Absolute convergence of Fourier series on totally disconnected groups

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Let G denote a totally disconnected locally compact metric abelian group with translation invariant metric d and character group Γ_G. The Lipschitz spaces are defined by

$$\mathrm{Lip}(\alpha;p)=\{f\in L^p(G)\colon \|\tau_a f-f\|_p=O(d(a,0)^\alpha),\ a\to 0\},$$

where $\tau_a f\colon x\to f(x-a)$ and $\alpha\in(0,1)$. For a suitable choice of metric it is shown that $\mathrm{Lip}(\alpha;p)^{\wedge}\subset L^r(\Gamma_G)$, where $\alpha>1/p+1/r-1\geqq 0$ and $1\leqq p\leqq 2$. In the case G is compact the corresponding result holds for $\alpha>1/r-1/2$ and $p>2$. In addition for G non-discrete the above result is shown to be sharp, in the sense that the range of values of α cannot be extended. The results include classical theorems of S. N. Bernstein, O. Szász and E. C. Titchmarsh.

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1. Introduction

In 1914 S. N. Bernstein ([3]) announced the result that functions in $\mathrm{Lip}(\alpha)$ with $\alpha>1/2$ have absolutely convergent Fourier series, while for $\alpha<1/2$ there are functions in $\mathrm{Lip}(\alpha)$ whose Fourier series do not converge absolutely. This was generalised by O. Szász ([16], [17]) who proved that if $f\in\mathrm{Lip}(\alpha;p)$ then $\hat{f}\in l^r$, where $\alpha>1/p+1/r-1$ if $1<p\leqq 2$ and $\alpha>1/r-1/2$ if $p>2$; he also gave examples to show that the range of values of α could not be extended. Here the generalised Lipschitz space $\mathrm{Lip}(\alpha;p)$ is defined by

$$\mathrm{Lip}(\alpha;p)=\{f\in L^p(T)\colon \|\tau_a f-f\|_p=O(|a|^\alpha),\ a\to 0\},$$

where $\tau_a f\colon x\to f(x-a)$ and T denotes the circle group; the notation $\mathrm{Lip}(\alpha)$ is standard for $\mathrm{Lip}(\alpha;\infty)$, in which case the functions are taken to be continuous.

Subsequently the result was extended to other groups; by E. C. Titchmarsh ([18]) to the real line $(1<p\leqq 2)$, by N. J. Fine ([9]) to the Cantor group $(p=\infty)$, by N. Ja. Vilenkin ([19]) to compact metric abelian groups with primary character groups $(p=\infty)$, by P. L. Walker ([21], [22]) to finite dimensional compact metric abelian groups $(p=\infty)$, by C. W. Onneweer ([13], [14]) to the so called bounded Vilenkin groups $(1\leqq p\leqq\infty)$, by T. S. Quek and L. Y. H. Yap ([15]) to Vilenkin groups $(1\leqq p\leqq 2)$, by G. Benke ([1], [2]) to totally disconnected compact non-abelian groups $(p=\infty)$, and by W. R. Bloom ([7]) to finite dimensional locally compact metric abelian groups $(1\leqq p\leqq\infty)$. Titchmarsh, Onneweer, Benke and Bloom gave examples to show that their results were best possible. Other extensions may be found in J. S. Bradley ([8]) and N. Ya. Vilenkin and A. I. Rubinshtein ([20]); both authors considered bounded Vilenkin groups.

Here we show that the statement of Bernstein's theorem, and its extension by Szász, is valid for all totally disconnected locally compact metric abelian groups; this will be the main result of Section 2. In Section 3 it will be shown that this result is sharp for all non-discrete totally disconnected locally compact metric abelian groups, by making use of a corresponding result (see Theorem 3) given here for a wide class of compact metric abelian groups. This completes part of the programme initiated by N. Ja. Vilenkin ([19]).

Throughout G will denote a non-discrete locally compact metric abelian group with translation invariant metric d and character group Γ_G. We shall choose Haar measures λ, θ for G, Γ_G respectively so that Plancherel's theorem is valid, with λ normalised in the usual way when G is compact (see [11], (31.1)). The real line R will be taken with its usual Euclidean metric, as will the circle group $T=R/Z$ (Z is the group of integers). For G totally disconnected we take a neighbourhood basis (V_n) at zero consisting of a strictly decreasing sequence of compact open subgroups of G (for the existence of such a basis see [11], (7.7)), (β_n) to be any strictly decreasing sequence of positive numbers tending to zero, and d defined on $G\times G$ by

$$d(x,y)=\begin{cases}\beta_{n+1}, & x-y\in V_n\setminus V_{n+1},\\ \beta_1, & x-y\notin V_1,\\ 0, & x=y\end{cases}$$

(see [21], Section 2). It is easily verified that d is a translation invariant metric on G compatible with the given topology. We follow [21], Section 2 and put $\beta_n=\lambda(V_n)$. This choice of metric agrees with that usually taken when G is a product of finite cyclic groups, and includes those considered by the authors above. It should be noted that our choice of the strictly decreasing sequence (V_n) is arbitrary.

Finally, the characteristic function of a set E will be denoted by ξ_E, and wherever C appears it denotes a positive constant, not necessarily the same from line to line.

2. Bernstein's theorem

We shall say that G has property $P(V_n, k_n, \Lambda_n)$ if there is a basis (V_n) of symmetric open neighbourhoods of zero and corresponding families (k_n), (Λ_n) of non-negative continuous functions and compact subsets of Γ_G respectively such that for each positive integer n, $\operatorname{supp}(k_n) \subset \mathscr{V}_n$ (the open subgroup of G generated by V_n), $\hat{k}_n(0)=1$, $\operatorname{supp}(\hat{k}_n) \subset \Lambda_n$ and

$$\int_{\mathscr{V}_n} m_{V_n} k_n \, d\lambda \leqq C;$$

here m_{V_n} is the integer valued function on $\mathscr{V}_n$ defined by

$$m_{V_n}(x) = \min \{m \in Z^+ \colon x \in mV_n\}.$$

It was shown in [5] that all metrisable locally compact abelian groups satisfy property $P(V_n, k_n, \Lambda_n)$ for suitable families (V_n), (k_n) and (Λ_n). However in many applications in approximation theory (for example, Theorem 1 below) it is important that the Λ_n do not increase too quickly as the V_n decrease. When $G=R$, T or is totally disconnected, or G is a group formed from these by taking quotients and finite products, families (V_n), (k_n) and (Λ_n) can be computed explicitly (see [6]), and the relation between V_n and Λ_n seems optimal.

By imposing a suitable growth condition on the families (V_n), (Λ_n), here given in terms of the radius $\delta(V_n)=\sup\{d(a,0)\colon a \in V_n\}$ of V_n and the Haar measure of Λ_n, we have obtained in [7] the following result on the Fourier transforms of functions in $\operatorname{Lip}(\alpha; p)$ (defined as in Section 1 with $|a|$ replaced by $d(a,0)$, where we assume throughout that $0<\alpha<1$; when $p=\infty$ the functions are taken to be continuous).

Theorem 1. *Suppose G satisfies property $P(V_n, k_n, \Lambda_n)$ where*

$$\sum_{n=1}^{\infty} \delta(V_n)^{\varepsilon} \theta(\Lambda_{n+1} \setminus \Lambda_n)^{\varepsilon'} < \infty$$

for $\varepsilon > \varepsilon'$. Then, for $1 \leqq p \leqq 2$ and $\alpha > 1/p + 1/r - 1 \geqq 0$ (with the convention that $\infty^{-1}=0$), $\operatorname{Lip}(\alpha; p)^{\wedge} \subset L^r(\Gamma_G)$. If, in addition, G is compact then $\operatorname{Lip}(\alpha; p)^{\wedge} \subset l^r(\Gamma_G)$ for $p>2$ and $\alpha > 1/r - 1/2$.

In the case when G is totally disconnected it suffices to take $k_n = \lambda(V_n)^{-1} \xi_{V_n}$ and $\Lambda_n = A(\Gamma_G, V_n)$, the annihilator of V_n in Γ_G. Then G has property $P(V_n, k_n, \Lambda_n)$ and $\delta(V_n) = \lambda(V_{n+1}) = \theta(\Lambda_{n+1})^{-1}$. Furthermore since (V_n) is strictly decreasing it follows that Λ_n is a proper subgroup of Λ_{n+1}, so that $\theta(\Lambda_n) \leqq 2^{-1}\theta(\Lambda_{n+1})$. Hence $\theta(\Lambda_{n+1}) \geqq 2^n \theta(\Lambda_1)$ and

$$\sum_{n=1}^{\infty} \delta(V_n)^{\varepsilon} \theta(\Lambda_{n+1} \setminus \Lambda_n)^{\varepsilon'} \leqq \sum_{n=1}^{\infty} \theta(\Lambda_{n+1})^{\varepsilon' - \varepsilon} < \infty$$

for $\varepsilon > \varepsilon'$. Thus the conditions of the theorem are satisfied and we have proved the following result:

Theorem 2. *Let G be a non-discrete totally disconnected locally compact metric abelian group, (V_n) any neighbourhood basis at zero consisting of a strictly decreasing sequence of compact open subgroups of G, and take the translation invariant metric d as in Section 1 with $\beta_n = \lambda(V_n)$. Then, for $1 \leqq p \leqq 2$ and $\alpha > 1/p + 1/r - 1 \geqq 0$, $\mathrm{Lip}(\alpha; p)^{\wedge} \subset L^r(\Gamma_G)$. If, in addition, G is compact then $\mathrm{Lip}(\alpha; p)^{\wedge} \subset l^r(\Gamma_G)$ for $p > 2$ and $\alpha > 1/r - 1/2$.*

It should be noted that a version of Theorem 2 is stated in [7], Theorem 3 for all finite dimensional groups. However the metric given there is unnecessarily small in the case when G is totally disconnected (0-dimensional). Also note that Theorem 2 continues to hold for G discrete, albeit rather trivially.

3. Sharpness of the results

In this section we give conditions on the group G under which Theorem 2 can be shown to be sharp (see Theorem 3) and we apply this result to the case where G is totally disconnected. We first state a definition.

Definition. Let (G, d) be a compact metric abelian group and assume the existence of a sequence (Λ_n) of non-empty finite subsets of Γ_G satisfying

(a) $\theta(\Lambda_n - \Lambda_n) \leqq C\theta(\Lambda_n)$;

(b) $\omega_{\Lambda_n}(a) \leqq C\theta(\Lambda_n) d(a, 0)$ for all $a \in G$, where

$\omega_{\Lambda_n}(a) = \max\{|\gamma(a) - 1| : \gamma \in \Lambda_n\}$;

(c) $\lim_{n\to\infty} \theta(\Lambda_n) = \infty$.

Let (P_n) be a sequence of trigonometric polynomials with $\Lambda_n = \mathrm{supp}(\hat{P}_n)$.

(i) (P_n) is said to be of type (F) if (Λ_n) satisfies (a)—(c) above,

$$P_n \geqq 0, \quad 0 \leqq \hat{P}_n \leqq 1, \quad \hat{P}_n(0) = 1$$

and

$$\theta[\{\gamma \in \Lambda_n \setminus \Lambda_n^{\#} : \hat{P}_n(\gamma) \geqq 1/2\}] \geqq C'\theta(\Lambda_n),$$

where $\Lambda_n^{\#} = \bigcup_{k<n} \Lambda_k$, $n > 1$, $\Lambda_1^{\#}$ is defined to be the empty set, and $C' \in (0, 1]$ is a constant.

(ii) (P_n) is said to be of type (RS) if (Λ_n) satisfies (a)—(c) above, $|\hat{P}_n(\gamma)| = 1$ for all $\gamma \in \Lambda_n$ and $\|P_n\|_\infty \leqq C\theta(\Lambda_n)^{1/2}$.

When G is the circle group the sequences (P_n) of type (F) and type (RS) are typified by the Fejér kernel and the sequence of Rudin—Shapiro polynomials respectively.

Theorem 3. *If a compact metric abelian group G admits a sequence of trigonometric polynomials of type (F) (respectively (RS)) then there exists $f\in \mathrm{Lip}\,(\alpha;\, p)$ with $\hat{f}\notin l^r(\Gamma_G)$, where $1\leqq p\leqq 2$ and $\alpha=1/p+1/r-1$ (respectively $p>2$ and $\alpha=1/r-1/2$).*

Proof. Suppose we have a sequence (Λ_n) satisfying (a)—(c) of the definition; in view of (c) we can, by choosing a subsequence of (Λ_n) if necessary, assume that $\theta(\Lambda_{n+1})\geqq 2^n\theta(\Lambda_n)$ for all n. Let $a\in G$ with $0<d(a,0)\leqq\theta(\Lambda_1)^{-1}$, and choose k such that $\theta(\Lambda_{k+1})^{-1}<d(a,0)\leqq\theta(\Lambda_k)^{-1}$. Let (P_n) be any sequence of trigonometric polynomials with $\Lambda_n=\mathrm{supp}\,(\hat{P}_n)$ and write $f=\sum_{n=1}^{\infty}\theta(\Lambda_n)^{-1/r}P_n$. Then

$$\|\tau_a f-f\|_p\leqq \sum_{n=1}^{k}\theta(\Lambda_n)^{-1/r}\,\|\tau_a P_n-P_n\|_p+2\sum_{n=k+1}^{\infty}\theta(\Lambda_n)^{-1/r}\,\|P_n\|_p.$$

Both of these sums will be estimated.

First, appealing to [4], Corollary 1.4,

$$\begin{aligned}\|\tau_a P_n-P_n\|_p &\leqq 3\left(\frac{\theta(\Lambda_n-\Lambda_n)}{\theta(\Lambda_n)}\right)^{1/2}\omega_{\Lambda_n-\Lambda_n}(a)\,\|P_n\|_p\\ &\leqq C\theta(\Lambda_n)\,d(a,0)\,\|P_n\|_p,\end{aligned}$$

using (a), (b) above. We now assume that (P_n) is of type (F) if $1\leqq p\leqq 2$ or of type (RS) if $p>2$. In either case

$$\|P_n\|_p\leqq C\theta(\Lambda_n)^{1/r-\alpha},$$

where α, p, r are related as in the statement of the theorem. Then

$$\begin{aligned}\sum_{n=1}^{k}\theta(\Lambda_n)^{-1/r}\|\tau_a P_n-P_n\|_p &\leqq C\sum_{n=1}^{k}\theta(\Lambda_n)^{-1/r}\theta(\Lambda_n)\,d(a,0)\,\|P_n\|_p\\ &\leqq C\sum_{n=1}^{k}\theta(\Lambda_n)^{1-\alpha}d(a,0)\\ &\leqq C\theta(\Lambda_k)^{1-\alpha}d(a,0)\\ &\leqq Cd(a,0)^{\alpha}.\end{aligned}$$

The second sum gives

$$\begin{aligned}\sum_{n=k+1}^{\infty}\theta(\Lambda_n)^{-1/r}\,\|P_n\|_p &\leqq C\sum_{n=k+1}^{\infty}\theta(\Lambda_n)^{-\alpha}\\ &\leqq C\theta(\Lambda_{k+1})^{-\alpha}\\ &\leqq Cd(a,0)^{\alpha}.\end{aligned}$$

Consequently $\|\tau_a f-f\|_p\leqq Cd(a,0)^{\alpha}$, so that $f\in\mathrm{Lip}\,(\alpha;p)$.

To show that $\hat{f}\notin l^r(\Gamma_G)$, consider

$$\sum_{\gamma\in\Gamma_G}|\hat{f}(\gamma)|^r = \sum_{m=1}^{\infty}\sum_{\gamma\in\Lambda_m\setminus\Lambda_m^{\#}}\Big|\sum_{n=1}^{\infty}\theta(\Lambda_n)^{-1/r}\hat{P}_n(\gamma)\Big|^r$$

$$= \sum_{m=1}^{\infty}\sum_{\gamma\in\Lambda_m\setminus\Lambda_m^{\#}}\Big|\sum_{n=m}^{\infty}\theta(\Lambda_n)^{-1/r}\hat{P}_n(\gamma)\Big|^r.$$

Now if $1\leqq p\leqq 2$ then, by assumption, (P_n) is of type (F) and for $\gamma\in\Lambda_m\setminus\Lambda_m^{\#}$ such that $\hat{P}_m(\gamma)\geqq 1/2$,

$$\Big|\sum_{n=m}^{\infty}\theta(\Lambda_n)^{-1/r}\hat{P}_n(\gamma)\Big|\geqq 2^{-1}\theta(\Lambda_m)^{-1/r}.$$

For $p>2$ we have that (P_n) is of type (RS) and

$$\Big|\sum_{n=m}^{\infty}\theta(\Lambda_n)^{-1/r}\hat{P}_n(\gamma)\Big| \geqq \theta(\Lambda_m)^{-1/r}-\sum_{n=m+1}^{\infty}\theta(\Lambda_n)^{-1/r}$$

$$\geqq \theta(\Lambda_m)^{-1/r}-\theta(\Lambda_m)^{-1/r}\sum_{n=m}^{\infty}2^{-(n+\cdots+m)/r}$$

$$\geqq \theta(\Lambda_m)^{-1/r}-C2^{-m/r}\theta(\Lambda_m)^{-1/r}$$

$$\geqq 2^{-1}\theta(\Lambda_m)^{-1/r}$$

for m sufficiently large. In either case

$$\sum_{m=1}^{\infty}\sum_{\gamma\in\Lambda_m\setminus\Lambda_m^{\#}}\Big|\sum_{n=m}^{\infty}\theta(\Lambda_n)^{-1/r}\hat{P}_n(\gamma)\Big|^r=\infty,$$

so that $\hat{f}\notin l^r(\Gamma_G)$. ||

The converse of Bernstein's theorem for $G=T$ (when $\alpha=1/2$, $p=\infty$ and $r=1$) follows from Theorem 3 on taking $\Lambda_n=\{0, 1, \dots, 2^n-1\}$ and (P_n) to be the sequence of Rudin—Shapiro polynomials (see also [12], Chapter 2, Section 7). For Szász's extension of this result ([16], [17]) the same choice of (Λ_n), (P_n) will suffice in the case $p>2$; and when $1\leqq p\leqq 2$, take $\Lambda_n=\{-3^n, -3^n+1, \dots, -1, 0, 1, \dots, 3^n-1, 3^n\}$ and $P_n=K_{3^n}$, where (K_n) is the Fejér kernel. The proof of Theorem 3 can be slightly modified to give an analogue of the converse of Bernstein's theorem for $G=R$. This case was first considered by Titchmarsh ([18]).

We shall now show that sequences of trigonometric polynomials of type (F) and type (RS) exist in all non-discrete totally disconnected compact metric abelian groups.

Lemma 1. *Let G be a non-discrete totally disconnected compact metric abelian group, (V_n) any neighbourhood basis at zero consisting of a strictly decreasing sequence of (compact) open subgroups of G, and take the translation invariant metric d as in Section 1 with $\beta_n=\lambda(V_n)$. Then G admits sequences of type (F) and type (RS) with $\Lambda_n=A(\Gamma_G, V_n)$.*

Proof. We first note that (Λ_n) satisfies (a)—(c) of the definition above. Indeed (a) and (c) obviously hold; and for (b) consider $a \in G \setminus V_n$. Then $d(a, 0) \geqq \theta(\Lambda_n)^{-1}$, so that

$$\omega_{\Lambda_n}(a) \leqq 2 \leqq 2\theta(\Lambda_n) d(a, 0).$$

For $a \in V_n$ there is nothing to prove.

Now $P_n = \lambda(V_n)^{-1} \xi_{V_n}$ gives a sequence of type (F) with $\Lambda_n = \mathrm{supp}\,(\hat{P}_n)$. Clearly $P_n \geqq 0$, and $\hat{P}_n = \xi_{\Lambda_n}$ satisfies $0 \leqq \hat{P}_n \leqq 1$ and $\hat{P}_n(0) = 1$. Also, for $n > 1$, Λ_{n-1} is a proper subgroup of Λ_n so that, given $\gamma \in \Lambda_n \setminus \Lambda_{n-1}$, the sets $\gamma + \Lambda_{n-1}$ and Λ_{n-1} are disjoint. Hence $\gamma + \Lambda_{n-1} \subset \Lambda_n \setminus \Lambda_{n-1}$ and

$$\theta(\Lambda_{n-1}) = \theta(\gamma + \Lambda_{n-1}) \leqq \theta(\Lambda_n \setminus \Lambda_{n-1}).$$

From this it follows that $\theta(\Lambda_n) \leqq 2\theta(\Lambda_n \setminus \Lambda_{n-1})$ and

$$\theta[\{\gamma \in \Lambda_n \setminus \Lambda_n^{\#} : \hat{P}_n(\gamma) \geqq 1/2\}] = \theta(\Lambda_n \setminus \Lambda_{n-1}) \geqq 2^{-1}\theta(\Lambda_n).$$

To show that G admits a sequence of type (RS) we appeal to the results in [10]. For each n consider the natural homomorphism

$$\pi_n \colon G \to G/V_n.$$

The character group of (the finite group) G/V_n is isomorphic with Λ_n, and [10] gives the existence of P_n^* on G/V_n such that $|\hat{P}_n^*| = 1$ and $\|P_n^*\|_\infty = \theta(\Lambda_n)^{1/2}$. Clearly $P_n = P_n^* \circ \pi_n$ is a trigonometric polynomial on G with $\mathrm{supp}\,(\hat{P}_n) = \Lambda_n$, $|\hat{P}_n| = 1$ on Λ_n and $\|P_n\|_\infty = \theta(\Lambda_n)^{1/2}$. ||

Theorem 4. *Let G be a non-discrete totally disconnected locally compact metric abelian group, (V_n) any neighbourhood basis at zero consisting of a strictly decreasing sequence of compact open subgroups of G, and take the translation invariant metric d as in Section* 1 *with $\beta_n = \lambda(V_n)$. Then, for $1 \leqq p \leqq 2$ and $\alpha > 1/p + 1/r - 1 \geqq 0$,* $\mathrm{Lip}\,(\alpha; p)^\wedge \subset L^r(\Gamma_G)$. *If, in addition, G is compact then* $\mathrm{Lip}\,(\alpha; p)^\wedge \subset l^r(\Gamma_G)$ *for $p > 2$ and $\alpha > 1/r - 1/2$.*

In neither case can the range of values of α be extended.

Proof. That the theorem holds for G compact follows by combining Theorem 3, Lemma 1 and Theorem 2. It remains to remove the restriction that G be compact for the last assertion.

Suppose that G is noncompact. We assume (see [11], (31.1)) that the Haar measures λ, θ on G, Γ_G respectively are normalised so that $\lambda(V) = \theta(A(\Gamma_G, V)) = 1$ (where, for typographical reasons, we have written V for V_1). Since V is non-discrete, totally disconnected and compact, the first part of the proof shows the existence of

$f' \in L^p(V)$ with $f' \in \mathrm{Lip}\,(\alpha; p)$ and $\hat{f}' \notin l^r(\Gamma_V)$, where $\alpha = 1/p + 1/r - 1$ for $1 \leqq p \leqq 2$, and $\alpha = 1/r - 1/2$ for $p > 2$. Now it is clear that the tensor product $\xi_{\{V\}} \otimes f'$ on $G/V \times V$ (defined by $\xi_{\{V\}} \otimes f'(x+V, v) = \xi_{\{V\}}(x+V) f'(v)$) belongs to $\mathrm{Lip}\,(\alpha; p)$ and, since $\Gamma_{G/V \times V} \cong \Gamma_{G/V} \times \Gamma_V \cong A(\Gamma_G, V) \times \Gamma_V$ and $A(\Gamma_G, V)$ is compact, it follows that $(\xi_{\{V\}} \otimes f')\hat{} \notin L^r(\Gamma_{G/V \times V})$.

Now $G/V \times V$ and G have the same $\mathrm{Lip}\,(\alpha; p)$ functions and $L^r(\Gamma_{G/V \times V}) = L^r(\Gamma_G)$; indeed the first assertion follows from the facts that $G/V \times V$ and G are isometric, locally isomorphic and have the same Haar measure, and for the second note that $\Gamma_{G/V \times V} \cong A(\Gamma_G, V) \times (\Gamma_G / A(\Gamma_G, V))$ and Γ_G have the same Haar measure (recall that $A(\Gamma_G, V)$ is compact and open). Hence we deduce the existence of $f \in L^p(G)$ with $f \in \mathrm{Lip}\,(\alpha; p)$ and $\hat{f} \notin L^r(\Gamma_G)$, with α, p, r related as above. ||

It is of interest to see that, in general, the restriction $1/p + 1/r - 1 \geqq 0$ in Theorem 4 cannot be removed. Just consider $G = Z \times H$, where H is an arbitrary totally disconnected group. If $1/p + 1/r - 1 < 0$, so that $p' < r$, then there exists $f_1 \in l^p(Z)$ with $\hat{f}_1 \notin L^r(T)$. Take any $f_2 \in L^p(H) \cap \mathrm{Lip}\,(\alpha; p)$ with $f_2 \neq 0$ ($\alpha > 0$ is arbitrary). Then $f_1 \otimes f_2 \in L^p(G) \cap \mathrm{Lip}\,(\alpha; p)$ but $(f_1 \otimes f_2)\hat{} \notin L^r(\Gamma_G)$.

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