Bounded point evaluations and balayage

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0. Introduction

The problem treated in this paper can be formulated as follows: Define W as the closure of $C_0^\infty(\mathbf{R}^d)$ in the norm $\{\int |\text{grad} f|^2 dx\}^{1/2}$. If E is open and bounded let $H(E)$ be the subspace of W consisting of functions harmonic in E. If E is compact let $H(E)$ be the closure in W of functions harmonic in some neighbourhood of E. (Here $d\geqq 3$; if $d=2$ we replace $C_0^\infty(\mathbf{R}^d)$ by $C_0^\infty(D)$ — D the open unit disc — and we assume that $E\subset\subset D$.)

A point $a\in\bar{E}$ for which the mapping $f\to f(a)$ is bounded on $H(E)$ is called a *bounded point evaluation* (BPE) for $H(E)$ and our aim is to characterize these points.

In Fernström—Polking [4] a similar problem is treated for a more general elliptic differential operator acting on $L^p(E)$, E a compact set in $\mathbf{R}^d$. (For a more detailed discussion on BPEs we refer to that paper and the references there.) Compared to [4] we are here dealing with a special case; we can then make use of other methods, specific to this problem. In particular we apply the operation of sweeping out a measure, balayage. We can also take care of the case when E is an open set.

We get several conditions characterizing the BPEs. Two of these are to be stressed cf. [4, theorems 1 and 3]): The first one is that the fundamental solution of the Laplace operator with a pole at a (the function $x\to\text{const}\,|x-a|^{2-d}$ if $d\geqq 3$) can be continued from E^c to $\mathbf{R}^d$ so as to be an element of $H(E)$. Moreover, this new function is the Newton potential of the Dirac measure δ_a at a swept out onto E^c. The second one is a Wiener type condition, the BPEs are precisely the points for which E^c is subject to a certain kind of thinness.

The main references are Cartan [2] and Landkof [6]. Also Hedberg [5] is a good reference for sections I and II.

We will use the following notation:

For a (Borel) set $B\subset\mathbf{R}^d$ ($d\geqq 3$ in sections I—IV, $d=2$ in section V) B^0 its interior, $\bar{B}$ its closure, ∂B its boundary and B^c its complement. $C_0^\infty(G)$ is the space of all infinitely differentiable functions $\mathbf{R}^d \to \mathbf{R}$ with compact support in the open set G. Δ is the Laplacean $(\partial/\partial x_1)^2+\ldots+(\partial/\partial x_d)^2$ and grad is the gradient $(\partial/\partial x_1, \ldots, \partial/\partial x_d)$. We will use the letter A to denote various positive constants (only depending on the dimension etc.). A may then denote different constants in the same chain of inequalities.

Before turning to the heart of the matter, I want to thank Lars Inge Hedberg who initiated this paper and who has been helpful during the the work on it.

I also want to thank Bent Fuglede for showing kind interest in this problem.

I. The space W

For $f, g \in C_0^\infty(\mathbf{R}^d)$ $(d \geqq 3)$ we define $(f|g)=\int \operatorname{grad} f \cdot \operatorname{grad} g\, dx$. This is an *inner product* and we define the real *Hilbert space* W as the completion of $C_0^\infty(\mathbf{R}^d)$ with respect to the corresponding norm. It is well known that any function in W can be represented as the *Riesz potential* of an L^2-function. This means that given $f \in W$ there exists a function $F \in L^2$ such that

$$f(x) = \int |x-y|^{1-d} F(y)\, dy. \tag{1}$$

We shall use the notation $R_\alpha(x)=|x|^{\alpha-d}$ $(0<\alpha<d)$ so (1) can be written $f=R_1 * F$.

The (outer) capacity of a set $M \subset \mathbf{R}^d$ is defined as follows,

$$C(M) = \inf \int f(x)^2\, dx,$$

where the infimum is taken over all $0 \leqq f \in L^2$ such that $R_1 * f(x) \geqq 1$ for all $x \in M$. We say that a statement holds true *quasi everywhere* (q.e.) or for *quasi all x* if it is true outside a set M with $C(M)=0$. The elements of W are equivalence classes of functions, two functions being equivalent if they coincide q.e.

W is *closed under truncations*. More explicitly: if $f \in W$ then f^+ and f^- are both in W. Furthermore, if also $f \geqq 0$ then $\inf(f, A) \in W$ for any positive constant A. (See Deny—Lions [3, p. 316])

Let G be an open and bounded set in $\mathbf{R}^d$. Define $W_0(G)$ as the closure in W of $C_0^\infty(G)$. By the *spectral synthesis* of Beurling and Deny,

$$W_0(G) = \{f \in W \colon f = 0 \text{ q.e. on } G^c\}.$$

See Hedberg [5] for further details. If f, an element of W, is orthogonal to all functions in $C_0^\infty(G)$ we get, by Green's formula and Weyl's lemma, that f is harmonic in G. Hence, denoting by $H(G)$ the orthogonal complement of $W_0(G)$. we get $H(G)=\{f \in W\colon \Delta f|_G=0\}$. The projection f_G onto $H(G)$ of a function $f \in W$ is the generalized solution of the *Dirichlet problem* with boundary value f in the sense that $f-f_G\in W_0(G)$ and thus equals zero q.e. off G.

If $F\subset\mathbf{R}^d$ is compact we put $W_0(F)=\bigcap_{G\supset F} W_0(G)$, G open. We also define $H(F)$ as the closure in W of functions harmonic in some neighbourhood of F; i.e. $H(F)=\overline{\bigcup_{G\supset F} H(G)}$, G open. $H(F)$ is then the orthogonal complement of $W_0(F)$. Let $(G_n)_1^\infty$ be an arbitrary decreasing sequence of open sets for which $\bigcap_1^\infty G_n=F$ and $\operatorname{dist}(F,\partial G_n)$ tends to zero as n approaches infinity. By using the parallelogram identity one can show that $f_F=\lim f_{G_n}$. (See [5, pp. 4—5].)

Hence, if $E\subset\mathbf{R}^d$ is a bounded open or closed set, W *splits into perpendicular subspaces*,

(2)
$$W=W_0(E)\oplus H(E).$$

II. The space $\mathscr{E}$

W is closely related to *Newton potentials* of signed measures μ, that is,

(3)
$$U^\mu(x)=\int|x-y|^{2-d}\,d\mu(y)=R_2*\mu(x).$$

We have $\Delta_x(|x-y|^{2-d})=-A\delta_y$ where δ_y is the Dirac measure supported by $\{y\}$. Thus the kernel $|x-y|^{2-d}$ is — except for a multiplicative constant — a fundamental solution of Δ with a pole at y.

If $\int U^{|\mu|}\,d|\mu|<\infty$ we define the *energy* of μ as the non-negative number $I(\mu)=\int U^\mu d\mu$ and we write $\mu\in\mathscr{E}$. If also $\operatorname{supp}\mu\subset M$ we write $\mu\in\mathscr{E}(M)$. We use the notation $\mathscr{E}^+$ for all positive measures in $\mathscr{E}$.

The kernels R_α satisfy the relation

(4)
$$A\cdot R_\alpha*R_\beta=R_{\alpha+\beta},\quad (0<\alpha,\beta,\alpha+\beta<d).$$

We can then write (3) as $U^\mu=R_2*\mu=AR_1*R_1*\mu=R_1*f$ where $f\in L^2$ if $\mu\in\mathscr{E}$. This describes the connection between $\mathscr{E}$ and W. In fact, potentials of measures in $\mathscr{E}$ are dense in W.

We can also define, for sets $M\subset\mathbf{R}^d$, an inner *capacity*, say C', as

$$C'(M)=\sup I(\mu),$$

the supremum being taken over all $\mu\in\mathscr{E}^+(M)$ such that $U^\mu\leqq 1$ on M. In view of (4) it is not surprising that $C'(M)=AC(M)$ for compact sets M and by Choquet's theorem this holds also for Borel sets. C and C' being equivalent we will not take the trouble to distinguish them by different notation so, throughout this paper C will denote any of these capacities.

If M is a bounded set there exists a measure $\lambda_M\in\mathscr{E}^+(\overline{M})$ such that $U^{\lambda_M}=1$ q.e. on M and $C(M)=I(\lambda_M)=\int d\lambda_M$. λ_M is called the *equilibrium measure* of M.

The form $(\mu, \nu)=\int U^\mu d\nu$ is an *inner product* on $\mathcal{E}$ so,

(5) $$|(\mu, \nu)|^2 \leqq I(\mu)\cdot I(\nu), \quad (\mu, \nu\in\mathcal{E}).$$

($\mathcal{E}$ is a pre-Hilbert space: it fulfills all the requirements for a Hilbert space except being complete.)

The inner products $(.,.)$ and $(.|.)$ are related by the identity

(6) $$(\mu, \nu) = A\cdot(U^\mu|U^\nu), \quad (\mu, \nu\in\mathcal{E}).$$

In particular,

(6′) $$I(\mu) = (\mu, \mu) = A\cdot(U^\mu|U^\mu), \quad (\mu\in\mathcal{E}).$$

(See [6, p. 97].)

The operation $W\ni f\to f_E\in H(E)$ has an analogue for measures. Assume that μ is a positive measure and that $U^\mu\not\equiv\infty$. Then there exists a measure $\mu_E\geqq 0$ supported by E^{oc} such that $U^{\mu_E}\leqq U^\mu$ everywhere and equality holds q.e. on E^c. If μ is of finite energy then so is μ_E. (If μ is a signed measure we treat μ^+ and μ^- separately.) The operation $\mu\to\mu_E$ is called *"sweeping out"* or *balayage*. One sweeps μ out onto E^c.

A function $V\colon \mathbf{R}^d\to(-\infty, \infty]$ is said to be *superharmonic* if it belongs to L^1_{loc} and if $\Delta V\leqq 0$. In particular, for any measure $\mu\geqq 0$ with $U^\mu\in L^1_{loc}$, U^μ is a superharmonic function. We denote by SH the *class of positive superharmonic functions.* The potentials U^μ, $\mu\geqq 0$ can then be characterized by

$$U^{\mu_E} = \inf\{V\in SH\colon V = U^\mu \text{ q.e. on } E^c\}, \quad (\mu\geqq 0).$$

(See [2, pp. 255 and 260].)

From now on we assume that E is a bounded open or closed set. By (6), $\mathcal{E}\subset_\to W$ so for any $\mu\in\mathcal{E}$ we can define U^μ_E, the projection of U^μ onto $H(E)$.

Consider the following diagram

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{E}(E^{oc}) \\ \downarrow & & \downarrow \\ W & \longrightarrow & H(E). \end{array}$$

Here the rows are the projections $\mu\to\mu_E$ and $f\to f_E$ respectively. The vertical arrows represent the injection map $\mu\to U^\mu$. Knowing that this *diagram is commutative* will be essential for what follows. For the reader's convenience we include a proof of this known fact.

We have to show that $U^\mu_E=U^{\mu_E}$ for any measure $\mu\in\mathcal{E}$ and we can assume that $\mu\geqq 0$. Suppose first that E is open. We have $\mu_E\in\mathcal{E}$ so $U^{\mu_E}\in W$. But supp $\mu_E\subset E^c$ implies that U^{μ_E} is harmonic in E and thus $U^{\mu_E}\in H(E)$. Since $U^\mu_E(x)=U^\mu(x)=U^{\mu_E}(x)$ for quasi all $x\in E^c$, $U^\mu_E-U^{\mu_E}\in W_0(E)$ whence $U^\mu_E-U^{\mu_E}\in W_0(E)\cap H(E)=\{0\}$.

If E is compact, choose a decreasing sequence $(E_n)_1^\infty$ of open sets such that $\lim \operatorname{dist}(E, \partial E_n)=0$, $n\to\infty$, and $\bigcap E_n=E$. Let us, for a more convenient notation, put $u_n=U^{\mu_{E_n}}$ and $u=U^{\mu_E}$. We know that $u_n=U^\mu_{E_n}$ and by what was said in section I, $U^{\mu_n}_E\to U^\mu_E$ in W. By selecting a subsequence if necessary we can assume that this holds also q.e. Thus it suffices to show that $u_n\to u$ q.e.

Letting $J_n=\{v\in SH:\ v=U^\mu \text{ q.e. on } E_n^c\}$, $(J_n)_1^\infty$ is a decreasing sequence and u_n is the infimum over J_n. Thus $(u_n(x))_1^\infty$ is, for quasi all x, an increasing sequence and therefore it has a limit q.e. This limit is q.e. equal to the infimum over $\cap J_n$. But since this also applies to u, $u=\lim u_n$ q.e. and the assertion follows.

Hence we get

$$U^\mu_E = \inf\{v\in SH:\ v = U^\mu \text{ q.e. on } E^c\}, \quad (\mu\in\mathcal{E}^+). \tag{7}$$

The relation (6) has the following important generalization

$$\int g\,d\mu = A(g|U^\mu), \quad (g\in W, \mu\in\mathcal{E}). \tag{8}$$

A consequence of this is that $\{U^\mu, \mu\in\mathcal{E}(E^c)\}$ is *dense in* $H(E)$ since the elements of $W_0(E)$ are precisely those in W which vanish q.e. off E.

III. Bounded point evaluations

Assume that $a\in\bar{E}$ and consider the measure δ_{aE}, δ_a swept out onto E^c. (If E is open this is the harmonic measure.) We use the notation $H_E(\cdot, a)$ for $U^{\delta_{aE}}$. This function, harmonic in E^0 and equal to $U^{\delta_a}(x)=|x-a|^{2-d}$ for quasi all $x\in E^c$, will be thoroughly examined. The case when $H_E(\cdot, a)\in H(E)$, and thus $\delta_{aE}\in\mathcal{E}$ by (6), will be of particular interest to us.

The swept out measure δ_{aE} fulfills, by [2, pp. 264—265] and Fubini's theorem,

$$U^\mu_E(a) = \int U^\mu\,d\delta_{aE} = \int H_E(\cdot, a)\,d\mu, \quad (\mu\in\mathcal{E}^+). \tag{9}$$

If $\mu\in\mathcal{E}(E^c)$, $U^\mu_E=U^{\mu_E}=U^\mu$ and we get, by (9),

$$U^\mu(a) = \int H_E(\cdot, a)\,d\mu^+ - \int H_E(\cdot, a)\,d\mu^-,$$

if U^{μ^+} and U^{μ^-} are both bounded. From what has been said above it is clear that such potentials form a dense subset of $H(E)$. Hence we can define the mapping $f\to f(a)$ densely in $H(E)$ and we say that a is a *bounded point evaluation* (BPE) for $H(E)$ if this mapping is bounded.

Thus, a is a BPE for $H(E)$ if and only if there exists a function $g\in H(E)$ such that,

$$\int H_E(\cdot, a)\,d\mu = (g|U^\mu), \tag{10}$$

for all measures of the above mentioned kind.

Theorem 1. *Assume that E is an open or closed relatively compact set. Then $a\in\bar{E}$ is a* BPE *for $H(E)$ if and only if $H_E(\cdot, a)\in H(E)$.*

Proof. Suppose $H_E(\cdot, a)\in H(E)$. Bearing in mind (8) and (9) a is then clearly a BPE for $H(E)$.

Conversely, suppose a is a BPE for $H(E)$. If u is a suitable multiple of the representing function g, we get by (8) and (10),

$$U^{\mu}(a) = \int H_E(\cdot, a)\,d\mu = \int u\,d\mu, \quad (\mu\in\mathcal{E}^+(E^c)). \tag{11}$$

By (11) $u=H_E(\cdot, a)$ q.e. on E^c.

We know that $\varphi\geqq 0$ implies $\varphi_E\geqq 0$ and for any $\varphi\in C_0^\infty(\mathbf{R}^d)$ we get, $\Delta u(\varphi)=u(\Delta\varphi)=-(u|\varphi)=-(u|\varphi_E)-(u|\varphi-\varphi_E)=-A\varphi_E(a)$ since u and $\varphi-\varphi_E$ are orthogonal. Thus $\Delta u\leqq 0$ and u is a superharmonic function.

It is well known (see e.g. the proof of theorem 6.4 in [6, pp. 360—362]) that functions in W fulfill

$$f = AU^{-\Delta f}, \quad (f\in W). \tag{12}$$

Applying this on u, we get $u\in SH$. Thus, by (7) and the above observation,

$$u = \inf\{v\in SH\colon\ v = H_E(\cdot, a) \text{ q.e. on } E^c\} \tag{13}$$

which means that $u=H_E(\cdot, a)$. Q.E.D.

Remarks 1. The theorem can be formulated in the following manner: a is a BPE for $H(E)$ if and only if the fundamental solution of Δ with a pole at a can be continued from E^c so as to be an element of $H(E)$.

This means that we can solve — in W — the Dirichlet problem with boundary data $x\to|x-a|^{2-d}$ given on ∂E, the boundary of E. Furthermore, this unique solution (or continuation) is nothing but the potential of δ_a swept out onto E^c.

Note also that by (6) a is a BPE for $H(E)$ iff δ_{aE} is of finite energy.

2. We assume, with no loss of generality, that $a=0$ is a BPE for $H(E)$. Put $\mu=\delta_{0E}$ and $Q=E^c$. Let $Q'=\{x'=x/|x|^2\colon x\in Q\}$ be the *inversion* of Q in the unit sphere S^{d-1}.

We can transform μ to a measure μ' on Q' by the formula $d\mu'(x')=|x-a|^{2-d}d\mu(x)$. Then $I(\mu)=I(\mu')$ and the latter quantity is (if $\delta_{0E}\neq\delta_0$ which is necessary in order that 0 be a BPE) equal to the capacity of Q'. (See [2, pp. 275—277].)

Hence a is a BPE for $H(E)$ iff $(E^c)'$ has finite capacity. In the terminology of Brelot [1, p. 31] a is a BPE iff $(E^c)'$ is *thin at the point of infinity.*

IV. A Wiener condition

E^c is by definition said to be *thin* at a point a if $\delta_{aE} \neq \delta_a$. Recalling what was said above it is clear that this condition will not guarantee that a be a BPE. (Thinness is actually a regularity condition for the Dirichlet problem.)

The points of this kind are completely characterized by Wiener's criterion:

$$\sum_1^\infty \frac{C(A_n(a)\setminus E)}{C(A_n(a))} < \infty \tag{14}$$

where we have denoted by $A_n(a)$ the *annulus* $\{x:\ 2^{-n-1} < |x-a| \leqq 2^{-n}\}$.

It is surprising that there is, in fact, a condition similar to (14) which turns out to be the proper one.

Theorem 2. *a is a* BPE *for $H(E)$ if and only if*

$$\sum_1^\infty \frac{C(A_n(a)\setminus E)}{C(A_n(a))^2} < \infty. \tag{15}$$

Proof. Note first of all that $C(A_n(a)) \approx 2^{-n(d-2)}$. We assume (15) for $a=0$. Then for any $\mu \in \mathscr{E}^+(E^c)$ we get, letting $A_n = A_n(0)$,

$$U^\mu(0) = \sum \int_{A_n\setminus E} |x|^{2-d}\, d\mu(x) \leqq A \sum 2^{n(d-2)} \int_{A_n\setminus E} d\mu$$
$$= A \sum 2^{n(d-2)} \int_{A_n\setminus E} U^{\lambda_n}\, d\mu,$$

where λ_n is the equilibrium measure for $A_n\setminus E$. Cauchy's inequality gives

$$U^\mu(0) \leqq A \sum \Big\{2^{n(d-2)} \Big(\int U^{\lambda_n}\, d\lambda_n\Big)^{1/2} \Big(\int_{A_n\setminus E} U^\mu\, d\mu\Big)^{1/2}\Big\}$$
$$= A \sum \Big\{\big(2^{2n(d-2)} C(A_n\setminus E)\big)^{1/2} \Big(\int_{A_n\setminus E} U^\mu\, d\mu\Big)^{1/2}\Big\}$$
$$\leqq A \Big\{\sum 2^{2n(d-2)} C(A_n\setminus E)\Big\}^{1/2} \Big\{\sum \int_{A_n\setminus E} U^\mu\, d\mu\Big\}^{1/2}$$
$$= A \Big\{\sum C(A_n\setminus E)/C(A_n)^2\Big\}^{1/2} \cdot I(\mu)^{1/2}.$$

Let (E_j) be a sequence of open sets decreasing to E, and apply this to $\mu = \mu_j = \delta_{0,E_j} \in \mathscr{E}$. Then $U^{\mu_j}(0) = I(\mu_j)$, so $I(\mu_j)$ has a uniform bound. It follows that $\delta_{0,E} \in \mathscr{E}$, so (15) is sufficient.

For the converse, supposing 0 to be a BPE for $H(E)$, we have by theorem 1, $H_E(x, 0) = R_1 * h(x)$ for some $h \in L^2$. This gives q.e.

$$\int R_1(x-y)|h(y)|\, dy \geqq H_E(x, 0) \geqq A 2^{n(d-2)}, \quad (x \in A_n\setminus E). \tag{16}$$

Letting $B_n=\{y\colon |y|<2^{-n-2} \text{ or } |y|>2^{-n+1}\}$, we get

$$\int_{B_n} R_1(x-y)|h(y)|\,dy \leqq \int_{|x-y|\geqq 2^{-n-2}} R_1(x-y)|h(y)|\,dy$$
$$\leqq A\left\{\int_{|y|\geqq 2^{-n-2}} R_1(y)^2\,dy\right\}^{1/2}\cdot\|h\|_2 \leqq A2^{n(d-2)/2},$$

for all $x\in A_n$. This estimate gives, invoking (16),

$$\int_{B_n^c} R_1(x-y)|h(y)|\,dy \geqq A2^{n(d-2)},$$

for quasi all $x\in A_n\setminus E$ which means that

$$C(A_n\setminus E) \leqq A\int_{B_n^c} (|h(y)|2^{-n(d-2)})^2\,dy.$$

Multiplying with $2^{2n(d-2)}$ and summing over n, one finally gets

$$\sum 2^{2n(d-2)}C(A_n\setminus E) \leqq A\sum\int_{B_n^c} |h(y)|^2\,dy \leqq A\int |h(y)|^2\,dy < \infty,$$

and (15) follows.

Remarks. 1. It so happens that one can use a result of D. R. Adams [7], on functions of finite capacity, to deduce theorem 2. In fact, the Adams theory can be used to obtain quite general extension theorems for non-linear potentials of positive measures. The result is in the spirit of theorem 2, but we omit the precise formulation.

2. In the language of probability theory, (15) is quite suggestive. Let P_x denote the probability measure of a Brownian motion $\{X(t),\ t\geqq 0\}$ starting at $x\in\mathbf{R}^d$ and let τ_M denote the hitting time of the set M, $\tau_M=\inf\{t>0\colon X(t)\in M\}$. Then (15) is equivalent to

$$\int\left\{\sum_1^\infty 2^{n(d-2)}P.\ (\tau_{A_n(a)\setminus E}<\infty)\right\}dS < \infty.$$

Here dS denotes the uniform probability measure on the unit sphere $S^{d-1}=\{|x|=1\}$.

On the subject of Brownian motion in this connexion, see e.g. Port and Stone [9].

3. In [8] B. Fuglede introduces the space $H(E)$ where E is allowed to be a quasi-coanalytic set, i.e. E^c differs from an analytic set by at most a set of capacity zero. Then the "H" in $H(E)$ refers to finely harmonic functions.

It turns out that the "BPE-points" of a set E are precisely the set of removable singularities for functions in $H(E)$. Hence this set forms, in a sense, a natural domain of definition for finely harmonic functions.

4. Let, for a given set E, E' denote the points where E^c is thin, and let E'' denote the set of BPE's for $H(E)$. If the dimension $d \geqq 4$, the counterexample in [4] can be seen to have E'' as a proper subset of E'. In fact, $E''=\emptyset$, and $E=E'$. This is however impossible in two or three dimensions. See [8] for a discussion of this phenomenon.

See also [2, p. 277] for an example of a point $a \in E' \setminus E''$.

V. The planar case

Most of what is said in sections I—IV remains true — after some necessary alterations — also in the case $d=2$. The functions considered are assumed to have their supports in the open unit disc $D=\{|x|<1\}$. Then W will be the closure of $C_0^\infty(D)$ with respect to the norm $\{\int |\text{grad} f|^2 dx\}^{1/2}$ and we still have, for E an open or closed relatively compact subset of D,

$$W = W_0(E) \oplus H(E).$$

Newton potentials are replaced by *Green potentials*, i.e.

$$U^\mu(x) = \int G(x, y)\, d\mu(y),$$

where μ is a signed measure with its support in D and G is *Green's function* for D,

$$G(x, y) = \log |(1-\bar{x}y)/(x-y)|, \quad (x, y \in D).$$

If we replace E^c by $D \setminus E$ and $H_E(\cdot, a)$ by the Green potential of δ_a then everything in sections I and II remains true. In section III we cannot perform the inversion under remark 2 but apart from this there are no changes.

Section IV can also be left almost as it is but we want to point out that for $d=2$, $C(A_n(a))$ behaves as n, so explicitly (15) reads

$$\sum_1^\infty n^2 C(A_n(a) \setminus E) < \infty.$$

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Received August 15, 1980

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