On the remainder term in the CLT

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1. Introduction and results

Let $\{X_n\}$ be a sequence of iid random variables with distribution function F and such that $EX_1=0$ and $EX_1^2=1$. Let $S_n=\sum_{k=1}^n X_k$, and define

$$\Delta_n = \sup_x \left|P\left[S_n \leqq x\sqrt{u}\right] - \Phi(x)\right|.$$

Where $\Phi(\cdot)$ is the distribution function of the standard normal variable.

The well-known Berry—Esséen theorem states that if $E|X_1|^3=\beta_3<\infty$, then $\Delta_n \leqq C\beta_3 n^{-1/2}$, where C is a generic constant. Among the many extensions and generalizations, Esséen (1969) proved that if

$$\varrho = \sup_{z>0} \left\{\left|\int_{|x|\leqq z} x^3\, dF(x)\right| + z\int_{|x|>z} x^2\, dF(x)\right\} < \infty,$$

then $\Delta_n \leqq C\varrho n^{-1/2}$. This result extends the Berry—Esséen theorem by relaxing the condition of finite third moment.

On the other hand, Paulauskas (1969) showed that if

$$\gamma_3 = \int |X|^3 \left|d(F(x)-\Phi(x))\right| < \infty,$$

then $\Delta_n \leqq C\max(\gamma, \gamma^{1/4})n^{-1/2}$, while later on, Zolotorov (1971), obtains an extension of Paulauskas' theorem assuming the finiteness of the third absolute difference moment. Precisely, if $\varkappa_3=\int |x|^2|F(x)-\Phi(x)|dx<\infty$, then $\Delta_n \leqq C\max(\varkappa_3, \varkappa_3^{1/4})n^{-1/2}$, see Zolotorov (1971). Since $\varkappa_3 \leqq \gamma_3$, Paulauskas' result follows from that of Zolotorov. Then the natural question to ask is whether we can extend the result of Zolotorov (1971) by assuming the finiteness of the truncated absolute moments. This, in fact, is the main goal of this note. Let

$$\gamma = \sup_{z>0} \left\{3\int_{|x|\leqq z} x^2|F(x)-\Phi(x)|\,dx + 2z\int_{|x|>z} |x|\,|F(x)-\Phi(x)|\,dx\right\}. \tag{1.1}$$

Then the following result is the main theorem of this note.

Theorem 1. *If* $\gamma<\infty$, *then*

$$\Delta_n \leqq C \max(\gamma, \gamma^{1/4}) n^{-1/2}.$$

Note that since $\gamma<\varrho$, then Esséen's result follows from the theorem, also note that if $\varkappa_3<\infty$ then $\gamma<\infty$ and thus Zolotorov's result follows from the theorem.

2. Proof of Theorem 1

We distinguish between two cases; $\gamma \geqq 1$ and $\gamma<1$. First assume that $\gamma \geqq 1$. By Esséen inequality (see Feller (1966), p. 512),

$$\Delta_n \leqq \frac{1}{\pi}\int_{-T}^{T} |\bar{f}_n(t)-e^{-t^2/2}|\,|t|^{-1}\,dt+\frac{24}{\pi\sqrt{2\pi}}T, \tag{2.1}$$

where $T>0$, is an arbitrary constant and $\bar{f}_n(t)=\left[f\left(\frac{t}{n}\right)\right]^n$ with $f(t)$ the characteristic function of X_1. From now on we shall denote all generic constants (which may be different) by C. The following lemmas are necessary for the proof.

Lemma 2.1. *For all real t and* $C>0$,

$$|f(t)-e^{-t^2/2}| \leqq C|t|^3\gamma. \tag{2.2}$$

Proof. Note that

$$\begin{aligned} & f(t)-e^{-t^2/2} \\ &= \int_{-\infty}^{\infty} e^{itx}\, d[F(x)-\Phi(x)] \\ &= \int_{-\infty}^{\infty} \left[e^{itx}-1-itx-\frac{(itx)^2}{2}\right] d[F(x)-\Phi(x)] \\ &= (it)\int_{-\infty}^{\infty} [e^{itx}-1-itx][F(x)-\Phi(x)]\,dx \\ &= (it)\left\{\int_{|x|\leqq|t|^{-1}} \left[\theta_1\frac{(itx)^2}{2}\right][F(x)-\Phi(x)]\,dx \right. \\ &\qquad \left. +\int_{|x|>|t|^{-1}}[-itx-\theta_2 itx][F(x)-\Phi(x)]\,dx\right\} \\ &= (it)\frac{t^2}{2}\int_{|x|\leqq|t|^{-1}} x^2[F(x)-\Phi(x)]\,dx(1-\theta_1) \\ &\qquad +(-it)\left(\int_{|x|>|t|^{-1}} x[F(x)-\Phi(x)]\,dx\right)(1+\theta_2), \end{aligned} \tag{2.3}$$

by application of Taylor expansion and integration by parts with $|\theta_1|$ and $|\theta_2|<1$. Thus

$$|f(t)-e^{-t^2/2}| \leqq \theta_3|t|^3 \Big(3\int_{|x|\leqq|t|^{-1}} x^2|F(x)-\Phi(x)|\,dx \tag{2.4}$$
$$+\theta_4 t^2\Big(2\int_{|x|>|t|^{-1}}|x|\,|F(x)-\Phi(x)|\,dx\Big) \leqq \theta_5|t|^3\gamma.$$

Lemma 2.2. *Let $\gamma<\infty$. Then for all $|t|\leqq(1/2C\gamma)$ and for some constant $C>0$,*

$$|f(t)| \leqq Ce^{-t^2/4}. \tag{2.5}$$

Proof. By Lemma 2.1 and the assumption we have

$$\begin{aligned}|f(t)| &\leqq |f(t)-e^{-t^2/2}|+e^{-t^2/2} \\ &\leqq C|t|^3\gamma+e^{-t^2/2} = e^{-t^2/2}(1+e^{t^2/2}C\cdot|t|^3\gamma) \\ &\leqq e^{-t^2/2}(1+Ce^{1/8\gamma^2}|t|^3\gamma) = e^{-t^2/2}(1+C\cdot\gamma\cdot|t|^3) \\ &\leqq e^{-t^2/2}e^{C_1|t|^3\gamma} = e^{-(t^2/2)(1-2C|t|\cdot\gamma)} \leqq e^{-t^2/4}.\end{aligned} \tag{2.6}$$

Lemma 2.3. *For some $C>0$, and all $|t|\leqq\sqrt{n}/2C\gamma$, we have*

$$\left|f^n\left(\frac{t}{\sqrt{n}}\right)-e^{-t^2/2}\right| \leqq C\gamma|t|^3 n^{-1/2}e^{-t^2/8}. \tag{2.7}$$

Proof. Note that

$$f^n\left(\frac{t}{\sqrt{n}}\right)-e^{-t^2/2} = \left(f\left(\frac{t}{\sqrt{n}}\right)-e^{-t^2/2n}\right)$$
$$\times\left(\sum_{j=0}^{n-1} f^{n-1-j}\left(\frac{t}{\sqrt{n}}\right)e^{-t^2j/2n}\right).$$

Thus by Lemma 2.1, $\left|f\left(\frac{t}{\sqrt{n}}\right)-e^{-t^2/2n}\right|\leqq C\cdot\gamma\cdot|t|^3n^{-3/2}$ while by Lemma 2.2 for $|t|\leqq\sqrt{n}/2C\gamma$, $\left|f\left(\frac{t}{\sqrt{n}}\right)\right|\leqq e^{-t^2/4n}$. Hence using these results,

$$\begin{aligned}\left|f^n\left(\frac{t}{\sqrt{n}}\right)-e^{-t^2/2}\right| &\leqq C\cdot\gamma\cdot\frac{|t|^3}{n^{3/2}}\sum_{j=0}^{n-1} e^{-\frac{(n-1-j)t^2}{4n}-\frac{jt^2}{2n}} \\ &\leqq C\cdot\gamma\cdot\left(\frac{|t|}{\sqrt{n}}\right)^3\sum_{j=0}^{n-1} e^{-t^2\left(\frac{n-1}{4n}\right)} \\ &\leqq C\cdot\gamma\cdot|t|^3n^{-1/2}e^{-t^2/8}.\end{aligned} \tag{2.8}$$

Now, choosing $T=\sqrt{n}/2C\gamma$ in (2.1) we get using Lemma 2.3,

$$\Delta_n \leqq \frac{C\cdot\gamma}{\sqrt{n}}\int_{-\sqrt{n}/2C\gamma}^{\sqrt{n}/2C\gamma} t^2e^{-t^2/8}\,dt+C\cdot\gamma/\sqrt{n} \leqq C\cdot\gamma\cdot n^{-1/2}. \tag{2.9}$$

Next assume that $\gamma<1$, thus $\gamma^{1/4}\geqq\gamma$ and it follows from Lemma 2.1 that $|f(t)-e^{-t^2/2}|\leqq C\gamma^{1/4}|t|^3$. If we put the assumption that $|t|\leqq 1/2C\gamma^{1/4}$ in Lemma 2.2 we get that for this choice $|f(t)|\leqq e^{-t^2/4}$. Thus for $|t|\leqq \sqrt{n}/2C\gamma^{1/4}$ we get the following modification of Lemma 2.3,

$$\left|f^n\left(\frac{t}{\sqrt{n}}\right)-e^{-t^2/2}\right|\leqq C\gamma^{1/4}|t|^3 n^{-1/2}e^{-t^2/8}. \tag{2.10}$$

Hence in (2.1) let $T_0=n/2C\gamma$ and $T_1=n/2C\gamma^{1/4}(T_0\geqq T_1)$ and thus

$$\begin{aligned}\Delta_n &\leqq \frac{1}{\pi}\int_{-T_1}^{T_1}\left|f^n\left(\frac{t}{\sqrt{n}}\right)-e^{-t^2/2}\right||t|^{-1}\,dt \\ &+\int_{T_1<t\leqq T_0}\left|f^n\left(\frac{t}{\sqrt{n}}\right)-e^{-t^2/2}\right||t|^{-1}\,dt+C\cdot\gamma n^{-1/2} \\ &\leqq C\gamma^{1/4}n^{-1/2}+C\gamma n^{-1/2}+C\gamma n^{1/2}\leqq C\gamma^{1/4}n^{-1/2}.\end{aligned} \tag{2.11}$$

Thus the theorem is completely proved.

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