A constructive method for L^p-approximation by analytic functions

Per Lindberg

In [3] Havin gave a necessary and sufficient condition for L^2-approximation by functions analytic in the neighbourhood of a Borel set. He also proved the possibility of approximation in L^p-norm when $1<p<2$. A condition necessary and sufficient for all p, $1<p<\infty$, was given by Bagby [1]. He formulated this condition for approximation on compact sets by rational functions. It was proved by a dual method depending on the behaviour of functions of a Sobolev space. Other results stating different conditions were obtained by Hedberg [4], [5]. In this paper we give a constructive method for L^p-approximation by analytic functions analogous to the Vituškin method for uniform approximation [7]. We obtain a theorem equivalent to that of Bagby, although it is here given a more general formulation. The necessary and sufficient conditions are here expressed in analytic p-capacity. For the interdependence between the analytic p-capacity and other capacities, including the one used by Bagby, see [5].

For any set E in $\mathbf{C}$ denote by $A^p(E)$ the $L^p(\mathbf{C})$-closure of the functions in $L^p(\mathbf{C})$ analytic in a neighbourhood of E.

Theorem

If E and D are sets in $\mathbf{C}$, and $1<p<\infty$, the following statements are equivalent.

(a) $$A^p(D) \subset A^p(E).$$

(b) For all open G and measurable Ω in $\mathbf{C}$. (Ω bounded if $p\leqq 2$)

$$\gamma_p(G\setminus D, \Omega) \leqq \gamma_p(G\setminus E, \Omega)$$

(c) There is a sequence $\delta_n\searrow 0$ and a constant $k>0$ such that for every open disc $B(z, \delta_n)$ with centre $z\in E\setminus D$ and radius δ_n

$$\gamma_p(B(z, \delta_n)\setminus D, B(z, 2\delta_n)) \leqq k\gamma_p(B(z, \delta_n)\setminus E, B(z, 2\delta_n)).$$

(d) There is a sequence $\delta_n \searrow 0$ such that for all $z \in E \setminus D$ except a set of capacity zero

$$\varliminf_{n\to\infty} \frac{\gamma_p\big(B(z,\delta_n)\setminus E,\, B(z,2\delta_n)\big)}{\gamma_p\big(B(z,\delta_n)\setminus D,\, B(z,2\delta_n)\big)} > 0.$$

γ_p is here the analytic p-capacity, defined for any set E by $\gamma_p(E,\Omega)=\sup_f |f'(\infty)| = \sup_f |\lim_{z\to\infty} zf(z)|$, where the supremum is taken over those functions f which are analytic off some compact subset of E, which satisfy $f(\infty)=0$ and

$$\|f\|_{p,\Omega} = \Big(\int_\Omega |f(z)|^p\,dx\,dy\Big)^{1/p} \leqq 1.$$

For $p>2$ there is no need to use the set Ω in the definition, it could everywhere be replaced by $\mathbf{C}$.

The conditions for approximation by rational functions follows immediately from the theorem.

Corollary

Let E be a compact set, and denote by $L^p_a(E)$ the functions in $L^p(E)$ which are analytic in the interior of E. Then the following are equivalent for $1<p<\infty$:

(a′) Rational functions with poles off E are dense in $L^p_a(E)$.

(b′) For all open G and measurable Ω in $\mathbf{C}$ (Ω bounded if $p\leqq 2$)

$$\gamma_p(G\setminus E^0, \Omega) = \gamma_p(G\setminus E, \Omega).$$

The corollary follows by use of Runge's theorem on rational approximation. Thus *(a′)* is true if and only if functions analytic in a neighbourhood of E are dense in $L^p_a(E)$, which is equivalent to $A^p(E)=A^p(E^0)$.

Remarks

1. When $p<2$ the situation is much simplified, since rational functions with simple poles are then locally in L^p. From this fact it follows that

$$k_1\cdot 2^{1-(2/p)} \leqq \gamma_p\big(B(z,\delta)\setminus E,\, B(z,2\delta)\big) \leqq k_2\cdot 2^{1-(2/p)}$$

as soon as $B(z,\delta)\setminus E$ is not empty. Thus the statement *(c)* is for $p<2$ equivalent to $E^0\subset D$.

The theorem is true also for $p=1$. It can be proved by a constructive method used in [6].

2. It is possible to give conditions for approximation of a single function similar to those for uniform approximation ([7] Ch. IV § 2 Lemma 1 or [2] Ch. VIII Th. 8:1).

The constructive method could also be used to give results for approximation by other linear classes of functions in the plane. The only important thing is that the treated classes should be closed under the T_φ-operator.

The T_φ-operator

If φ is a Lipschitz continuous function with compact support in the plane, we can define an operator T_φ by

$$T_\varphi f(\zeta) = \frac{1}{\pi}\int \frac{f(z)-f(\zeta)}{z-\zeta}\frac{\partial\varphi}{\partial\bar z}\,dx\,dy = \varphi(\zeta)f(\zeta)+\frac{1}{\pi}\int\frac{f(z)}{z-\zeta}\frac{\partial\varphi}{\partial\bar z}\,dx\,dy.$$

The operator will be defined on any locally integrable function f. We will use the following properties:

(a) $T_\varphi f$ is analytic where f is analytic.

(b) $T_\varphi f$ is analytic outside the support of φ.

(c) $T_\varphi f(\infty)=0$.

(d) $f-T_\varphi f$ is analytic both where f is analytic and in the interior of the set where $\varphi(z)=1$.

(e) If $B(z,\delta)$ contains the support of φ, then for any constant a:

$$\|T_\varphi f\|_{p, B(z, 2\delta)} \leqq 8\delta\left\|\frac{\partial\varphi}{\partial\bar z}\right\|_\infty \cdot \|f-a\|_{p, B(z,\delta)}.$$

The statements (a)—(d) are well known, see [2]. To prove (e) let $z=0$ and let $B=B(0,\delta)$, $2B=B(0,2\delta)$ etc. Now

$$\|T_\varphi f\|_{p,2B} \leqq \|\varphi\cdot f\|_{p,2B}+\frac{1}{\pi}\left\|\int\frac{f(z)}{z-\zeta}\frac{\partial\varphi(z)}{\partial\bar z}\,dx\,dy\right\|_{p,2B}$$

$$\leqq \|\varphi\|_\infty \|f\|_{p,B}+\frac{1}{\pi}\left\|f\frac{\partial\varphi}{\partial\bar z}*\frac{1}{z}\right\|_{p,2B}.$$

For the first term use that $\|\varphi\|_\infty \leqq 2\delta\left\|\frac{\partial\varphi}{\delta\bar z}\right\|_\infty$ since φ has its support in B. To estimate the second term use Young's inequality. This gives

$$\|T_\varphi f\|_{p,2B} \leqq 2\delta\left\|\frac{\partial\varphi}{\partial\bar z}\right\|_\infty \cdot\|f\|_{p,B}$$

$$+\frac{1}{\pi}\left\|f\frac{\partial\varphi}{\partial\bar z}\right\|_{p,B}\cdot\left\|\frac{1}{z}\right\|_{1,3B} \leqq 8\delta\left\|\frac{\partial\varphi}{\partial\bar z}\right\|_\infty \|f\|_{p,B}.$$

Now observe that $T_\varphi(f-a)=T_\varphi f$ for any constant a. Therefore (e) is obtained when f is replaced by $f-a$ in the inequality above.

The Vituškin constructive method for uniform approximation of analytic functions is briefly the following. Construct a partition of unity by test functions φ_j each with its support in a small disc B_j. Now $f=\sum T_{\varphi_j} f$ and each $T_{\varphi_j} f$ is analytic outside the disc B_j. For each $T_{\varphi_j} f$ then choose a function g_j from the desired class of approximating functions, in such a way that $\sum g_j$ approximate f. For uniform approximation the crucial point is to choose g_j such that $T_{\varphi_j} f-g_j$ has a zero of order three at infinity. For L^p-approximation however it will suffice to have a zero of order two, due to the following lemma.

Lemma

Let B_j, $j=1, 2, \dots$, be discs with identical radii, such that no point in the plane lies in more than M discs. Let f_j be analytic outside $\bar{B}_j$ and have a double zero at infinity. Then for $1<p<\infty$,

$$\Big\|\sum\nolimits_j f_j\Big\|_p^p \leqq k_{p,M} \sum\nolimits_j \|f_j\|_p^p .$$

Proof of the lemma

The first part is to prove the inequality when every f_j has a zero of order three at infinity. Let $B_j=B(z_j, \delta)$ and $A_n(w)=\{z|(n-1)\delta\leqq|z-w|<n\delta\}$.

If f_j is analytic off $\bar{B}_j$ and has a zero of order three at infinity, we have the inequality,

$$\|f_j\|_{p, A_n(z_j)} \leqq k_p n^{(1/p)-3} \|f_j\|_{p, A_2(z_j)}, \quad \text{for} \quad n \geqq 2.$$

This inequality is easily deduced from the fact that the integral

$$\int_0^{2\pi} r^{3p} |f_j(re^{i\theta}+z_j)|^p \, d\theta$$

decreases when r increases, which is a consequence of the fact that $|(z-z_j)^3 f(z)|^p$ is a subharmonic function on the Riemann sphere outside $\bar{B}_j$.

Thus for $n\geqq 1$

$$\|f_j\|_{p, A_n(z_j)}^p \leqq k_p n^{1-3p} \|f_j\|_{p, 2B_j}^p,$$

where $2B_j=B(z_j, 2\delta)$.

Let M_n be the number of points z_j that lie in $A_n(w)$. Since the discs with these points as centers are contained in the annulus $(n-2)\delta\leqq|z-w|\leqq(n+1)\delta$, we get

$$M_n \pi\delta^2 \leqq M\big(\pi(n+1)^2\delta^2-\pi(n-2)^2\delta^2\big),$$

which gives $M_n \leqq 6M \cdot n$.

By use of Minkowski's and Hölder's inequalities and the inequalities above we get

$$\begin{aligned}
\Big\|\sum_j f_j(z)\Big\|_p &\leqq \Big(\int \big(\sum_{n=1}^{\infty} \sum_{j\atop z_j\in A_n(z)} |f_j(z)|\big)^p\,dx\,dy\Big)^{1/p} \\
&\leqq \sum_{n=1}^{\infty}\Big(\int \big(\sum_{j\atop z_j\in A_n(z)} |f_j(z)|\big)^p\,dx\,dy\Big)^{1/p} \\
&\leqq \sum_{n=1}^{\infty}\Big(\int M_n^{p/q}\big(\sum_{j\atop z_j\in A_n(z)} |f_j(z)|^p\big)\,dx\,dy\Big)^{1/p} \\
&\leqq \sum_{n=1}^{\infty}(6Mn)^{1/q}\Big(\sum_j \int_{z\in A_n(z_j)} |f_j(z)|^p\,dx\,dy\Big)^{1/p} \\
&\leqq \sum_{n=1}^{\infty}(6M\cdot n)^{1/q}\big(\sum_j \|f_j\|_{p,2B_j}^p\cdot k_p n^{1-3p}\big)^{1/p} \\
&\leqq (6M)^{1/q}\cdot k_p^{1/p}\big(\sum_j \|f_j\|_{p,2B_j}^p\big)^{1/p}\sum_{n=1}^{\infty} n^{-2} \\
&\leqq k_p'\cdot M^{1/q}\big(\sum_j \|f_j\|_p^p\big)^{1/p}.
\end{aligned}$$

Thus we have done the first part of the proof. We will now extend this result. Assume in the following that the functions f_j have zeros of order two at infinity.

Let $f_j''(\infty)=\lim_{z\to\infty} z^2 f_j(z)$, and define the functions g_j by

$$g_j(z)=\begin{cases} f_j''(\infty)/\pi\delta^2 & \text{when } |z-z_j|<\delta. \\ 0 & \text{otherwise.}\end{cases}$$

Since $\gamma_p\big(B(z,\delta),\ B(z,2\delta)\big)\leqq k_p\delta^{1-(2/p)}$ we get

$$\begin{aligned}
|f_j''(\infty)| &\leqq \gamma_p(B_j, 2B_j)\cdot\|(z-z_j)f_j\|_{p,2B_j} \\
&\leqq k_p\delta^{2-(2/p)}\|f_j\|_{p,2B_j},
\end{aligned}$$

and therefore

$\|g_j\|_p\leqq k_p\|f_j\|_p$ for some constant k_p only depending on p.

Now define a singular integral operator S by

$$Sg(\zeta)=\lim_{r\to 0}\int_{|z|>r}\frac{g(\zeta-z)}{z^2}\,dx\,dy.$$

From the theory on singular integral operators we know that $\|Sg\|_p\leqq k_p\|g\|_p$. The functions g_j are constructed so that f_j-Sg_j has a zero of order 3 at infinity, and we can apply the first part of the proof to these functions.

$$\begin{aligned}
\Big\|\sum_j f_j\Big\|_p &\leqq \Big\|\sum_j Sg_j\Big\|_p+\Big\|\sum_j (f_j-Sg_j)\Big\|_p \\
&\leqq \Big\|S\big(\sum_j g_j\big)\Big\|_p+k_{p,M}\big(\sum_j \|f_j-Sg_j\|_p^p\big)^{1/p} \\
&\leqq k_p\Big\|\sum_j g_j\Big\|_p+k_{p,M}\big(\sum_j 2^p(\|f_j\|_p^p+k_p\|g_j\|_p^p)\big)^{1/p} \\
&\leqq k_p\Big\|\sum_j g_j\Big\|_p+k_{p,M}'\big(\sum \|f_j\|_p^p\big)^{1/p}.
\end{aligned}$$

As no point in the plane is in the support of more than the number M of functions g_j,

$$\Big|\sum_j g_j(z)\Big|^p\leqq M^p\sum_j |g_j(z)|^p.$$

This gives

$$\left\|\sum_j g_j\right\|_p \leqq M\left(\sum_j \|g_j\|_p^p\right)^{1/p} \leqq M\cdot k_p\left(\sum_j \|f_j\|_p^p\right)^{1/p}.$$

Inserted in the inequality above this gives the lemma.

Proof of the theorem

We begin by proving that *(c)* gives *(a)*. Let f in L^p be analytic in a neighbourhood of D. Assume that *(c)* is true. Let δ be an element of the sequence δ_n. Cover $E\setminus D$ with open discs $B_j=B(z_j, \delta)$ of radii δ and with centers in $E\setminus D$, such that also the half scale discs $B(z_j, \delta/2)$ cover $E\setminus D$, and such that no point in $\mathbf{C}$ lies in more than M discs. The bound M is made independent of δ.

It is possible to choose test functions $\varphi_j\in C_0^\infty(B_j)$ such that

$$\left\|\frac{\partial\varphi_j}{\partial\bar z}\right\|_\infty \leqq \frac{k}{\delta} \quad\text{and}\quad \sum_j \varphi_j(z)=1 \quad\text{in a neighbourhood of } E\setminus D.$$

The function $T_{\varphi_j}f$ is now analytic outside a compact subset of $B_j\setminus D$, and $T_{\varphi_j}f(\infty)=0$. Therefore according to the assumption *(c)* there is a function g_j, analytic outside $B_j\setminus E$ with $g_j(\infty)=0$ such that

$$g_j'(\infty) = T_{\varphi_j}f'(\infty)$$

and

$$\|g_j\|_{p, B(z_j, 2\delta)} \leqq k\|T_{\varphi_j}f\|_{p, B(z_j, 2\delta)}.$$

The function $f-\sum_j(T_{\varphi_j}f-g_j)$ is analytic in a neighbourhood of E. Applying the lemma we get, $k_{p,M}$ denoting various constants:

$$\begin{aligned}\left\|\sum_j(T_{\varphi_j}f-g_j)\right\|_p^p &\leqq k_{p,M}\sum_j \|T_{\varphi_j}f-g_j\|_p^p\\ &\leqq k_{p,M}\sum_j \|T_{\varphi_j}f-g_j\|_{p,B(z_j,2\delta)}^p\\ &\leqq k_{p,M}\sum_j \|T_{\varphi_j}f\|_{p,B(z_j,2\delta)}^p\\ &\leqq k_{p,M}\sum_j \|f-a_j\|_{p,B(z_j,\delta)}^p.\end{aligned}$$

Let now $a_j=(\pi\delta^2)^{-1}\int_{B(z_j,\delta)} f\,dx\,dy$.

This will imply that $\sum_j \|f-a_j\|_{p,B(z_j,\delta)}^p\to 0$ when $\delta\to 0$. This is in fact true for a continuous function f, and follows generally since continuous functions are dense in L^p.

Thus we have obtained *(a)*.

To prove that *(a)* gives *(b)*, let $\varepsilon>0$, G open and Ω be given. We can assume that $\gamma_p(G\setminus D, \Omega)>0$, otherwise there is nothing to prove. Therefore $G\setminus D$ is not empty. We will assume that $\gamma_p(G\setminus D, \Omega)$ is finite. If it is not, the proof only needs a few adjustments. Now there exists a compact set $K\subset G\setminus D$ such that

$$\gamma_p(G\setminus D, \Omega) \leqq \gamma_p(K, \Omega)+\varepsilon.$$

Let $\varphi\in C_0^\infty(G)$ with $\varphi(z)=1$ in a neighbourhood of K. Choose a function f analytic off K with $f(\infty)=0$, $\|f\|_{p,\Omega}\leqq 1$ and $|f'(\infty)|\geqq\gamma_p(K, \Omega)-\varepsilon$. Since $\varphi(z)=1$ where

f is not analytic, $T_\varphi f=f$. Assume now that *(a)* is true. Then there is a sequence of functions $(g_n)_1^\infty$ analytic in neighbourhoods of E, which tend to f in L^p-norm.

We will now get

$$\|T_\varphi(g_n-f)\|_{p,\Omega} \to 0.$$

When Ω is bounded, this can be seen directly from the property *(e)* of the T_φ-*operator*. When $p>2$, also observe that there exists a bounded set Ω' only dependent on the support of φ, such that

$$\|T_\varphi(g_n-f)\|_{p,\mathbf{C}} \leqq k_p\|T_\varphi(g_n-f)\|_{p,\Omega'}.$$

We will also get

$$T_\varphi g_n'(\infty) \to f'(\infty).$$

This follows from the fact that

$$T_\varphi(g_n-f)'(\infty) = -\frac{1}{\pi}\int (g_n-f)\frac{\partial\varphi}{\partial\bar z}\,dx\,dy.$$

Therefore, when n is large enough, we have

$$\|T_\varphi g_n\|_{p,\Omega} \leqq \|f\|_{p,\Omega}+\varepsilon \leqq 1+\varepsilon$$

and

$$|T_\varphi g_n'(\infty)| \geqq |f'(\infty)|-\varepsilon$$
$$\geqq \gamma_p(K,\Omega)-2\varepsilon \geqq \gamma_p(G\setminus D,\Omega)-3\varepsilon.$$

This gives

$$\gamma_p(G\setminus E,\Omega) \geqq \frac{\gamma_p(G\setminus D,\Omega)-3\varepsilon}{1-\varepsilon},$$

which implies *(b)*.

So far we have proved that *(a)*, *(b)* and *(c)* are equivalent. This we will use to prove that *(d)* implies *(a)*. We will also need the following three properties of the analytic p-capacity.

1) If E_n for $n=1,2,\ldots$ is a decreasing sequence of arbitrary sets, then

$$\gamma_p(E_n,\Omega) \to \gamma_p\left(\bigcap_1^\infty E_n,\Omega\right).$$

2) For any sets E_1 and E_2

$$\gamma_p(E_1\cup E_2,\Omega) \leqq k_p(\gamma_p(E_1,\Omega)+\gamma_p^*(E_2,\Omega))$$

where γ_p^* is the outer capacity.

3) If E is a Borel set then

$$\gamma_p^*(E,\Omega) \leqq k_p\gamma_p(E,\Omega).$$

The first statement follows for arbitrary sets since it is true for compact sets. The other two follows by a comparison with the capacity used by Bagby, since this is subadditive and makes all Borel sets capacitable.

The constants k_p only depend on p, as long as there is a disc B containing E, E_1 and E_2 with $2B$ contained in Ω.

Assume now that *(d)* is true. Define a sequence of sets E_N by

$$E_N = \{z|\gamma_p(B(z, \delta_n)\setminus E,\ B(z, 2\delta_n)) < \frac{1}{N}\gamma_p(B(z, \delta_n)\setminus D, B(z, 2\delta_n)) \quad \text{for some} \quad n \geqq N\}.$$

Since the condition *(c)* is sufficient we know that $A^p(D\cup E_N)\subset A^p(E)$ for every N. From *(b)* we get

$$\gamma_p(B\setminus(D\cup E_N), 2B) \leqq \gamma_p(B\setminus E, 2B)$$

for all open discs B.

The inequality *2)* gives

$$\gamma_p(B\setminus D, 2B) \leqq k_p\gamma_p((B\setminus D)\setminus E_N, 2B) + k_p\gamma_p^*(B\cap E_N, 2B) \leqq k_p\gamma_p(B\setminus E, 2B) + k_p\gamma_p^*(B\cap E_N, 2B).$$

Since the set $\{z|\gamma_p(B(z, \delta)\setminus E, 2B)\leqq a\}$ is closed for any constant a, E_N is a Borel set. By *3)* this implies

$$\gamma_p^*(B\cap E_N, 2B) \leqq k_p\gamma_p(B\cap E_N, 2B) \to \gamma_p\left(B\cap \bigcap_{N=1}^{\infty} E_N, 2B\right) = 0.$$

Thus we get for all discs B, $\gamma_p(B\setminus D, 2B)\leqq k_p\gamma_p(B\setminus E, 2B)$ and therefore $A^p(D)\subset A^p(E)$.

References

1. BAGBY, T., Quasi topologies and rational approximation. *J. Funct. Analysis* **10** (1972), 259—268.
2. GAMELIN, T. W., *Uniform Algebras*. Prentice Hall 1969.
3. HAVIN, V. P., Approximation in the mean by analytic functions. *Doklady Akad. Nauk SSSR* **178** (1968), 1025—1028 (Soviet Math. Doklady **9** (1968), 245—248).
4. HEDBERG, L. I., Approximation in the mean by analytic functions. *Trans. of the Am. Math. Soc.* **163** (1972), 157—171.
5. HEDBERG, L. I., Non-linear potentials and approximation in the mean by analytic functions. *Math. Z.* **129** (1972), 299—319.
6. LINDBERG, P., L^p-approximation by analytic functions in an open region. *Uppsala Univ. Dep. of Math.* Report No 1977:7.
7. VITUŠKIN, A. G., Analytic capacity of sets in problems of approximation theory. *Uspehi Mat. Nauk* **22** (1967), 141—199 (*Russian Math. Surveys* **22** (1967), 139—200).

Received September 24, 1980

Dr. Per Lindberg
Televerkets Centralförvaltning
S-123 86 Farsta
Sweden