A Hilbert—Schmidt norm inequality associated with the Fuglede—Putnam theorem

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The familiar Fuglede—Putnam theorem asserts that $AX=XB$ implies $A^*X=XB^*$ when A and B are normal. We prove that let A and B^* be hyponormal operators and let C be hyponormal commuting with A^* and also let D^* be a hyponormal operator commuting with B respectively, then for every Hilbert—Schmidt operator X, the Hilbert—Schmidt norm of $AXD+CXB$ is greater than or equal to the Hilbert—Schmidt norm of $A^*XD^*+C^*XB^*$. In particular, $AXD=CXB$ implies $A^*XD^*=C^*XB^*$. If we strengthen the hyponormality conditions on A, B^*, C and D^* to quasinormality, we can relax Hilbert—Schmidt operator of the hypothesis on X to be every operator in $B(H)$ and still retain the inequality under hypotheses that C commutes with A and satisfies an operator equation and also D^* commutes withB^* and satisfies another similar operator equation respectively.

1.

An operator means a bounded linear operator on a separable infinite dimensional Hilbert space H. Let $B(H)$ and C_2 denote the class of all bounded linear operators acting on H and the Hilbert—Schmidt class in $B(H)$ respectively. C_2 forms a two-sided ideal in the algebra $B(H)$ and C_2 is itself a Hilbert space for the inner product

$$(X,Y)=\Sigma(Xe_j,Ye_j)=Tr(Y^*X)=Tr(XY^*)$$

where $\{e_j\}$ is any orthonormal basis of H and $Tr(T)$ denotes the trace. In what follows, $\|T\|_2$ denotes the Hilbert—Schmidt norm.

An operator T is called *quasinormal* if T commutes with T^*T, *subnormal* if T has a normal extension and *hyponormal* if $[T^*,T]\geqq 0$ where $[S,T]=ST-TS$. The inclusion relation of the classes of non-normal operator listed above is as follows:

$$\text{Normal} \subsetneqq \text{Quasinormal} \subsetneqq \text{Subnormal} \subsetneqq \text{Hyponormal}$$

the above inclusions are all proper [6, Problem 160, p. 101].

In [2], Berberian shows the following result.

Theorem A [2]. *If A and B^* are hyponormal, then $AX=XB$ implies $A^*X=XB^*$ for an operator X in the Hilbert—Schmidt class.*

On the other hand, in [3] we have shown Theorem B which is an extension of the Fuglede—Putnam theorem.

Theorem B [3]. *If A and B^* are subnormal and if X is an operator such that $AX=XB$, then $A^*X=XB^*$.*

Recently Weiss has obtained the following result.

Theorem C [11]. *Let $\{A_1, A_2\}$ and $\{B_1, B_2\}$ denote commuting pairs of normal operators and let $X \in B(H)$. Then*

$$\|A_1XB_1+A_2XB_2\|_2 = \|A_1^*XB_2^*+A_2^*XB_2^*\|_2.$$

In this paper we prove Theorem 1 which is an extension of Theorem A and also we prove a slightly stronger Theorem 2 by integrating Theorem B and Theorem C.

2.

First of all we show the following theorem.

Theorem 1. *Let A and B^* be hyponormal on H. Let C be hyponormal commuting with A^* and also let D^* be hyponormal commuting with B respectively. Then*

(i) $$(*) \quad \|AXD+CXD\|_2 \geqq \|A^*XD^*+C^*XB^*\|_2$$

holds for every X in Hilbert—Schmidt class. Equality in $(*)$ holds for every X in Hilbert—Schmidt class when A, B, C and D are all normal.

(ii) *If X is an operator in Hilbert—Schmidt class such that $AXD=CXB$, then $A^*XD^*=C^*XB^*$.*

Proof. Define an operator $\mathscr{I}$ on C_2 as follows:

$$\mathscr{I}X = AXD+CXB.$$

Then, if we view C_2 as an underlying Hilbert space, then $\mathcal{J}^*$ exists and $\mathcal{J}^*$ is given by the formula $\mathcal{J}^*X=A^*XD^*+C^*XB^*$ which we easily see from

$$\begin{aligned}(\mathcal{J}^*X, Y) &= (X, \mathcal{J}Y) = (X, AYD+CYB) = Tr(XD^*Y^*A^*)+Tr(XB^*Y^*C^*)\\ &= Tr(A^*XD^*Y^*)+Tr(C^*XB^*Y^*) = Tr\big((A^*XD^*+C^*XB^*)Y^*\big)\\ &= (A^*XD^*+C^*XB^*, Y).\end{aligned}$$

Also

$$\begin{aligned}(\mathcal{J}^*\mathcal{J}-\mathcal{J}\mathcal{J}^*)X &= A^*(AXD+CXB)D^*+C^*(AXD+CXB)B^*\\ &\quad -A(A^*XD^*+C^*XB^*)D-C(A^*XD^*+C^*XB^*)B\\ &= (A^*AXDD^*-AA^*XD^*D)+(C^*CXBB^*-CC^*XB^*B)\\ &\quad +A^*CXBD^*-AC^*XB^*D+C^*AXDB^*-CA^*XD^*B\\ &= (A^*A-AA^*)XDD^*+AA^*X(DD^*-D^*D)\\ &\quad +(C^*C-CC^*)XBB^*+CC^*X(BB^*-B^*B)\\ &\quad +(A^*CXBD^*-CA^*XD^*B)+(C^*AXDB^*-AC^*XB^*D)\end{aligned}$$

and fifth and sixth terms in the above formula are both zero since the hypotheses $CA^*=A^*C$ and $D^*B=BD^*$ hold, so that

$$(\mathcal{J}^*\mathcal{J}-\mathcal{J}\mathcal{J}^*)X = (A^*A-AA^*)XDD^*+AA^*X(DD^*-D^*D) +(C^*C-CC^*)XBB^*+CC^*X(BB^*-B^*B). \tag{1}$$

Left and right multiplication acting on C_2 as the Hilbert space by a positive operator is itself a positive operator. Since $\mathcal{J}^*\mathcal{J}-\mathcal{J}\mathcal{J}^*$ is the sum of four positive operators by the hyponormality of A, B^* C and D^*, $\mathcal{J}$ is hyponormal. Therefore

$$\|\mathcal{J}X\|_2 \geqq \|\mathcal{J}^*X\|_2$$

that is,

$$\|AXD+CXB\|_2 \geqq \|A^*XD^*+C^*XB^*\|_2 \tag{2}$$

and the proof of equality easily follows by (1) and (2). If an operator T is hyponormal, then $-T$ is also hyponormal, so the proof of (ii) easily follows by $(*)$ in (i).

Corollary 1. *Let A and B^* be hyponormal on H. Let C be normal commuting with A and also let D be normal commuting with B respectively. Then*

(i) $$(*)\quad \|AXD+CXB\|_2 \geqq \|A^*XD^*+C^*XB^*\|_2$$

holds for every X in Hilbert—Schmidt class. Equality in $(*)$ holds for every X in Hilbert—Schmidt class when A and B are both normal.

(ii) *If X is an operator in Hilbert—Schmidt class such that $AXD=CXB$, then $A^*XD^*=C^*XB^*$.*

Proof. The hypotheses $CA=AC$ and $DB=BD$ imply $CA^*=A^*C$ and $DB^*=B^*D$, that is, $D^*B=BD^*$ by the original Fuglede—Putnam theorem [1], [6], [7], [8], so the proof follows by Theorem 1.

Remark 1. We remark that Weiss [10, Theorem 3] shows the case of the equality in (i) of Corollary 1 when $A=B$ is normal and $C=D=I$ the identity operator on H, by a different method and also Corollary 1 is an extension of Theorem A.

3.

If we strengthen the hyponormality conditions to quasinormality, then we can relax Hilbert—Schmidt operator of the hypothesis on X to be every operator in $B(H)$ in Theorem 1 and still retain the inequality under suitable hypotheses.

Definition 1. Let N_T denote a normal extension on $H\oplus H$ of a subnormal operator T on H. In fact, for every subnormal operator T, there exists a normal extension N_T on $H\oplus H$ whose restriction to $H\oplus\{0\}$ is T [5].

Lemma. *Let A and B^* be subnormal on H. Let C be subnormal such that N_C commutes with N_A and also D^* be subnormal such that N_{D^*} commutes with N_{B^*} respectively. Then*

(i) $$(**)\quad \|AXD+CXB\|_2 \geqq \|A^*XD^*+C^*XB^*\|_2$$

holds for every X in $B(H)$. Equality in $(**)$ holds for every X in $B(H)$ when A, B, C and D are all normal.

(ii) *If X is an operator such that $AXD=CXB$, then $A^*XD^*=C^*XB^*$.*

Proof. By Definition 1, N_A and N_C are given by

$$N_A=\begin{pmatrix} A & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad\text{and}\quad N_C=\begin{pmatrix} C & C_{12} \\ 0 & C_{22} \end{pmatrix}$$

acting on $H\oplus H$ whose restrictions to $H\oplus\{0\}$ are A and C respectively and also N_{B^*} and N_{D^*} are given by the same reason as follows on $H\oplus H$

$$N_{B^*}=\begin{pmatrix} B^* & B_{12} \\ 0 & B_{22} \end{pmatrix} \quad\text{and}\quad N_{D^*}=\begin{pmatrix} D^* & D_{12} \\ 0 & D_{22} \end{pmatrix}.$$

For X acting on H, we consider $\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$ acting on $H\oplus H$. $\{N_A, N_C\}$ and $\{N_{D^*}^*, N_{B^*}^*\}$

are commuting pairs of normal operators on $H\oplus H$. Then by Theorem C, we have

$$\left\|\begin{pmatrix} A & A_{12}\\ 0 & A_{22}\end{pmatrix}\begin{pmatrix} X & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} D & 0\\ D_{12}^* & D_{22}^*\end{pmatrix}+\begin{pmatrix} C & C_{12}\\ 0 & C_{22}\end{pmatrix}\begin{pmatrix} X & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} B & 0\\ B_{12}^* & B_{22}^*\end{pmatrix}\right\|_2$$

$$=\left\|\begin{pmatrix} A^* & 0\\ A_{12}^* & A_{22}^*\end{pmatrix}\begin{pmatrix} X & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} D^* & D_{12}\\ 0 & D_{22}\end{pmatrix}+\begin{pmatrix} C^* & 0\\ C_{12}^* & C_{22}^*\end{pmatrix}\begin{pmatrix} X & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} B^* & B_{12}\\ 0 & B_{22}\end{pmatrix}\right\|_2$$

that is,

$$\left\|\begin{pmatrix} AXD+CXB & 0\\ 0 & 0\end{pmatrix}\right\|_2=\left\|\begin{pmatrix} A^*XD^*+C^*XB^* & A^*XD_{12}+C^*XB_{12}\\ A_{12}^*XD^*+C_{12}^*XB^* & A_{12}^*XD_{12}+C_{12}^*XB_{12}\end{pmatrix}\right\|_2$$

so that

$$\|AXD+CXB\|_2^2=\|A^*XD^*+C^*XB^*\|_2^2+\|A^*XD_{12}+C^*XB_{12}\|_2^2 +\|A_{12}^*XD^*+C_{12}^*XB^*\|_2^2+\|A_{12}^*XD_{12}+C_{12}^*XB_{12}\|_2^2 \tag{3}$$

whence we have

$$\|AXD+CXB\|_2\geqq\|A^*XD^*+C^*XB^*\|_2$$

which is the desired norm inequality $(**)$. When A, B, C and D are all normal, then $A_{12}=0$, $B_{12}=0$, $C_{12}=0$ and $D_{12}=0$ in (3), so that equality in $(**)$ holds and the proof is complete.

We remark that sum of second, third and fourth terms of the right hand in (3) can be considered as a "*perturbed terms*" measures the deviation of subnormality from normality.

Definition 2. Let $[S,T]_*$ denote the following "**-commutator*":

$$[S,T]_*=ST-TS^*$$

this *-commutator is completely different from usual commutator $[S,T]$.

Theorem 2. *Let A and B^* be quasinormal on H. Let C be quasinormal such that it commutes with A and satisfies $[A,S_C]_*=[C,S_A]_*$ and also let D^* be quasinormal such that it commutes with B^* and satisfies $[B^*,S_{D^*}]_*=[D^*,S_{B^*}]_*$ respectively, where S_T denotes the positive square root of $[T^*,T]$ for a quasinormal T. Then*

(i) $$(**)\quad \|AXD+CXB\|_2\geqq\|A^*XD^*+C^*XB^*\|_2$$

holds for every X in $B(H)$ when A, B, C and D are all normal.

(ii) *If X is an operator such that $AXD=CXB$, then $A^*XD^*=C^*XB^*$.*

Proof. Let $A=UP$ be the polar decomposition of A, where U is a partial isometry and P is a positive operator such that $P^2=A^*A$. A normal extension N_A of A can be written as follows [6, p. 308]:

$$N_A=\begin{pmatrix} A & S(A)\\ 0 & A^*\end{pmatrix}$$

acting on $H\oplus H$, where $S(A)=(I-UU^*)P$. Since A is quasinormal, then $A=UP=PU$ [6, Problem 108]. As UU^* is projection and P commutes with U and U^*, then

$$(4)\qquad S(A)=(I-UU^*)P=[(I-UU^*)P^2]^{1/2}$$
$$=(P^2-UPU^*P)^{1/2}=(A^*A-AA^*)^{1/2}=S_A.$$

Similarly normal extensions of C, B^* and D^* are also given as follows:

$$N_C=\begin{pmatrix}C & S_C\\ 0 & C^*\end{pmatrix}\quad N_{B^*}=\begin{pmatrix}B^* & S_{B^*}\\ 0 & B\end{pmatrix}\quad\text{and}\quad N_{D^*}=\begin{pmatrix}D^* & S_{D^*}\\ 0 & D\end{pmatrix}.$$

Hypotheses imply that $\{N_A, N_C\}$ and $\{N^*_{D^*}, N^*_{B_*}\}$ are pairs of commuting normal operators, so that the desired relations follow by Lemma.

Corollary 2. *Let A and B^* be quasinormal on H. Let C be normal commuting with A and also D be normal commuting with B respectively. Then*

(i) $$(**)\quad \|AXD+CXB\|_2\geqq\|A^*XD^*+C^*XB^*\|_2$$

holds for every X in $B(H)$. Equality in $(**)$ holds for every X in $B(H)$ when A, B, C and D are all normal.

(ii) If X is an operator such that $AXD=CXB$, then $A^*XD^*=C^*XB^*$

Proof. Take $N_C=\begin{pmatrix}C & 0\\ 0 & C\end{pmatrix}$ in the proof of Theorem 2 since C is normal. Then the hypothesis $CA=AC$ implies $CA^*=A^*C$ by the original Fuglede—Putnam theorem [1], [6], [7], [8], so that we have $CS_A^2=S_A^2C$ since (4) holds, that is, $CS_A=S_AC$ holds, whence N_A commutes with $N_C=\begin{pmatrix}C & 0\\ 0 & C\end{pmatrix}$. Similarly $N^*_{D^*}=\begin{pmatrix}D^* & 0\\ 0 & D^*\end{pmatrix}$ commutes with $N^*_{B^*}$, so that the proof is complete by Lemma.

Remark 2. If we strengthen on X to be in Hilbert—Schmidt class in Corollary 2, then we can relax quasinormality of the hypotheses on A and B^* to hyponormality and still retain the inequality, this is just Corollary 1.

Corollary 3. *Let A and B^* be hyponormal satisfying $[A^*, S_A]_*=0$ and $[B, S_{B^*}]_*=0$ respectively. Let C be hyponormal which commutes with A and satisfies $[C^*, S_C]_*=0$ and $[A, S_C]_*=[C, S_A]_*$ and also let D^* be hyponormal which commutes with B^* and satisfies $[D, S_{D^*}]_*=0$ and $[B^*, S_{D^*}]_*=[D^*, S_{B^*}]_*$ respectively. Then*

(i) $$(**)\quad \|AXD+CXB\|_2\geqq\|A^*XD^*+C^*XB^*\|_2$$

holds for every X in $B(H)$. Equality in $(**)$ holds for every X in $B(H)$ when A, B, C and D are all normal.

(ii) *If X is an operator such that $AXD=CXB$, then $A^*XD^*=C^*XB^*$.*

Proof. The hypotheses imply that A, B^*, C and D^* are all subnormal and $N_A=\begin{pmatrix} A & S_A \\ 0 & A^* \end{pmatrix}$ and similarly N_{B^*}, N_C and N_{D^*} are also given in the similar forms [4, Theorem 1]. As stated in the proof of Theorem 2, the hypotheses imply that $\{N_A, N_C\}$ and $\{N^*_{D^*}, N^*_{B^*}\}$ are pairs of commuting normal operators, so that the proof is complete by Lemma.

Can quasinormality be replaced by subnormality (or further hyponormality) in Theorem 2 and Corollary 2? Partial and modest answers to this question are cited in [2], [3], [9]. Theorem 1 is a modest result and Corollary 3 is in this direction.

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