to have the property that each p-harmonic function u in Ω, satisfying a set of boundary conditions (1.2) can be continued into a function $u^{\sim}\in H^p(\Omega_1)$ by means of the formula

$$u^{\sim}(x_1, x') = \begin{cases} u(x_1, x'), & x\in\Omega \\ \lim\limits_{x_1\to 0} u(x_1, x'), & x\in\omega \\ Qu(-x_1, x'), & x\in\underline{\Omega} \end{cases} \tag{5.1}$$

Throughout the rest of the paper, we shall make use of the fact that a necessary conditions for (5.1) to constitute a p-harmonic continuation of u into Ω_1 is that for all $u\in H^p(R^n)$ satisfying the same boundary conditions (1.2), we must have

$$u(-x_1, x') = Qu(x_1, x'), \quad x\in R^n. \tag{5.2}$$

We shall call (5.2) a reflexion formula. It will be seen that the set $H^p(R^n)$ is so large that for our purpose the condition (5.2) is also sufficient.

In this chapter we assume the polynomials $q_i(D_1)$ in (1.2) to be homogeneous, that is $q_i(D_1)D_1^{\nu_i}$, $i=1, \ldots, p$.

Theorem 5.1. *Let y be a fixed point with $y_1 \neq 0$ and assume that there is a differential operator $Q(D)$ with constant coefficients, and with the property that*

$$u(-y_1, y') = Q(D)\, u(y_1, y') \tag{5.3}$$

for all $u \in H^p(R^n)$ such that u satisfies the boundary conditions

$$D_1^{\nu_i} u = 0, \quad x_1 = 0, \quad i = 1, \ldots, p, \tag{5.4}$$

where $0 \leqq \nu_1 < \nu_2 < \ldots < \nu_p$. Then there is a differential operator Q_1 defined by

$$Q_1(x_1, D)\, u = \sum_{i=0}^{p-1} B_i x_1^{p+i} \Delta^i \left(\frac{u}{x_1^{p-i}} \right) \tag{5.5}$$

with constant B_i such that

$$u(-x_1, x') = Q_1(x_1, D)\, u(x_1, x') \tag{5.6}$$

for all x and all $u \in H^p(R^n)$ satisfying (5.4).

The proof will be given by means of a few lemmas.

Lemma 5.1. *If the assumptions of Theorem* 5.1 *are fulfilled, then there is a differential operator $Q_1(x_1, D)$ of the form*

$$Q_1(x_1, D) \equiv \sum_\alpha \sum_\beta A_{\alpha,\beta} x_1^{\alpha+2\beta} D_1^\alpha \Delta^\beta, \quad \beta < p, \tag{5.7}$$

where the $A_{\alpha,\beta}$ are constants, and such that (5.6) *holds.*

Proof. First we observe that any orthonormal transformation O in the boundary variables x', which keeps y' fix, transforms Q into an operator Q^0 which also satisfies the condition (5.3). Since the set of all orthonormal transformations in the x'-variables is a compact group, it can be equipped with a Haar-measure. See e.g. Weil [14] p. 34. Therefore, if we take the mean value of Q^0 over the set of all orthonormal transformations in the x'-variables by means of an integration with respect to this Haar measure, we obtain an operator Q' which also satisfies (5.3) and which is invariant under orthonormal transformations in the boundary variables.

It is clear that a function whose values depend only on x_1 and $r' = (x_2^2 + \ldots + x_n^2)^{1/2}$ is a function of x_1 and r'. Therefore, a polynomial which is invariant for all orthonormal transformations in the boundary variables x_ι, $\iota = 2, \ldots, n$ is a polynomial in x_1 and r'^2, hence a polynomial in x_1 and $r^2 = x_1^2 + r'^2$. This shows that the mean value operator Q' must be of the form

$$Q'u(y_1, y') = \sum_\alpha \sum_\beta A'_{\alpha,\beta} D_1^\alpha \Delta^\beta u(y_1, y') \tag{5.8}$$

and satisfy (5.3).

We observe further that the conditions (5.4) are invariant for translations in the x'-variables and for contractions. This means that if $u(x_1, x') \in H^p(R^n)$, and u satisfies (5.4), then, for any fixed (x_1, x') with $x_1 \neq 0$, the function $v(z)$ defined by

$$v(z_1, z') = u\left(z_1 \frac{x_1}{y_1}, (z'-y')\frac{x_1}{y_1} + x'\right)$$

is also in $H^p(R^n)$ and satisfies (5.4). Hence we can apply (5.3) and (5.8) to v with $z=y$, and get for any x with $x_1 \neq 0$

$$u(-x_1, x') = v(-y_1, y') = Q'v(y_1, y') = \sum_\alpha \sum_\beta A'_{\alpha,\beta} \left(\frac{x_1}{y_1}\right)^{\alpha+2\beta} D_1^\alpha \Delta^\beta u(x_1, x'),$$

since $\dfrac{\partial v}{\partial z_\iota} = \left(\dfrac{x_1}{y_1}\right) D_\iota u$, $\iota = 1, \ldots, n$. Since y was a fixed point, this proves the lemma.

Remark. It is obvious that the lemma remains true if the boundary conditions (5.4) contain differentials in the boundary variables if only these conditions remain invariant for contractions, for translations and orthogonal transformations in the boundary variables x', e.g., if each condition (5.4) is of the form

$$q_i(D) = \sum_{k+2l=i} a_{k,l}^i D_1^k \Delta^l.$$

We shall now introduce a set of p operators defined by

$$u \to u_i = x_1^{p+i} \Delta^i \left(\frac{u}{x_1^{p-i}}\right), \quad i = 0, 1, \ldots, p-1 \tag{5.9}$$

which were used by Huber in formula (1.6).

Lemma 5.2. *If $u \in H^p(R^n)$, then $u_i \in H^p(R^n)$.*

Proof. Let $u = x_1^j v$ where v is simply harmonic. For $p-1 \geqq j \geqq p-i$ we get

$$u_i = x_1^{p+i} \Delta^i \left(\frac{x_1^j v}{x_1^{p-i}}\right) = x_1^{p+i} \Delta^i (x_1^{j-p+i} v) = 0$$

since for such a j, $i-1 \geqq j-p+i \geqq 0$, and hence $x_1^{j-p+i} v \in H^i$. For $p-i > j \geqq 0$ we get from a repeated use of formula (2.7)

$$u_i = x_1^{p+i} \Delta^i \left(\frac{v}{x_1^{p-i-j}}\right) = x_1^{p+i} \sum_{k=0}^i a_k \frac{1}{x_1^{p-j+k}} D_1^{i-k} v = \sum_{k=0}^i a_k x_1^{i+j-k} D_1^{i-k} v,$$

where a_k are constants. Since $0 \leqq k \leqq i$, $0 \leqq j < p-i$ and $D_1^{i-k} v \in H^1$, we get $0 \leqq i + j - k \leqq p-1$, hence the right hand sum belongs to H^p.

Since the Huber operators (5.9) are linear, the lemma now follows from the Almansi representation of u.

It is evident that the Huber operators (5.9) are linearly independent, and a straightforward computation by means of (2.7) shows that they can be written in the form

$$u_i = \sum_{0\leqq\alpha+\beta\leqq i} A^i_{\alpha,\beta} x_1^{\alpha+2\beta} D_1^\alpha \Delta^\beta u, \quad i = 0, 1, \ldots, p-1, \tag{5.10}$$

where $A^i_{0,i}=1$.

Our third lemma is the converse of this statement.

Lemma 5.3. *If a differential operator of the form* (5.7) *maps* $H^p(R^n)$ *into* $H^p(R^n)$, *it is equivalent to an operator written in the form* (5.5), *using Huber operators only.*

Proof. Take $\xi=(\xi_1, \xi')$ with $\xi_1\neq 0$ and $\langle\xi, \xi\rangle=0$ and set

$$u(x) = \frac{x_1^k}{k!} e^{\langle x, \xi\rangle}. \tag{5.11}$$

We shall prove the lemma by applying the operator Q_1 in (5.7) to the function u in (5.11) for different values of k. We observe that for $k=k_0$, $u\in H^{k_0+1}$, and $u\notin H^{k_0}$, so that a necessary condition for u to belong to H^p is that $k<p$. Let Q_1 defined by (5.7) transform H^p into H^p. We may suppose that $\beta<p$. We denote the upper bound of $(\alpha+\beta)$ in (5.7) by j. Applying (5.7) to (5.11) we get

$$Q_1 u = x_1^{j+k} e^{\langle x, \xi\rangle} \xi_1^j \sum_{\alpha+\beta=j} A_{\alpha,\beta} \frac{2^\beta}{(k-\beta)!} + R(x_1, \xi_1) e^{\langle x, \xi\rangle}, \tag{5.12}$$

where R is a polynomial in x_1 of degree less than $j+k$ in x_1. We shall interpret $2^\beta/(k-\beta)!$ as 0 when $k-\beta<0$. If $k<p$ we have $u\in H^p$, hence by assumption $Q_1 u\in H^p$. This implies that if $j+k\geqq p$, the coefficient of $x_1^{j+k} e^{\langle x, \xi\rangle}$ must be zero, that is

$$\sum_{\alpha+\beta=j} A_{\alpha,\beta} \frac{2^\beta}{(k-\beta)!} = 0. \tag{5.13}$$

First suppose that $j\geqq p$. Putting $k=0$, we get that the sum (5.13) reduces to one term with $\beta=0$. Hence $A_{j,0}=0$, and continuing with $k=1, \ldots, p-1$, we get recursively that all $A_{\alpha,\beta}=0$, $\alpha+\beta=j\geqq p$. Hence we may assume that $j<p$. Applying (5.7) to the functions (5.11) for $k=p-j, p-j+1, \ldots, p-1$, we again infer that the coefficient (5.13) of $\xi_1^j x_1^{j+k} e^{\langle x, \xi\rangle}$ in (5.12) must be zero for each $k\geqq p-j$, that is, we get a system of j linear equations in the $j+1$ unknowns $A_{\alpha,\beta}$, $\alpha+\beta=j$. The

matrix of the coefficients has the rank j. Indeed the matrix is

$$\begin{pmatrix} \frac{2^0}{(p-j)!} & \cdots & \frac{2^j}{(p-j-j)!} \\ \vdots & \cdots & \\ \frac{2^0}{(p-1)!} & \cdots & \frac{2^j}{(p-1-j)!} \end{pmatrix}.$$

The determinant of the first j columns is easily transformed to

$$\frac{2^{1+2+\ldots+(j-1)}}{(p-1)!\ldots(p-j)!}\cdot\begin{vmatrix} 1 & p-j & \ldots & (p-j)^{j-1} \\ 1 & p-j+1 & & (p-j+1)^{j-1} \\ \vdots & \vdots & & \vdots \\ 1 & p-1 & & (p-1)^{j-1} \end{vmatrix}$$

which is a Van der Monde determinant. Thus the determinant is $\neq 0$. This means that there is exactly one degree of freedom among the $A_{\alpha,\beta}$, $\alpha+\beta=j$, for each $j=p-1, p-2, \ldots, 1, 0$.

In other words: once we have choosen the value of e.g. $A_{0,j}$, the value of all $A_{\alpha,\beta}$, $\alpha+\beta=j$ are determined.

Now consider the Huber operator $B_j u_j = B_j x_1^{p+j} \Delta^j (u/x_1^{p-j})$.

(5.10) shows that $B_j u_j$ can be written in the form (5.7) with $A_{\alpha,\beta}=0$ if $\alpha+\beta>j$, and we have $A_{0,j}=B_j$.

Since, by Lemma 5.2, a Huber operator transforms H^p into H^p, we obtain from (5.12).

$$Q_1 u = A_{0,j} x_1^{p+j} \Delta^j (u/x_1^{p-j}) + Q_1' u,$$

where Q_1' is also of the form (5.7), transforms H^p into H^p and only contains terms with $\alpha+\beta<j$. We can therefore iterate the procedure for $j=p-1, p-2, \ldots, 0$, and finally get, with $B_i=A_{0,i}$,

$$Q_1 u = \sum_{i=0}^{p-1} B_i x_1^{p+i} \Delta^i (u/x_1^{p-i})$$

which was to be proved.

To prove Theorem 5.1 it is now sufficient to note that it follows from Theorem 4.1 that if $Qu \in H^p$ for those $u \in H^p$ which satisfy the boundary conditions (5.4), then $Qu \in H^p$ for all $u \in H^p$, that is Q maps $H^p(R^n)$ into $H^p(R^n)$.

It is readily seen that Q_1 is invariant if x_1 is replaced by $-x_1$. Using (5.6) twice, we obtain

$$Q^2 u = u, \quad u \in H^p, \tag{5.14}$$

provided that u satisfies (5.4). Because of Theorem 4.1, (5.14) must then hold for all p-harmonic functions.

Definition 5.1. To each operator Q of the form (5.5) we define p numbers $C_\alpha(Q)$ by

$$C_\alpha(Q) = \sum_{i=0}^{\alpha} \frac{(\alpha+i)!}{(\alpha-i)!} B_i, \quad \alpha = 0, 1, \ldots, p-1.$$

Theorem 5.2. *A differential operator Q of the form* (5.5) *has the property* (5.14) *if and only if*

$$C_\alpha(Q) = \pm 1, \quad \alpha = 0, \quad 1, \ldots, p-1. \tag{5.15}$$

Proof. For the proof we shall consider the p-harmonic functions x_1^k $0 \leqq k < 2p$. We first prove the "only if". To do so, note that an elementary computation gives

$$Q(x_1, D) x_1^{p+\alpha} = C_\alpha(Q) x_1^{p+\alpha}, \quad 0 \leqq \alpha < p, \tag{5.16}$$

and

$$Q(x_1, D) x_1^{p-1-\alpha} = C_\alpha(Q) x_1^{p-1-\alpha}, \quad 0 \leqq \alpha < p. \tag{5.17}$$

Hence, applying either (5.16) or (5.17) in (5.14) we obtain the necessity of (5.15).

Next we prove the sufficiency. Since $Q^2 u - u \in H^p$ for all $u \in H^p$, and since $Q^2 u - u$ is obviously of the form (5.7) except for terms containing Δ^p as a factor, it follows from Lemma 5.3 that with constant A_i

$$Q^2 u - u = \sum_{i=0}^{p-1} A_i x_1^{p+i} \Delta^i \left(\frac{u}{x_1^{p-i}} \right), \quad u \in H^p. \tag{5.18}$$

Now we have $Q^2 u - u = 0$ if $u(x) = x_1^{p-1-\alpha}$ $0 \leqq \alpha < p$, in view of (5.17). Applying (5.18), to $x_1^{p-1-\alpha}$ for $\alpha = 0, 1, \ldots, p-1$ we get successively that $A_0 = 0, A_1 = 0, \ldots, A_{p-1} = 0$. This completes the proof.

The theorem implies that there are exactly 2^p sets of coefficients B_i, each defining an operator Q of the form (5.5), satisfying (5.14), and transforming H^p into H^p. For each p, the set of 2^p operators mentioned in this theorem will be denoted T^p.

6. Reflection formulas and the corresponding boundary conditions

Let Ω and $\underline{\Omega}$ be domains as defined in the introduction. Let $u \in H^p(\Omega)$ and let Q be an operator such that $Qu \in H^p(\Omega)$ for all $u \in H^p(\Omega)$. Define a function $u^\sim$ in $\Omega \cup \underline{\Omega}$ by

$$u^\sim(x) = \begin{cases} u(x_1, x') & x \in \Omega \\ Qu(-x_1, x') & x \in \underline{\Omega} \end{cases} \tag{6.1}$$

Then $u^\sim$ is p-harmonic in Ω and in $\underline{\Omega}$, and it is clear that a necessary and sufficient condition for $u^\sim$ to be p-harmonic in $\Omega_1 = \Omega \cup \omega \cup \underline{\Omega}$ is that $u^\sim$ is continuous over ω together with the $2p-1$ first normal derivatives.

If we denote

$$q_j(D)u(0,x') = \lim_{x_1\to 0}[D_1^j u(x_1,x') - (-1)^j D_1^j Qu(x_1,x')], \quad j = 0, 1, \ldots \tag{6.2}$$

then this condition becomes

$$q_j(D)u(0,x') = 0, \quad j = 0, 1, \ldots, 2p-1 \quad \text{and} \quad u \in C^{2p}(\Omega\cup\omega). \tag{6.3}$$

When (6.3) is satisfied, $u^{\sim}$ is a p-harmonic extension of u into Ω_1. Since $u^{\sim}$ is then analytic in Ω_1, it follows that $q_j(D)u(0,x')=0$ for $j \geqq 2p$ also.

From (6.2) we get $2p$ conditions on u over ω and since the $q_j(D)$ obviously depend only on Q, we shall call them the boundary conditions corresponding to Q.

We shall here only be concerned with operators $Q \in T^p$.

Lemma 6.1. *Let $Q \in T^p$. Then a boundary condition $q_j(D)$, corresponding to Q, is a homogeneous differential polynomial of order j, $j=0, 1, \ldots$. It can be written in the form:*

$$q_j(D) = \sum_{l=0}^{[j/2]} a_{j,l} D_1^{j-2l} \Delta'^l \tag{6.4}$$

where $[j/2]$ denotes the integer part of $j/2$.

If $C_\alpha(Q)(-1)^{p+\alpha} = -1$, then $a_{p+\alpha,0}=2$ and $a_{p-\alpha-1,0}=0$.

If $C_\alpha(Q)(-1)^{p-\alpha-1} = -1$, then $a_{p+\alpha,0}=0$ and $a_{p-\alpha-1,0}=2$.

Proof. We note that

$$\lim_{x_1\to 0} D_1^j x_1^{\alpha+2\beta} D_1^\alpha \Delta^\beta u = \begin{cases} 0 & \text{if } j < \alpha+2\beta \\ (\alpha+2\beta)!\binom{j}{\alpha+2\beta} D_1^{j-2\beta}\Delta^\beta u & \text{if } j \geqq \alpha+2\beta, \end{cases}$$

which is homogeneous of order j. Since all terms of $Q \in T^p$ are of this type, it is clear that $q_j(D)$ is also homogeneous of order j. And since the boundary differentials only appear as $\Delta' = (\Delta - D_1^2)$ we see that $q_i(D)$ has the form (6.4).

To show the second part of the lemma, we use the p-harmonic functions

$$u_\alpha = x_1^{p+\alpha}, \qquad \alpha = 0, 1, \ldots, p-1$$

and

$$v_\alpha = x_1^{p-\alpha-1}, \quad \alpha = 0, 1, \ldots, p-1.$$

Since $Q \in T^p$, (5.16) shows that

$$q_{p+\alpha}(D)u_\alpha(0) = \lim_{x_1\to 0}[D_1^{p+\alpha}x_1^{p+\alpha} - (-1)^{p+\alpha} C_\alpha(Q) D_1^{p+\alpha} x_1^{p+\alpha}].$$

Let $C_\alpha(Q)(-1)^{p+\alpha} = -1$. Then

$$q_{p+\alpha}(D)u_\alpha(0) = 2\cdot(p+\alpha)!$$

On the other hand we obtain from (6.4)

$$q_{p+\alpha}(D)u_\alpha(0) = a_{p+\alpha,0}\cdot(p+\alpha)!$$

Hence $a_{p+\alpha,0}=2$. Repeating the argument for $C_\alpha(Q)(-1)^{p-\alpha-1}=-1$, (i.e. $C_\alpha(Q)(-1)^{p+\alpha}=+1$), we obtain

$$q_{p+\alpha}(D)u_\alpha(0) = a_{p+\alpha,0}\cdot(p+\alpha)! = 0.$$

Hence $a_{p+\alpha,0}=0$.

The statement about $a_{p-\alpha-1,0}$ is proved in the same way by means of the functions v_α.

For a special Q then number of conditions (6.3) may be reduced by two reasons. It may be that for some j, q_j reduces to identically zero. It may also be that some of the conditions (6.3) are consequences of the others in the following way. Suppose that there are differential polynomials $s_{j,k}(D')$ in the boundary differentials only, such that for, $0\leqq j<2p$, we have

$$q_j(D) = \sum\nolimits_{k\in N} s_{j,k}(D')q_k(D) \tag{6.5}$$

where N is a set of integers with at least some $j\notin N$. In that case $q_j(D)u(0, x')=0$ for all u such that $q_k(D)u(0, x')=0$, all $k\in N$, since $s_{j,k}(D')$ differentiates in the boundary variables only.

These two cases may be treated as one by permitting the polynomials $s_{j,k}$ to vanish identically.

Since, by Lemma 6.1, the boundary differentials always appear as Δ', the $s_{j,k}(D')$ can be regarded as polynomials in the single variable Δ', i.e., $s_{j,k}(\Delta')$.

Definition 6.1. An operator $Q\in T^p$ is said to be in S^p if there are p numbers v_i, $i=1, \dots, p$ and differential polynomials $s_{j,v_i}(\Delta')$ such that the boundary conditions corresponding to Q satisfy

$$q_j(D) = \sum\nolimits_{i=1}^{p} s_{j,v_i}(\Delta')q_{v_i}(D), \quad 0\leqq j<2p.$$

The set $\{v_i|i=1, \dots, p\}$, will be denoted $N_1(Q)$.

Lemma 6.2. *Let $Q\in S^p$ and let $u\in H^p(R^n)$ satisfy the corresponding p boundary conditions:*

$$q_j(D)u(0, x') = 0 \quad j\in N_1(Q). \tag{6.6}$$

Then

$$u(-x_1, x') = Qu(x_1, x'). \tag{6.7}$$

Proof. Let $u\in H^p(R^n)$ satisfy (6.6). Let the restriction of u to R^n_+ be continued into the whole of R^n by means of (6.1). Since $Q\in S^p$, the conditions (6.3) are fulfilled. (6.7) is then a consequence of the uniqueness of the analytic continuation.

Example 6.1. Let $Qu\equiv -u$. We get from (6.2) that the corresponding boundary conditions are

$$q_j(D)u = \lim_{x_1\to 0}[D_1^j u+(-1)^j D_1^j u].$$

That is: $q_j(D)\equiv 2D_1^j$ for $j=2j', j'=0, \dots, p-1$ and $q_j(D)\equiv 0$ for $j=2j'+1$, $j'=0, \dots, p-1$.

Hence $Q\equiv -1$ belongs to S^p, all p, and $N_1(Q)$ consists of the first p even numbers $\{j=2j' \mid j'=0, \dots, p-1\}$.

For $Qu\equiv u$, we get in the same way that the corresponding boundary conditions are $q_j(D)=0$ for $j=2j', j'=0, \dots, p-1$, and $q_j(D)=2D_1^j$ for $j=2j'+1$, $j'=0, \dots, p-1$. $Q\equiv +1$ is also in S^p for all p and $N_1(Q)$ is the set

$$\{j = 2j'+1 \mid j' = 0, \dots, p-1\}.$$

The boundary conditions in Example 6.1 suggest the following definition.

Definition 6.2. For each p we define a set of operators M^p by the following condition. $Q\in T^p$ belongs to M^p if there are p numbers ν_i, such that the boundary conditions corresponding to Q can be written

$$q_j(D) = \sum_{i=1}^{p} s_{j,\nu_i}(\Delta')D_1^{\nu_i} \quad 0 \leqq j < 2p \tag{6.8}$$

where $s_{j,\nu_i}(\Delta')$ are zero or non-zero polynomials in Δ'.

The set $\{\nu_i | i=1, \dots, p\}$, will be denoted $N_2(Q)$.

M^p is not void, since at least the two operators in Example 6.1 belong to M^p. The operator given in (1.6) is also in M^p.

There is of course a lemma for M^p corresponding to Lemma 6.2.

Lemma 6.3. *Let $Q\in M^p$ and let $u\in H^p(R^n)$ satisfy the p conditions*

$$D_1^j u(0, x') = 0, \quad j\in N_2(Q).$$

Then $u(-x_1, x')=Qu(x_1, x')$, $x\in R^n$.

The proof is obvious.

We state the next lemma both for $Q\in S^p$ and, within parenthesis, for $Q\in M^p$.

Lemma 6.4. *Let $Q\in S^p$ (respectively $Q\in M^p$).*

Then

$$C_\alpha(Q)\cdot(-1)^{p+\alpha} = -1 \Leftrightarrow p+\alpha\in N_1(Q),\ \big(p+\alpha\in N_2(Q)\big), \quad \alpha = 0, \dots, p-1 \tag{6.9}$$

and

$$C_\alpha(Q)\cdot(-1)^{p-1-\alpha} = -1 \Leftrightarrow p-1-\alpha\in N_1(Q),\ \big(p-1-\alpha\in N_2(Q)\big), \quad \alpha = 0, \dots, p-1. \tag{6.10}$$

According to this lemma, only one of the two numbers $p+\alpha$ and $p-1-\alpha$ can belong to $N_1(Q)$, $\big(N_2(Q)\big)$.

Proof. As in Lemma 6.1, we consider the functions $u_\alpha = x_1^{p+\alpha}$ and obtain by means of (5.16)

$$q_j(D)u_\alpha(0) = \lim_{x_1 \to 0} [D_1^j x_1^{p+\alpha} - (-1)^j D_1^j C_\alpha(Q) x_1^{p+\alpha}]$$

Let $C_\alpha(Q)(-1)^{p+\alpha} = -1$. Then

$$\begin{cases} q_j(D)\,u_\alpha(0) = 0 & j \neq p+\alpha \\ q_{p+\alpha}(D)u_\alpha(0) = 2\cdot(p+\alpha)! \end{cases} \tag{6.11}$$

Let $Q \in S^p$ and assume that $p+\alpha \notin N_1(Q)$. Then the conditions of Lemma 6.2 are fulfilled, but

$$Qu_\alpha(x_1) = C_\alpha(Q)x_1^{p+\alpha} = -(-x_1)^{p+\alpha} \neq u_\alpha(-x_1).$$

Hence

$$C_\alpha(Q)(-1)^{p+\alpha} = -1 \Rightarrow p+\alpha \in N_1(Q), \quad \alpha = 0, \dots, p-1.$$

The same argument applied to the functions $v_\alpha = x_1^{p-\alpha-1}$, shows that

$$C_\alpha(Q)(-1)^{p-\alpha-1} = -1 \Rightarrow p-1-\alpha \in N_1(Q), \quad \alpha = 0, \dots, p-1.$$

Since this already makes p elements in $N_1(Q)$, the assumption $Q \in S^p$ proves the implication from right to left in (6.9) and (6.10).

This proves the Lemma for $Q \in S^p$.

To prove the lemma for $Q \in M^p$, it is sufficient to note that

$$D_1^j u_\alpha(0) = 0, \quad j \neq p+\alpha.$$

The rest of the argument will be the same as above, except that we appeal to Lemma 6.3 instead of Lemma 6.2.

It is an immediate consequence of Lemma 6.4 that if Q belongs to both S^p and M^p, then $N_1(Q) = N_2(Q)$.

A comparison of Lemma 6.4 and 6.1 shows

Corollary 6.1. *Let $Q \in S^p$ and $j \in N_1(Q)$. Then in* (6.4), $a_{j,0} = 2$.

Proof. Let $j = p+\alpha \in N_1(Q)$. Then by Lemma 6.4, $C_\alpha(Q)(-1)^{p+\alpha} = -1$ and by Lemma 6.1, $a_{p+\alpha,0} = 2$. The argument holds also for $j = p-\alpha-1$.

Lemma 6.5. *Let $Q \in S^p$. Then there are polynomials $s_{j,k}(\Delta')$ in Δ' only and $s_j(D_1, \Delta')$ such that the boundary conditions $q_j(D)$ corresponding to Q satisfy*

$$q_j(D) = \textstyle\sum_{k \in N_1(Q)} s_{j,k}(\Delta')q_k(D) + s_j(D_1, \Delta')\Delta^p, \quad \text{all} \quad j \geqq 0. \tag{6.12}$$

Proof. Let $u \in H^p(R_+^n)$. Define $u^\sim$ by means of (6.1). Let u satisfy the p boundary conditions

$$q_k(D)\,u(0, x') = 0, \quad k \in N_1(Q).$$

Then $u^{\sim}$ is analytic in R^n. Hence

$$q_j(D)u(0, x') = 0 \quad \text{all} \quad j \geqq 0.$$

Because of Lemma 6.1 and Corollary 6.1, Theorem 4.2 is applicable and proves the result.

In the same way we can prove the corresponding lemma for $Q \in M^p$ by means of Corollary 4.1.

We only state the lemma.

Lemma 6.6. *Let $Q \in M^p$. Then there are differential polynomials $s_{j,k}(\Delta')$ and $s_j(D_1, \Delta')$ such that*

$$q_j(D) = \sum_{k \in N_2(Q)} s_{j,k}(\Delta') D_1^k + s_j(D_1, \Delta')\Delta^p, \quad \text{all} \quad j \geqq 0.$$

The form (5.5) of the operators in T^p suggests the following definition 6.3.

Definition 6.3. To each sum of Huber operators

$$Q(u) = \sum_{i=0}^{p-1} B_i x_1^{p+i} \Delta^i \left(\frac{u}{x_1^{p-i}} \right) \tag{6.13}$$

i.e. transforming H^p into H^p, we define the operator Q^* by

$$Q^* u = -\sum_{i=0}^{p-2} B_i x_1^{p-1+i} \Delta^i \left(\frac{u}{x_1^{p-1-i}} \right). \tag{6.14}$$

It is clear that Q^* is an operator transforming H^{p-1} into H^{p-1}. The boundary conditions (6.2) corresponding to Q^* will be denoted $q_j^*(D)$. If Q^* belongs to S^{p-1} or M^{p-1}, the meaning of $N_1(Q^*)$ and $N_2(Q^*)$ is obvious. Note however that $N_1(Q^*)$, $(N_2(Q^*))$ contains only $p-1$ elements.

Lemma 6.7. *Let $Q \in T^p$. Then $Q^* \in T^{p-1}$. Let $Q \in S^p$ and $Q^* \in S^{p-1}$. Then $k \in N_1(Q^*)$ implies $k+1 \in N_1(Q)$. Similarly if $Q \in M^p$ and $Q^* \in M^{p-1}$, then $k \in N_2(Q^*)$ implies $k+1 \in N_2(Q)$.*

Proof. It is an immediate consequence of Definition 6.3 that for $u = x_1 v$, $v \in H^{p-1}$,

$$Q(x_1 v) = -x_1 Q^* v. \tag{6.15}$$

Hence if $Q \in T^p$,

$$(Q^*)^2 v = \frac{1}{x_1} Q^2(x_1 v) = \frac{1}{x_1} x_1 v = v$$

which proves that $Q^* \in T^{p-1}$.

Let $Q^* \in S^{p-1}$ and $Q \in S^p$, ($Q^* \in M^{p-1}$ and $Q \in M^p$). Suppose that $k = p-1+\alpha \in N_1(Q^*)(N_2(Q^*))$, $\alpha = 0, \ldots, p-2$. Then by Lemma 6.4

$$C_\alpha(Q^*)(-1)^{p-1+\alpha} = -1, \quad \alpha = 0, \ldots, p-2.$$

The definition of Q^* and Definition 5.1 applied to $C_\alpha(Q^*)$ show that

$$C_\alpha(Q^*) = -C_\alpha(Q), \qquad \alpha = 0, \dots, p-2.$$

Hence

$$C_\alpha(Q)(-1)^{p+\alpha} = -1, \qquad \alpha = 0, \dots, p-2$$

which by Lemma 6.4 implies that $k+1=p+\alpha\in N_1(Q)(p+\alpha\in N_2(Q))$.

The proof is similar for $k=p-1-\alpha-1$. To each $Q_1\in T^{p-1}$, there are two operators $Q\in T^p$ such that $Q^*=Q_1$. For one $C_{p-1}(Q)=+1$ and for the other $C_{p-1}(Q)=-1$.

Lemma 6.8. *Let Q be a Huber operator* (6.13) *transforming H^p into H^p. Let $q_j(D)$ be the boundary conditions corresponding to Q and $q_j^*(D)$ be the boundary conditions corresponding to Q^*. Then*

$$\frac{\partial q_j(\xi)}{\partial \xi_1} = j\cdot q_{j-1}^*(\xi), \quad j = 1, \dots, 2p-2$$

and

$$\frac{\partial q_{2p-1}(\xi)}{\partial \xi_1} = (2p-1)\cdot q_{2p-2}^*(\xi)+K\cdot \Delta^{p-1}, \tag{6.17}$$

where K is a constant.

Proof. Take a function $f(x_1, x')$ such that $u=x_1 f\in H^p(R^n)$. Then (6.15) shows, in view of Theorem 4.1 that

$$Q(x_1 f)+x_1 Q^* f = P(x_1, D)\Delta^{p-1} f$$

since the left hand side is zero for $f\in H^{p-1}$. To determine $P(x_1, D)$ we write $Q(x_1 f)$ and $x_1 Q^* f$ in the form (5.10). Since in (5.10) the summation only goes to $\alpha+\beta\leqq p-1$ for Q and $\alpha+\beta\leqq p-2$ for Q^*, we see that there will only be one term containing $\Delta^{p-1}f$, the coefficient of which is $K\cdot x_1^{2p-1}$ where K is a constant.

Hence

$$Q(x_1 f)+x_1 Q^* f = K\cdot x_1^{2p-1}\Delta^{p-1} f \tag{6.18}$$

Now (6.2) gives in view of (6.18)

$$\lim_{x_1\to 0} q_j(D) x_1 f(x_1, x') = \lim_{x_1\to 0}\left[D_1^j x_1 f(x_1, x')-(-1)^j D_1^j Q\big(x_1 f(x_1, x')\big)\right] =$$
$$= \lim_{x_1\to 0}[D_1^j x_1 f(x_1, x')-(-1)^j D_1^j\{-x_1 Q^* f(x_1, x')+K x_1^{2p-1}\Delta^{p-1} f(x_1, x')\}].$$

For $j<2p-1$ this becomes by Leibniz' formula

$$\lim_{x_1\to 0} j\,[D_1^{j-1} f(x_1, x')-(-1)^{j-1} D_1^{j-1} Q^* f(x_1, x')] = j\cdot q_{j-1}^*(D) f(0, x'),$$

and for $j=2p-1$ we obtain

$$(2p-1)\cdot q^*_{2p-2}(D)f(0,x')+K_1\Delta^{p-1}f(0,x')$$

where K_1 is a constant.

Since by Leibniz' formula

$$\lim_{x_1\to 0} q_j(D)x_1 f(x_1,x') = q_j'(D)f(0,x'), \quad j>0$$

where $q_j'(\xi)=\dfrac{\partial q_j(\xi)}{\partial \xi_1}$, the lemma is proved.

Lemma 6.9. *Let Q be an operator in T^p with $C_{p-1}(Q)=-1$ and such that $Q^*\in S^{p-1}$. Then $Q\in S^p$.*

Proof. Since $Q^*\in S^{p-1}$ there are by definition a set $N_1(Q^*)$ of $p-1$ numbers and polynomials $s_{j,k}(\Delta')$ such that the boundary conditions corresponding to Q satisfy

$$j\cdot q^*_{j-1}(D) = \sum\nolimits_{k-1\in N_1(Q^*)} s_{j,k}(\Delta')k\cdot q^*_{k-1}(D), \quad 0\leqq j-1\leqq 2p-3.$$

Hence by Lemma 6.8, the boundary conditions corresponding to Q satisfy

$$q_j(D) = s_j(\Delta')+\sum\nolimits_{k-1\in N_1(Q^*)} s_{j,k}(\Delta')q_k(D), \quad 1\leqq j\leqq 2p-2. \tag{6.19}$$

For $j=0$, $q_0(D)$ is by Lemma 6.1 a constant $a_{0,0}=2$.

Hence, in (6.19), $s_j(\Delta')$ may be written $s_j(\Delta')=s_{j,0}(\Delta')\cdot q_0(D)$. Thus there is a set N of p numbers such that $0\in N$ and $k\in N$ if $k-1\in N_1(Q^*)$ (cf. Lemma 6.7) and such that

$$q_j(D) = \sum\nolimits_{k\in N} s_{j,k}(\Delta')q_k(D) \quad 0\leqq j\leqq 2p-2. \tag{6.20}$$

It remains to show that (6.20) holds for $j=2p-1$, for then $N=N_1(Q)$ and $Q\in S^p$.

From Lemma 6.5 applied to $Q^*\in S^{p-1}$ and $j=2p-2$ we obtain that

$$(2p-1)\cdot q^*_{2p-2}(D) = \sum\nolimits_{k-1\in N_1(Q^*)} s_{2p-2,k}(\Delta')kq^*_{k-1}(D)+K_2\Delta^{p-1}$$

where K_2 is a constant. Then by (6.17)

$$q_{2p-1}(D) = \sum\nolimits_{k\in N} s_{j,k}(\Delta')q_k(D)+K_3\cdot q(D) \tag{6.21}$$

where

$$\frac{\partial q(\xi)}{\partial \xi_1} = (2p-1)\cdot\Delta^{p-1}.$$

Since the order of $q_k(D)$ is $k<2p-1$, $q(D)$ is the only term in (6.21) that contains the term D_1^{2p-1}. Hence, writing $q_{2p-1}(D)$ in the form (6.4), we see that

$$K_3 = a_{2p-1,0}.$$

Since $C_{p-1}(Q)=-1$, $a_{2p-1,0}=K_3=0$ by Lemma 6.1, which proves that (6.20) holds for $0\leqq j\leqq 2p-1$. Hence $Q\in S^p$.

There is of course a corresponding lemma for M^p.

Lemma 6.10. *Let Q be an operator in T^p with $C_{p-1}(Q)=-1$ and such that $Q^*\in M^{p-1}$. Then Q^*M^p.*

Proof. By the definition of M^{p-1}, there is a set $N_2(Q^*)$ of $p-1$ numbers and polynomials $s_{j,k}(\Delta')$ such that the boundary conditions corresponding to Q^* satisfy

$$j\cdot q_{j-1}^*(D)=\sum\nolimits_{k-1\in N_2(Q^*)} s_{j,k}(\Delta')\cdot k\cdot D_1^{k-1}\quad 0\leqq j-1\leqq 2p-3.$$

The proof of Lemma 6.9 can now be repeated to show that the boundary conditions corresponding to Q satisfy

$$q_j(D)=\sum\nolimits_{k\in N_2(Q)} s_{j,k}(\Delta')D_1^k\quad 0\leqq j\leqq 2p-1.$$

Hence $Q\in M^p$.

7. Sufficient conditions on reflection formulas

Up to now the coefficients B_i in (5.5) for an operator $Q\in T^p$ have only been defined implicitly by means of $C_\alpha(Q)$ and the condition (5.15). We shall begin this paragraph by studying them more closely.

Lemma 7.1. *Define a set of Huber operators O_m by*

$$O_m u=(-1)^p\sum\nolimits_{i=m}^{p-1}(-1)^i\frac{1}{(2i)!}\left[\binom{2i}{i-m}-\binom{2i}{i-m-1}\right]x_1^{p+i}\Delta^i\left(\frac{u}{x_1^{p-i}}\right).\tag{7.1}$$

Then each operator Q defined by

$$Qu=\sum\nolimits_{m=0}^{p-1}a_m O_m u,\ a_m=\pm 1,\quad m=0,\dots,p-1\tag{7.2}$$

is one of the operators in T^p. Furthermore

$$a_m=+1\Rightarrow C_m(Q)(-1)^{p+m}=1.\tag{7.3}$$

and

$$a_m=-1\Rightarrow C_m(Q)(-1)^{p-1-m}=1.\tag{7.4}$$

Proof. In the proof of Theorem 5.2, both the necessity and the sufficiency of (5.15) was proved by means of the p-harmonic functions x_1^k, $0\leqq k<2p$. Hence it follows from (5.16) and (5.17) that it is enough to prove that

$$O_m(x_1^{p+\alpha})=\delta_{m,\alpha}(-x_1)^{p+\alpha},\tag{7.5}$$

and

$$O_m(x_1^{p-1-\alpha}) = -\delta_{m,\alpha}(x_1)^{p-1-\alpha}, \tag{7.6}$$

where $\delta_{m,\alpha}=1$ if $m=\alpha$ and $\delta_{m,\alpha}=0$ otherwise.

Indeed, if this is proved, then for a Q defined by (7.2)

$$Qx_1^{p+\alpha} = \sum_{m=0}^{p-1} a_m O_m(x_1^{p+\alpha}) = a_\alpha(-x_1)^{p+\alpha}, \quad \alpha = 0, \ldots, p-1 \tag{7.7}$$

and

$$Qx_1^{p-1-\alpha} = \sum_{m=0}^{p-1} a_m O_m(x_1^{p-1-\alpha}) = -a_\alpha(-x_1)^{p-1-\alpha}, \quad \alpha = 0, \ldots, p-1 \tag{7.8}$$

an comparison with (5.16) and (5.17) proves (7.3) and (7.4) respectively.

A direct computation in (7.1) with $u=x_1^{p+\alpha}$ shows that

$$O_m(x_1^{p+\alpha}) = (-1)^p \sum_{i=m}^{p-1} (-1)^i \frac{2m+1}{(i-m)!(i+m+1)!} \frac{(\alpha+i)!}{(\alpha-i)!} x_1^{p+\alpha}, \quad \alpha = 0, \ldots, p-1 \tag{7.9}$$

where we define $a!=(a!)^{-1}=0$ for a negative. Hence

$$O_m(x_1^{p+\alpha}) = 0 \quad \text{for} \quad m > \alpha, \tag{7.10}$$

since then $i \geqq m > \alpha$ and $(\alpha-i)!^{-1}=0$.

For $m \leqq \alpha$ we get from (7.9, putting $i-m=j$

$$O_m(x_1^{p+\alpha}) = (-1)^{p+m} \frac{2m+1}{(\alpha-m)!} \sum_{j=0}^{p-m-1} (-1)^j \binom{\alpha-m}{j} \frac{(\alpha+j+m)!}{(j+2m+1)!} x_1^{p+\alpha}. \tag{7.11}$$

The coefficient in (7.11) can also be obtained in the following way. Differentiate, by means of Leibniz' rule for the differentiation of a product, the (constant) function

$$K = (-1)^{p+m} \frac{2m+1}{(\alpha-m)!} \cdot \frac{(\alpha+m)!}{(\alpha+m+1)!} \cdot \frac{1}{x_1^{\alpha+m+1}} \cdot x_1^{\alpha+m+1}$$

$(\alpha-m)$ times. Hence

$$O_m(x_1^{p+\alpha}) = 0 \quad \text{for} \quad \alpha > m.$$

From (7.11) we obtain for $\alpha=m$

$$O_\alpha(x_1^{p+\alpha}) = (-1)^{p+\alpha} x_1^{p+\alpha},$$

since $\binom{0}{j}=0$ for $j>0$. This proves (7.5).

If we take $u=x_1^{p-\alpha-1}$, $\alpha=0, \ldots, p-1$, the coefficients in the computations above will remain unchanged. Hence (7.6) and the lemma are proved.

Note. Since there are only 2^p elements in T^p, (7.2) exhausts T^p. If Q defined by (7.2) belongs to S^p, (M^p), then by Lemma 6.4, $a_m=+1$ implies $p-1-m \in N_1(Q)$, $(N_2(Q))$, $m=0, \ldots, p-1$ and $a_m=-1$ implies $p+m \in N_1(Q)$, $(N_2(Q))$.

Lemma 7.2. *Let $O_m u$ be defined by* (7.1). *Then*

$$Qu = u+2\cdot \sum_{m=0}^{p-1}(-1)^{p-1-m} b_m O_m u, \quad b_m = 0 \text{ or } 1, \quad m = 0, \dots, p-1, \tag{7.12}$$

is an operator in T^p.

If $b_m=1$, then $C_m(Q)=-1$, $m=0, \dots, p-1$.

If $b_m=0$, then $C_m(Q)=+1$, $m=0, \dots, p-1$.

Proof. (7.5) shows that with $u=x_1^{p+\alpha}$,

$$Q(x_1^{p+\alpha}) = x^{p+\alpha}+2(-1)^{p-1-\alpha}(-x_1)^{p+\alpha} b_\alpha, \quad \alpha = 0, \dots, p-1.$$

Hence

$$Q(x_1^{p+\alpha}) = x^{p+\alpha} \quad \text{if} \quad b_\alpha = 0$$

and

$$Q(x_1^{p+\alpha}) = -x_1^{p+\alpha} \quad \text{if} \quad b_\alpha = 1.$$

Similarly we obtain from (7.6)

$$Q(x_1^{p-1-\alpha}) = x_1^{p-1-\alpha} \quad \text{if} \quad b_\alpha = 0$$

and

$$Q(x_1^{p-1-\alpha}) = -x_1^{p-1-\alpha} \quad \text{if} \quad b_\alpha = 1.$$

That $b_m=0$ implies $C_m(Q)=+1$ and $b_m=1$ implies $C_m(Q)=-1$ is clear from a comparison with (5.16) and (5.17).

In Theorem 7.1, when we shall characterize the set M^p, we need the coefficients of the operators $Q\in M^p$ for which $N_2(Q)$ consists of the p elements:

$$0, 1, \dots, k, \quad \text{and then} \quad k+2, k+4, \dots, k+2(p-1-k), \qquad 0 \leqq k < p. \tag{7.13}$$

It is seen that this is consistent with the fact that only one of the two numbers $p+m$ and $p-1-m$ can belong to $N_2(Q)$, (Lemma 6.4). Formula (6.10) shows that in order to compute the coefficients B_i of the operator Q corresponding to (7.13) we shall for $0\leqq \nu \leqq k$ take

$$C_{p-\nu-1}(Q)(-1)^\nu = -1.$$

Hence, if we substitute $p-\nu-1=m$, we shall in (7.2) choose

$$a_m = +1, \quad m = p-1, p-2, \dots, p-k-1.$$

For $k<\nu<p$ we shall because of (6.10) take

$$C_{p-\nu-1}(Q)(-1)^\nu = (-1)^{\nu-1-k},$$

hence by (7.3) and (7.4)

$$a_m = (-1)^{p-m-k} = (-1)^{\nu-k}, \quad m = p-k-2, \dots, 0.$$

With these a_m we obtain from (7.1) and (7.2) that the coefficient B_i of

$$x_1^{p+i}\Delta^i\left(\frac{u}{x_1^{p-i}}\right)$$

is

$$B_i = \sum_{m=0}^{p-1} a_m(-1)^{p+i}\frac{1}{(2i)!}\left[\binom{2i}{i-m}-\binom{2i}{i-m-1}\right] =$$

$$=(-1)^{p+i}\frac{1}{(2i)!}\left[\sum_{m=0}^{p-k-1}(-1)^{p-m-k-1}\binom{2i}{i-m}-\sum_{m=1}^{p-k-2}(-1)^{p-m-k-1}\binom{2i}{i-m-1}\right],$$

since the terms for $m>p-k-1$ cancel each other because of the signs of a_m. Put $p-k-1=-l$ in the first sum. In the second sum put $p-k-1=l$ and $m=m'-1$. Then

$$B_i=(-1)^{p+i}\frac{1}{(2i)!}\left[\sum_{m=0}^{-l}(-1)^{m+l}\binom{2i}{i-m}+\sum_{m'=0}^{l}(-1)^{m'+l}\binom{2i}{i-m'}\right] =$$

$$=(-1)^{p+i}\frac{1}{(2i)!}\sum_{m=-l}^{l}(-1)^{m+l}\binom{2i}{i-m} =$$

$$=(-1)^{p+i}\frac{1}{(2i)!}\sum_{m=-l}^{l}\left[\binom{2i-1}{i-m}+\binom{2i-1}{i-m-1}\right](-1)^{m+l} = (-1)^{p+i}\frac{2}{(2i)!}\binom{2i-1}{i+l} \tag{7.14}$$

for $i\neq 0$ and $B_0=(-1)^{1+k}$. Since $2i-1<i+l$ for $i<l+1$, $B_i=0$ for $0<i\leqq l$.

For the special case $k=p-1$, (i.e. $l=0$) this becomes

$$B_i=(-1)^{p+i}\frac{2}{(2i)!}\binom{2i-1}{i}=(-1)^{p+i}\frac{1}{(i!)^2},$$

which we recognize as the coefficients in formula (1.6).

We shall now come to our first main theorem which completely characterizes the set M^p of operators whose boundary conditions are of the form (6.8).

Theorem 7.1. *Let $Q\in M^p$. Then either*

a) $Qu=u$ and $N_2(Q)=\{v_i|v_i=2i+1,\ i=0, \ldots, p-1\}$,

b) or else for some k with $0\leqq k<p$ and $l=p-1-k$

$$Qu=(-1)^p\left[(-1)^l u+\sum_{i=l+1}^{p-1}(-1)^i\frac{2}{(2i)!}\binom{2i-1}{i+l}x_1^{p+i}\Delta^i\left(\frac{u}{x_1^{p-i}}\right)\right], \tag{7.15}$$

and the corresponding $N_2(Q)$ is

$$N_2(Q)=\{v_i|v_i=i,\ i=0,\ldots,k;\ v_i=k+2(i-k),\ i=k+1,\ldots,p-1\}.$$

According to the theorem, M^p as defined above contains $p+1$ elements.

The proof will be split up into a few lemmas. We shall first prove the negative result, that is, M^p cannot contain more than $p+1$ elements.

Lemma 7.3. *For each $Q \in M^p$ we have either $C_{p-1}(Q) = -1$ or $Qu \equiv u$.*

Proof. Suppose that $Q \in M^p$ and let $C_{p-1}(Q) = +1$. Take j such that with Q expressed in the form (5.5), $B_i = 0$ for $i > j$, but $B_j \neq 0$. Of course j can be $p-1$. Let $u(x') \in H^p(R^n)$ be a function of the x' only and such that $\Delta^j u \neq 0$. Then

$$D_1^\nu u(x') = 0 \quad \text{for} \quad \nu > 0.$$

Since, by Lemma 6.4, $C_{p-1}(Q) = 1$ implies $0 \notin N_2(Q)$, the conditions of Lemma 6.3 are fulfilled. Hence

$$Qu(x') = u(x')$$

is independent of x_1.

A direct calculation however, shows that

$$Qu(x') = \sum_{i=0}^{j} B_i x_1^{p+i} \Delta^i \left(\frac{u(x')}{x_1^{p-i}} \right) = B_j x_1^{2j} \Delta^j u(x') + R(x_1, D) u(x'),$$

where R is a polynomial in x_1 and D of degree less than $2j$ in x_1. Hence $Qu(x')$ is not independent of x_1 if $j > 0$. Thus $j = 0$ and $Qu = \pm u$. Since for $Qu \equiv -u$, $C_{p-1}(Q) = -1$, the lemma is proved.

Lemma 7.4. *If $Q \in M^p$, then $Q^* \in M^{p-1}$. (Cf. Lemma 6.10).*

Proof. Let $Q \in M^p$. If $Qu \equiv u$, then $Q^* v \equiv -v$. Example 6.1 shows that the lemma is then valid. In view of Lemma 7.3, we may thus assume that $0 \in N_2(Q)$. By Definition 6.2 there are $p-1$ numbers $\nu_i > 0$, $i = 1, \dots, p-1$ such that the boundary conditions corresponding to Q satisfy

$$q_j(D) = s_{j,0}(\Delta') + \sum_{i=1}^{p-1} s_{j,\nu_i}(\Delta') D_1^{\nu_i}, \quad j = 0, \dots, 2p-1.$$

Lemma 6.8 then shows that the boundary conditions corresponding to Q^* satisfy

$$j \cdot q_{j-1}^*(D) = \sum_{i=1}^{p-1} s_{j,\nu_i}(\Delta') \cdot \nu_i \cdot D_1^{\nu_i - 1}, \quad j-1 = 0, \dots, 2p-2.$$

Hence $Q^* \in M^{p-1}$.

Proof of Theorem 7.1. From these two lemmas it follows by induction that M^p contains at most $p+1$ elements. Indeed, $M^p \subset T^p$ and T^p contains 2^p elements. For $p=1$ we have $p+1 = 2^p$. Hence the assertion is true for $p=1$. Suppose that M^j contains $j+1$ elements. Then by Lemma 7.4, M^{j+1} cannot contain more than $2(j+1)$ elements. However, only $j+1$ of these Q have $C_j(Q) = -1$. Therefore by Lemma 7.3, M^{j+1} contains at most $j+2$ elements. The positive result is also proved by induction. For $p=1$, $M^p = T^p$ consists of the two elements $Qu=$

$\pm u$, hence the theorem is true for $p=1$. Case a of the theorem is true for all p by Example 6.1. Suppose that case b has been proved for $p=j$. Then Lemma 6.7 and Lemma 6.10 show that the statement about the sets $N_2(Q)$ is true for $p=j+1$ also, since this implies that $C_j(Q)=-1$ and $Q^*\in M^j$. The computation of the coefficients B_i of the corresponding Q was effected in (7.14).

The results now obtained may be used to prove the existence of an analytic continuation of a function in $H^p(\Omega)$.

Theorem 7.2. *Let Ω and ω be sets as defined in the introduction. Let Q be an operator in M^p, and let $u\in H^p(\Omega)$ satisfy in the limit the p boundary conditions*

$$\lim_{x_1\to+0} D_1^{\nu_i}u(x_1, x') = 0,\ (0, x')\in\omega,\ \nu_i\in N_2(Q),\quad i = 1, \ldots, p-1.$$

Then the function $u\tilde{}(x_1, x')$ defined by

$$u\tilde{}(x_1, x') = \begin{cases} u(x_1, x'), & x\in\Omega \\ \lim\limits_{x_1\to+0} u(x_1, x'), & x\in\omega \\ Qu(-x_1, x'), & x\in\underline{\Omega} \end{cases}$$

is a polyharmonic extension of u into Ω_1.

Proof. It follows from Theorem 3.1 that u can be analytically continued across $x_1=0$. Hence $u\tilde{}(x_1, x')\in C^\infty(\Omega\cup\omega)$. Furthermore it follows that $u\tilde{}$ is p-harmonic in Ω and in $\underline{\Omega}$. Because of Theorem 7.1, $u\tilde{}$ has 2^{p-1} continuous derivatives over ω. Hence $u\tilde{}\in H^p(\Omega_1)$.

Remark. The formula (7.15) may be transformed into the following form, more similar to formula (1.6)

$$Qu = (-1)^p \sum_{i=0}^{p-1} \frac{(-1)^i}{(i!)^2} \prod_{j=0}^{l-1} \frac{(i-j)}{(i+j)} x_1^{p+i} \Delta^i\left(\frac{u}{x_1^{p-i}}\right).$$

It is seen immediately that (1.4) corresponds to the special case $l=1$.

The operators in T^p which are not in M^p have boundary conditions of more complicated structure. Still we have that they are in S^p, that is, p boundary conditions on $u\in H^p(\Omega\cup\omega)$ are enough to ensure that u can be continued into Ω_1 by Q.

Theorem 7.3. *Let $Q\in T^p$. Then $Q\in S^p$.*

We shall prove the theorem by induction over p.

Since the result is trivial when $p=1$, it is, in view of Lemma 6.9, enough to prove.

Lemma 7.5. *Let Q be an operator in T^p with $C_{p-1}(Q)=+1$ and such that $Q^*\in S^{p-1}$. Then $Q\in S^p$.*

Proof. Following the proof of Lemma 6.9, we know that the assumptions of the lemma imply that the boundary conditions $q_j(D)$ corresponding to Q satisfy (6.19) for $1 \leqq j \leqq 2p-2$.

From Lemma 6.1 it follows that $q_0(D) \equiv a_{0,0} = 0$, since $C_{p-1}(Q) = +1$. We need not bother with $q_{2p-1}(D)$, since if $Q \in S^p$ and $C_{p-1}(Q) = +1$, then by Lemma 6.4, $2p-1 \in N_1(Q)$.

Hence it is enough to prove that in (6.19), $s_{j,0}(\Delta') \equiv 0$, $1 \leqq j \leqq 2p-2$, for then by Definition 6.1, $Q \in S^p$.

Since Δ' is of order two, and since $q_j(D)$ is homogeneous of order j, a term $s_{j,0}(\Delta')$ can only appear for j even, $j = 2h$. Formula (6.2) and Lemma 7.2 show that for $j = 2h$

$$q_{2h}(D)u = \lim_{x_1 \to 0} \left[-2 \sum_{m=0}^{p-1} (-1)^{p-1-m} b_m D_1^{2h} O_m u\right] \tag{7.16}$$

where we may stop the summation at $m = p-2$, since $C_{p-1}(Q) = +1$ implies $b_{p-1} = 0$.

By the same argument as was used in Lemma 6.1, $o_{m,j}(D)$, defined by

$$o_{m,j}(D)u = \lim_{x_1 \to 0} D_1^j O_m u, \tag{7.17}$$

is a homogeneous differential polynomial of order j. Hence we may write (for $j = 2h$),

$$o_{m,2h}(D) = \sum_{l=0}^{h} a_{2h,l}^{m} \Delta'^{l} D_1^{2(h-l)}, \tag{7.18}$$

where the $a_{2h,l}^m$ are constants. We obtain from (7.16),

$$q_{2h}(D) = -2 \sum_{m=0}^{p-2} (-1)^{p-1-m} b_m o_{m,2h}(D).$$

For the coefficients $a_{2h,l}^m$ in (7.18), we shall now prove

Lemma 7.6. *For each operator O_m, $m \neq p-1$, and all $j = 2h$, $h > 0$,*

$$a_{2h,h}^m + K_m \cdot a_{2h,h-1}^m = 0$$

where $K_m = \frac{1}{2}(p+m)(p-1-m)$ is independent of h, $h > 0$.

We shall first see how this lemma proves Lemma 7.5.

Since $K_m \neq 0$, $m \neq p-1$, the following can be written as one term in the expression (7.18) of $o_{m,2h}$

$$a_{2h,h-1}^m \Delta'^{h-1}(D_1^2 - K_m \Delta')$$

for K_m is independent of h. Since this is true for each $m \neq p-1$, it must also be true for the sum

$$-2 \sum_{m=0}^{p-2} (-1)^{p-1-m} b_m o_{m,2h}(D) = q_{2h}(D)$$

that, with the same notation, no term Δ'^h will appear alone. Hence in the expression (6.19) of $q_{2h}(D)$, $s_{2h,0}(\Delta') \equiv 0$ for all h, since $s_{2h,0}(\Delta')$ does not contain D_1^2.

Proof of Lemma 7.6. The verification of the lemma consists of a trivial, though a bit lengthy computation.

First, let us determine $a^m_{2h,h}$, that is the coefficient of $\Delta'^h D^0_1$ in (7.18). If we write the operator $B_i x_1^{p+i}\Delta^i\left(\frac{u}{x_1^{p-i}}\right)$ in the form (5.10), we see that the coefficient $A^i_{0,h}$ of $x_1^{2h}D_1^0\Delta^h u$ is

$$A^i_{0,h} = B_i\frac{(p+i-2h-1)!}{(p-i-1)!}\binom{i}{h} \tag{7.19}$$

Then we observe that

$$\lim_{x_1\to 0} D_1^{2h}A^i_{0,h}x_1^{2h}\Delta^h u = (2h)!\,A^i_{0,h}\Delta^h u$$

and observe that in the binomial expression of $C\cdot\Delta^h = C(D_1^2+\Delta')^h$, the coefficients of Δ'^h and Δ^h are equal. Hence (7.1), (7.17) and (7.19) show that in (7.18)

$$a^m_{2h,h} = (-1)^p\sum_{i=m}^{p-1}(-1)^i\frac{(2h)!}{(2i)!}\left[\binom{2i}{i-m}-\binom{2i}{i-m-1}\right]\binom{i}{h}\frac{(p+i-2h-1)!}{(p-i-1)!}.$$

To compute $a^m_{2h,h-1}$ we may proceed as follows.

Define O^*_m and O^{**}_m by means of a repetition of Definition 6.3.

Note that Definition 6.3 and Lemma 6.8 do not require that $Q\in T^p$, only that Q is a sum of Huber operators transforming H^p into H^p.

We then define $o^*_{m,2h-1}(D)$ by

$$o^*_{m,2h-1}(D)u = \lim_{x_1\to 0} D_1^{2h-1}O^*_m u$$

and $o^{**}_{m,2h-2}(D)$ correspondingly.

A reproduction of the proof of Lemma 6.8 shows that

$$\frac{\partial}{\partial\xi_1}o_{m,2h}(\xi) = -2h\,o^*_{m,2h-1}(\xi),\quad 1\leqq 2h < 2p-1.$$

(The only difference in the definition of $q_{2h}(D)$ and $o_{m,2h}(D)$ is the term D_1^{2h} and the sign $(-1)^{2h}$.)

Hence by repetition

$$\frac{\partial^2}{\partial\xi_1^2}o_{m,2h}(\xi) = 2h(2h-1)o^{**}_{m,2h-2}(\xi),\quad 2\leqq 2h < 2p-2. \tag{7.20}$$

Since the coefficient of $x_1^{2h-2}D_1^0\Delta^{h-1}$ in the expansion (5.10) of

$$\left(x_1^{p+i}\Delta^i\left(\frac{u}{x_1^{p-i}}\right)\right)^{**} = x_1^{p+i-2}\Delta^i\left(\frac{u}{x_1^{p-i-2}}\right)$$

is

$$\binom{i}{h-1}\frac{(p+i-1-2h)!}{(p-i-3)!},$$

we obtain that the coefficient of Δ'^{h-1} in $o^{**}_{m,2h-2}(D)$ is

$$(-1)^p \sum_{i=m}^{p-1}(-1)^i \frac{(2h-2)!}{(2i)!}\left[\binom{2i}{i-m}-\binom{2i}{i-m-1}\right]\binom{i}{h-1}\frac{(p+i-1-2h)!}{(p-i-3)!}$$

Because of (7.20) this gives for the coefficient $a^m_{2h,h-1}$ of $\Delta'^{h-1}D_1^2$ in $o_{m,2h}(D)$

$$a^m_{2h,h-1} = (-1)^p \sum_{i=m}^{p-1}(-1)^i \frac{(2h)!}{2(2i)!}\left[\binom{2i}{i-m}-\binom{2i}{i-m-1}\right]\binom{i}{h-1}\frac{(p+i-1-2h)!}{(p-i-3)!}$$

where we use the convention that $(a!)^{-1}=0$ for $a<0$.

To show the lemma, we compute

$$S = a^m_{2h,h-1}+\frac{1}{2}(p+m)(p-1-m)a^m_{2h,h}$$

$$= (-1)^p \sum_{i=m}^{p-1}(-1)^i \frac{(2m+1)(p+i-2h-1)!2h!}{(i+m+1)!(i-m)!(p-i-3)!\cdot 2}$$

$$\times\left[\binom{i}{h-1}+\binom{i}{h}\frac{(p+m)(p-1-m)}{(p-i-2)(p-i-1)}\right]$$

for $\binom{2i}{i-m}-\binom{2i}{i-m-1} = \frac{(2m+1)(2i)!}{(i+m+1)!(i-m)!}$. Since also

$$\frac{(p+m)(p-1-m)}{(p-i-2)(p-i-1)} = 1+2\frac{i+1}{p-i-2}+\frac{(i+1+m)(i-m)}{(p-i-1)(p-i-2)},$$

we obtain from the expression in the brackets

$$\binom{i}{h-1}+\binom{i}{h}\left\{1+2\frac{i+1}{p-i-2}+\frac{(i+1+m)(i-m)}{(p-i-1)(p-i-2)}\right\} = \binom{i+1}{h}+2\binom{i+1}{h}\frac{i+1-h}{p-i-2}$$

$$+\binom{i}{h}\frac{(i+1+m)(i-m)}{(p-i-1)(p-i-2)} = \binom{i+1}{h}\frac{p+i-2h}{p-i-2}+\binom{i}{h}\frac{(i+1+m)(i-m)}{(p-i-1)(p-i-2)}.$$

Hence

$$S = (-1)^p\left[\sum_{i=m}^{p-2}(-1)^i\frac{(2m+1)(p+i-2h)!(2h)!}{(i+m+1)!(i-m)!(p-i-2)!2}\binom{i+1}{h}\right.$$

$$\left.+\sum_{i=m+1}^{p-1}(-1)^i\frac{(2m+1)(p+i-2h-1)!(2h)!}{(i+m)!(i-m-1)!(p-i-1)!2}\binom{i}{h}\right]$$

The deleted terms $i=p-1$ and $i=m$ are zero, since $(p-i-2)!^{-1}=0$ for $i=p-1$, and similarly $(i-m-1)!^{-1}=0$ for $i=m$.

Substituting $i\to i'+1$ in the last sum we obtain $S=0$, which proves the lemma.

There is also in this case a Theorem corresponding to Theorem 7.2.

Theorem 7.4. *Let Ω and ω be sets as defined in the introduction. Let Q be an operator in T^p (and hence in S^p), and let $q_j(D)$ be the boundary conditions corresponding to Q. Further let $u\in C^{2p-1}(\Omega\cup\omega)$ and $u\in H^p(\Omega)$ satisfy the p boundary conditions:*

$$q_j(D)u(0, x') = 0 \quad j\in N_1(Q).$$

Then $u^{\sim}(x_1, x')$ defined by

$$u^{\sim}(x_1, x') = \begin{cases} u(x_1, x') & x\in\Omega\cup\omega \\ Qu(-x_1, x') & x\in\underline{\Omega} \end{cases}$$

is a polyharmonic extension of u into Ω_1.

The proof is evident. It follows the proof of Theorem 7.2.

8. The MacLaurin expansion and continuation formulas

Let Ω and ω be defined as in the introduction. Let $u\in H^p(\Omega)$ satisfy the set of boundary conditions

$$\lim_{x_1\to 0} D_1^{\nu_i} u(0, x') = 0, \quad i = 1, \dots, p. \tag{8.1}$$

If the set $\{\nu_i\}$ is one of the sets mentioned in Theorem 7.1, then we have already obtained the reflection formula which continues u into the whole of Ω_1. We shall in this section give a method to obtain continuation formulas for other sets $\{\nu_i\}$ in (8.1). Since the corresponding continuation formulas cannot be of purely differential type, the continuation will not necessarily be possible into more than Ω_2.

For all natural numbers p, let Q_p be the operator in formula (1.6). This operator continues any $u\in H^p(\Omega)$ which satisfies on ω the Dirichlet boundary conditions.

Define the operator Q_p^k by

$$Q_p^k(x_1, D)u = \frac{1}{x_1^{k+1}} Q_{p+1}\left[x_1^{k+1} D_1\left(\frac{u}{x_1^k}\right)\right] \tag{8.2}$$

Theorem 8.1. *Let $u\in H^p(\Omega)$ satisfy the p boundary conditions*

$$\lim_{x_1\to 0} D_1^i u(x_1, x') = 0, \quad x\in\Omega, \quad i = 0, 1, \dots, k-1, \quad k+1, \dots, p, \quad 0 \leqq k < p. \tag{8.3}$$

Then the operator Q defined by

$$Qu = x_1^k \int_0^{x_1} Q_p^k(t, D)u\,dt + \frac{(-x_1)^k}{k!} \cdot [\lim_{x_1 \to 0} D_1^k u] \tag{8.4}$$

has the property that

$$u(-x_1, x') = Qu(x_1, x') \quad x \in \Omega'$$

continues u analytically into Ω_2.

Proof. The boundary conditions (8.3) imply by Theorem 3.1 that there exists a continuation of u into Ω_2. Hence u is analytic on ω. We can therefore expand u in MacLaurin series in x_1 (with $x_2, \ldots, x_n$ as parameters) with a positive radius of convergence

$$u(x_1, x') = \sum_{j=0}^{\infty} x_1^j g_j(x'). \tag{8.5}$$

The conditions (8.3) imply that

$$g_j(x') \equiv 0 \quad \text{for} \quad j = 0, 1, k-1, k+1, \ldots, p.$$

Using the expansion (8.5) we see that the function u_1 defined by

$$u_1 = x_1^{k+1} D_1 \left(\frac{u}{x_1^k}\right) \tag{8.6}$$

satisfies the $p+1$ Dirichlet boundary conditions

$$D_1^i u_1(0, x') \equiv 0, \quad i = 0, 1, \ldots, p.$$

Since $u_1 = x_1 D_1 u - ku$, $u_1 \in H^{p+1}(\Omega)$ because of Corollary 2.1.

Theorem 7.2 then shows that the continuation of u_1 into Ω_2 may be effected by

$$u_1(-x_1, x') = Q_{p+1} u_1(x_1, x') \quad x \in \Omega. \tag{8.7}$$

It follows from (8.6) that the MacLaurin expansion of u_1 is

$$u_1(x_1, x') = \sum_{j=0}^{\infty} (j-k)(+x_1)^j g_j(x'),$$

where the series converges uniformly inside the radius of convergence. We obtain from (8.7)

$$\sum_{j=0}^{\infty} (j-k)(-x_1)^j g_j(x') = u_1(-x_1, x') = Q_p u_1(x_1, x').$$

Divide both sides by x_1^{k+1} and integrate. In the left hand side we integrate term by term, which is permissible because of the uniform convergence. A multiplication with x_1^k then gives with the notation (8.2).

$$\sum_{\substack{j=0\\ j\neq k}}^{\infty}(-x_1)^j g_j(x') = x_1^k \int_0^{x_1} Q_p^k(t, D)u(t, x')\,dt.$$

Finally we add

$$(-x_1)^k g_k(x') = \frac{(-x_1)^k}{k!}\left[\lim_{x_1\to 0} D_1^k u\right]$$

to both sides and obtain (8.4). The unicity of the analytic continuation then proves the theorem.

The case $k=p-1$ is of course included in Theorem 7.2.

The method described in this section may be used for other sets (8.1) of boundary conditions as well.

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