Ark. Mat., **39** (2001), 27–64

Optimal decompositions for the K-functional for a couple of Banach lattices

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Abstract. Let $f = g_t + h_t$ be the optimal decomposition for calculating the exact value of the K-functional $K(t, f; \overline{X})$ of an element f with respect to a couple $\overline{X} = (X_0, X_1)$ of Banach lattices of measurable functions. It is shown that this decomposition has a rather simple form in many cases where one of the spaces X_0 and X_1 is either L^∞ or L^1. Many examples are given of couples of lattices $\overline{X}$ for which $|g_t|$ increases monotonically a.e. with respect to t. It is shown that this property implies a sharpened estimate from above for the Brudnyi–Krugljak K-divisibility constant $\gamma(\overline{X})$ for the couple. But it is also shown that certain couples $\overline{X}$ do not have this property. These also provide examples of couples of lattices for which $\gamma(\overline{X}) > 1$.

1. Introduction

Let X_0 and X_1 be Banach lattices of (equivalence classes of) real valued measurable functions on the same measure space (Ω, Σ, μ). It is well known (see e.g. [13], pp. 40–42 or Remark 1.41 of [10]) that X_0 and X_1 form a Banach couple $\overline{X} = (X_0, X_1)$ in the sense of interpolation theory ([4], p. 24, [5], p. 91).

A basic notion in the study of interpolation spaces with respect to any Banach couple $\bar{A} = (A_0, A_1)$ is the Peetre K-functional, defined for each $f \in A_0 + A_1$ and each $t > 0$ by

$$K(t, f; \bar{A}) = \inf\{\|g\|_{A_0} + t\|h\|_{A_1} : g \in A_0,\ h \in A_1,\ g + h = f\}. \tag{1}$$

The norms of many interpolation spaces are obtained by composing the K-functional with suitable lattice norms defined on functions on $(0, \infty)$. For many couples $\bar{A}$, *all* interpolation space norms with respect to $\bar{A}$ can be obtained in this way.

There is a rather extensive literature devoted to the calculation of K-functionals for particular couples. In many cases there are concrete formulæ for functionals

(1) Research supported by the Technion V. P. R. Fund.

which are equivalent to $K(t,f;\bar{A})$, i.e. the constants of equivalence are independent of f and t. Furthermore, for a number of specific couples, an explicit and exact formula has been obtained for the K-functional for each element $f \in A_0 + A_1$ and it is also possible to describe elements g_t and h_t for which the infimum in (1) is attained, i.e.

$$(2) \qquad f = g_t + h_t, \quad g_t \in A_0, \quad h_t \in A_1 \quad \text{and} \quad K(t,f;\bar{A}) = \|g_t\|_{A_0} + t\|h_t\|_{A_1}.$$

See e.g. [1], [2], [11], Lemma 4.1, [16] and [19]. It will be convenient to refer to any pair of families $\{g_t\}_{t>0}$ and $\{h_t\}_{t>0}$ satisfying $f = g_t + h_t$, $g_t \in A_0$ and $h_t \in A_1$ for some fixed $f \in A_0 + A_1$ and for each $t>0$ as a *decomposition of* f. We shall also use the notation $\{f = g_t + h_t\}_{t>0}$. Such a decomposition will be called an *optimal decomposition of* f if it satisfies (2) for each $t>0$.

Every optimal decomposition $\{f = g_t + h_t\}_{t>0}$ of any given element f has the property that

$$(3) \quad t \longmapsto \|g_t\|_{A_0} \text{ is non-decreasing and } t \longmapsto \|h_t\|_{A_1} \text{ is non-increasing on } (0,\infty).$$

Let us describe a slightly more general result which implies (3) and which holds also if f does not have an optimal decomposition: For each fixed $t>0$ there always exist sequences of functions $\{g_{n,t}\}_{n\in\mathbf{N}}$ and $\{h_{n,t}\}_{n\in\mathbf{N}}$, in A_0 and A_1 respectively such that $f = g_{n,t} + h_{n,t}$ and $K(t,f;\bar{A}) \le \|g_{n,t}\|_{A_0} + t\|h_{n,t}\|_{A_1} \le (1+1/n)K(t,f;\bar{A})$. By passing if necessary to subsequences, we can suppose that the limits $x(t) = \lim_{n\to\infty} \|g_{n,t}\|_{A_0}$ and $y(t) = \lim_{n\to\infty} \|h_{n,t}\|_{A_1}$ both exist. Then $x(t) + ty(t) = K(t,f;\bar{A})$. Every pair of functions $x(t)$ and $y(t)$ obtained for each $t>0$ in this way satisfies

$$(4) \qquad x(t) \text{ is non-decreasing and } y(t) \text{ is non-increasing on } (0,\infty).$$

The validity of the condition (4) and so also of (3) is rather well known. It can be deduced from an examination of the Gagliardo diagram (cf. e.g. [4], p. 39). For the reader's convenience, we also provide an explicit proof at the end of this section. (See Remark 1.9.)

For quite a number of previously studied particular Banach couples which are couples of lattices, there always exist optimal decompositions which have a certain monotonicity property, which is in some sense a "refinement" of (3). This property, which will be our main object of study here, is described precisely in the following definition.

Definition 1.1. Let $\overline{X} = (X_0, X_1)$ be a couple of Banach lattices of measurable functions on the measure space (Ω, Σ, μ). A decomposition $\{f = g_t + h_t\}_{t>0}$ of an element $f \in X_0 + X_1$ is said to be *monotone* if, for a.e. $\omega \in \Omega$, it satisfies

$$|g_s(\omega)| \le |g_t(\omega)|, \quad \text{whenever } 0 < s < t.$$

The couple $\overline{X}$ is said to be *exactly monotone* if every $f \in X_0 + X_1$ has a monotone optimal decomposition.

In this paper we shall identify a number of exactly monotone couples. These include couples of L^p spaces (in Sections 2 and 6), and of certain Lorentz spaces, and also couples of the form (B, L^∞) for "most" Banach lattices B. (Section 2). They also include the couple (L^1, X) for "most" rearrangement invariant spaces X (Section 5). We also show (Section 3) that (X_0, X_1) is exactly monotone whenever the dimension of $X_0 + X_1$ is no greater than 2. On the other hand we give examples (Section 4) of couples (X_0, X_1) which are not exactly monotone. These, too, can be finite dimensional. In fact, in our examples, the dimension of $X_0 + X_1$ is 3.

In some of our examples in Sections 2 and 6 we will also consider *weighted* Banach lattices.

Definition 1.2. Given any measure space (Ω, Σ, μ), we shall use the usual terminology *weight function* for any measurable $u: \Omega \to (0, \infty)$. For each Banach lattice X of measurable functions on (Ω, Σ, μ) and each weight function u, we shall use the usual notation X_u for the weighted Banach lattice consisting of all measurable functions f on Ω such that $fu \in X$. It is normed by $\|f\|_{X_u} = \|fu\|_X$.

Remark 1.3. If $p, q \in [1, \infty]$ with $p \neq q$, then many results about the couple of weighted L^p spaces (L^p_u, L^q_v) on a given measure space (Ω, Σ, μ) can be deduced from corresponding results for the "unweighted" couple $(L^p(\nu), L^q(\nu))$ on the same measurable space (Ω, Σ) equipped with a suitably chosen different measure ν. This can be done using a positive one-to-one linear mapping introduced by Stein and Weiss (see [21], pp. 162–163, Lemma 2.6) which is simultaneously an isometry of $L^p_u(\mu)$ onto $L^p(\nu)$ and of $L^q_v(\mu)$ onto $L^q(\nu)$. (Cf. also [7], Corollary 2, p. 234.)

The exact monotonicity of a couple implies that it has other special properties. We give one explicit illustration of this in Section 7, where we investigate the relationship between exact monotonicity and the size of the *K-divisibility constant.* This is the constant $\gamma = \gamma(\overline{X})$ which is the infimum of all values of the constant appearing in the important "K-divisibility theorem" of Brudnyi and Krugljak (see [5], p. 325, or the beginning of Section 7 below). Moreover, $\gamma(\overline{X})$ is also the infimum of all values of the constant appearing in the strong form of the "fundamental lemma of interpolation theory" (see [9] and also Remarks 1.34 and 1.36 and Proposition 1.40 of [10]). We show that $\gamma(\overline{X}) \leq 4$ whenever $\overline{X}$ is exactly monotone. This is an improvement (for such couples) of the sharpest result obtained thus far for general couples, namely that $\gamma(\overline{X}) \leq 3 + 2\sqrt{2}$ (see [9]). It is relevant to note that, on p. 492 of [5], Brudnyi and Krugljak claim that there are sound reasons to believe that $\gamma(\overline{X}) \leq 4$ for all couples $\overline{X}$.

In some cases, rather than using exact monotonicity to obtain better estimates for the constant $\gamma(\overline{X})$, we can, conversely, use information about $\gamma(\overline{X})$ to deduce that $\overline{X}$ has a property related to exact monotonicity. In particular, if $\gamma(\overline{X})=1$ for some couple $\overline{X}$ of Banach lattices, then $\overline{X}$ is "almost exactly monotone" in a sense which we will define now, via a slight generalization of the notion of an exactly monotone couple.

Definition 1.4. A couple $\overline{X}$ of Banach lattices of measurable functions on a measure space (Ω, Σ, μ) is *λ-monotone* for some number $\lambda \geq 1$ if, for each $f \in X_0+X_1$, there exists a decomposition $\{f=g_t+h_t\}_{t>0}$ such that, at almost every $\omega \in \Omega$, the function $t \mapsto |g_t(\omega)|$ is non-decreasing and

$$(5) \qquad \|g_t\|_{X_0}+t\|h_t\|_{X_1} \leq \lambda K(t, f; \overline{X})$$

for all $t \in (0, \infty)$.

The couple $\overline{X}$ is *almost exactly monotone* if it is λ-monotone for every $\lambda>1$.

Remark 1.5. It is very easy to see that a couple (X_0, X_1) is λ-monotone if and only if the corresponding weighted couple $(X_{0,u}, X_{1,u})$ is λ-monotone for any, or every, weight function u.

The property of λ-monotonicity is also related (see Proposition 7.5 below) to another property of the K-functional for arbitrary couples of Banach lattices.

$(*)$ *For some constant $C=C(\overline{X})$ and each $f \in X_0+X_1$, there exists an increasing family $\{E_t\}_{t>0}$ of measurable subsets of Ω (depending on f) such that*

$$(6) \qquad K(t, f; \overline{X}) \leq \|f\chi_{E_t}\|_{X_0}+t\|f(1-\chi_{E_t})\|_{X_1} \leq CK(t, f; \overline{X}) \quad \textit{for each } t>0.$$

This property is established in Theorem 4.1 of [10] and plays an important rôle in the general results of [10]. It has also been obtained independently by Brudnyi and Krugljak ([5], Lemma 4.4.30, pp. 599, 603–605).

Remark 1.6. If f is a non-negative function in X_0+X_1 and it has a decomposition $\{f=g_t+h_t\}_{t>0}$, then the new decomposition $\{f=G_t+H_t\}_{t>0}$ obtained by setting $G_t=\min\{f, |g_t|\}$ and $H_t=f-G_t$ satisfies $\|G_t\|_{X_0} \leq \|g_t\|_{X_0}$ and $\|H_t\|_{X_1} \leq \|h_t\|_{X_1}$ and also $0 \leq G_t \leq f$. Using this and other obvious facts, it is easy to see that a couple $\overline{X}$ of Banach lattices is λ-monotone if and only if for each *non-negative* function $f \in X_0+X_1$ and each $t>0$ there exist *non-negative* functions g_t and h_t such that $f=g_t+h_t$ and, at almost every point of the underlying measure space, g_t is non-decreasing with respect to t and (5) holds.

For such a decomposition we also, of course, have that h_t is non-increasing with respect to t at almost every point of the underlying measure space. This observation enables us to immediately see that the couple (X_0, X_1) is λ-monotone if and only if the "reversed" couple (X_1, X_0) is λ-monotone.

It follows almost immediately from the definition that $\gamma(\bar{X}) \geq 1$ for all couples $\bar{X}$. It is also known that $\gamma(\bar{X})=1$ for certain special couples. In particular the couples (L^p_u, L^q_v), where u and v denote arbitrary weight functions and the exponents p and q are each either 1 or ∞, satisfy $\gamma(L^p_u, L^q_v)=1$. (We refer to [5], p. 335, Proposition 3.2.13, for the proof in the cases where $p=q$. The case where $p\neq q$ and both u and v are identically 1 is proved in [12] or by (an obvious generalization of) the proof of Lemma 5.2 of [10], p. 44. To extend this case to general u and v we use the mapping of Stein–Weiss mentioned in Remark 1.3.)

It is also easy to show (see Section 2 for details) that these same couples (L^p_u, L^q_v), for p and q as above, are all exactly monotone. We shall extend this latter result (in Section 6) by showing that (L^p_u, L^q_v) is exactly monotone for *all* values of p and q in $[1, \infty]$.

Remark 1.7. It is known that $\gamma(\bar{A})>1$ for certain couples $\bar{A}=(A_0, A_1)$ of Banach spaces (which apparently cannot be represented as couples of Banach lattices on a measure space). This was first shown in [14] for the couple $\bar{A}=(C, C^1)$ and it was subsequently shown in [17] that this same couple satisfies $\gamma(\bar{A}) \geq (3+2\sqrt{2})/(1+2\sqrt{2})$. A different approach in [20] produced a couple $\bar{A}=(A_0, A_1)$ for which $\gamma(\bar{A})=(3+2\sqrt{2})/(1+2\sqrt{2})$. Here A_0 is $\mathbf{R}^2$ equipped with the l^∞ norm and A_1 is a one-dimensional subspace of $\mathbf{R}^2$ whose unit ball is a line segment which makes an angle of $\frac{1}{8}\pi$ with one of the coordinate axes. Furthermore, it was shown in [20] that $\gamma(\bar{A}) \leq (3+2\sqrt{2})/(1+2\sqrt{2})$ for all couples $\bar{A}$ such that $A_0 \subset \mathbf{R}^2$ and $A_1 \subset \mathbf{R}^2$. Our results here enable us to produce the apparently first known examples of couples of *lattices* $\bar{X}$ which satisfy $\gamma(\bar{X})>1$. (See Corollary 7.3.)

Let us recall one more notion which will be needed later.

Definition 1.8. Let $\bar{A}=(A_0, A_1)$ be a Banach couple. For $j=0,1$ the *Gagliardo completion* of A_j, which we denote by $A_j^\sim$ is the set of elements $a \in A_0+A_1$ which are limits in the A_0+A_1 norm of bounded sequences in A_j or, equivalently, for which $\|a\|_{A_j^\sim} = \sup_{t>0} K(t, a; \bar{A})/t^j$ is finite.

We refer, e.g., to [11] and also [10] for examples and more details concerning Gagliardo completions.

Remark 1.9. As promised above, we close this section with a proof of (4) and (3).

For each $t>0$ let P_t be the point $(x(t), y(t)) \in \mathbf{R}^2$ and let L_t be the line $\{(x,y) \in \mathbf{R}^2 : x+ty=K(t, f; \bar{A})\}$ which passes through P_t. Now let us make an arbitrary choice of s and t such that $0<s<t$ and show that

$$x(t) \geq x(s) \quad \text{and} \quad y(t) \leq y(s). \tag{7}$$

We first claim that

$$P_t \text{ lies on or above } L_s \quad \text{and} \quad P_s \text{ lies on or above } L_t. \tag{8}$$

If, on the contrary, P_t lies strictly below L_s then

$$K(s,f) \leq \lim_{n\to\infty} (\|g_{n,t}\|_{A_0} + s\|h_{n,t}\|_{A_1}) = x(t) + sy(t) < x(s) + sy(s) = K(s,f)$$

which is, of course, impossible. Similarly, if P_s lies strictly below L_t, then

$$K(t,f) \leq \lim_{n\to\infty} (\|g_{n,s}\|_{A_0} + t\|h_{n,s}\|_{A_1}) = x(s) + ty(s) < x(t) + ty(t) = K(t,f)$$

which is again impossible, and we have established (8).

Since L_t passes through the points $(K(t,f),0)$ and $(0,K(t,f)/t)$, since $K(s,f) \leq K(t,f)$ and $K(s,f)/s \geq K(t,f)/t$, and since $s \neq t$, we see that $L_s \cap L_t$ is a single point (x,y) with $x \geq 0$ and $y \geq 0$. In view of the slopes of these two lines and (8), we obtain that P_s cannot lie strictly to the right of (x,y) and P_t cannot lie strictly to the left of (x,y). Consequently $x(s) \leq x \leq x(t)$ and (again using the slopes) $y(t) \leq y \leq y(s)$. This establishes (7) and so also (4) and (3).

2. Some previously known examples of exactly monotone couples

In many, but not all, of the couples $\overline{X}$ which we shall show to be exactly monotone, this is a consequence of the fact that each non-negative $f \in X_0 + X_1$ has an optimal decomposition $\{f = g_t + h_t\}_{t>0}$, where for each $t>0$ the function h_t is of the form $h_t = \min\{f, \lambda_t\}$ for some constant $\lambda_t \in [0,\infty]$. The most obvious instance of this phenomenon is the next theorem.

Theorem 2.1. *Let B be any Banach lattice of real valued measurable functions on a measure space (Ω, Σ, μ) and let L^∞ denote the space $L^\infty(\mu)$ of essentially bounded measurable functions on Ω. Then the couple (B, L^∞) is almost exactly monotone. Furthermore, this couple is exactly monotone if*

(i) *B has the Fatou property, or*

(ii) *B coincides isometrically with its Gagliardo completion $B^\sim$ with respect to the couple (B, L^∞).*

Remark 2.2. In fact, (i) implies (ii). (See [10], Corollary 1.17.)

Proof. Let f be a non-negative function in $B+L^\infty$. If we know that every such f has some optimal decomposition $\{f=g_t+h_t\}_{t>0}$ into non-negative functions, then it is simple and immediate to show that (B,L^∞) is exactly monotone: We use the decomposition $\{f=G_t+H_t\}_{t>0}$, where $H_t=\min\{f,\|h_t\|_{L^\infty}\}$. Clearly this decomposition must also be optimal, and $G_t=f-H_t$ must be pointwise non-decreasing as a function of t because, by (3), $\|h_t\|_{L^\infty}$ is a non-increasing function of t. The general proof uses an elaboration of the same simple idea. For each $t>0$ and $n\in\mathbf{N}$, we can (cf. Remark 1.6) express the above function f as the sum of two non-negative functions $f=g_{n,t}+h_{n,t}$ such that $g_{n,t}\in B$ and $h_{n,t}\in L^\infty$ and $K(t,f)\le\|g_{n,t}\|_B+t\|h_{n,t}\|_{L^\infty}\le(1+1/n)K(t,f)$. As in the formulation of (4), we can suppose that $\lim_{n\to\infty}\|g_{n,t}\|_B=x(t)$ and $\lim_{n\to\infty}\|h_{n,t}\|_{L^\infty}=y(t)$, where, by (4), $y(t)$ is a non-increasing function of t. Given any $\lambda>1$, we define $H_t=\min\{f,y(t)+(\lambda-1)K(t,f)/4t\}$ and we choose $n\in\mathbf{N}$ sufficiently large so that $(1+1/n)\le1+\frac{1}{2}(\lambda-1)$ and also

$$y(t)-\frac{\lambda-1}{4t}K(t,f)\le\|h_{n,t}\|_{L^\infty}\le y(t)+\frac{\lambda-1}{4t}K(t,f).$$

Then, since $0\le h_{n,t}\le f$, we have that $h_{n,t}\le H_t$. Consequently, $0\le f-H_t\le f-h_{n,t}=g_{n,t}$ and so

$$\begin{aligned}\|f-H_t\|_B+t\|H_t\|_{L^\infty}&\le\|g_{n,t}\|_B+t\left(y(t)+\frac{\lambda-1}{4t}K(t,f)\right)\\&\le\|g_{n,t}\|_B+t\left(y(t)-\frac{\lambda-1}{4t}K(t,f)\right)+\frac{\lambda-1}{2}K(t,f)\\&\le\|g_{n,t}\|_B+t\|h_{n,t}\|_{L^\infty}+\frac{\lambda-1}{2}K(t,f)\\&\le\left(\left(1+\frac{\lambda-1}{2}\right)+\frac{\lambda-1}{2}\right)K(t,f)=\lambda K(t,f).\end{aligned}\tag{9}$$

Since $y(t)+(\lambda-1)K(t,f)/4t$ is a non-increasing function of t, this shows that (B,L^∞) is λ-monotone. Now suppose that B satisfies condition (i) or (ii). For any fixed $t>0$, consider the sequence of functions $H_{m,t}=\min\{f,y(t)+K(t,f)/mt\}$. If we choose $\lambda=1+4/m$, then $\frac{1}{4}(\lambda-1)=1/m$ and we obtain from (9) that

$$\|f-H_{m,t}\|_B+t\|H_{m,t}\|_{L^\infty}\le\left(1+\frac{4}{m}\right)K(t,f).\tag{10}$$

Obviously, $H_{m,t}$ converges pointwise and in L^∞ norm to $H_{*,t}=\min\{f,y(t)\}$. So the sequence $G_{m,t}=f-H_{m,t}$ is pointwise non-decreasing and converges pointwise and also in $B+L^\infty$ to $G_{*,t}=f-\min\{f,y(t)\}$. Thus, using either the Fatou property, or the condition $B^\sim=B$, we deduce that $G_{*,t}\in B$ and $\|G_{*,t}\|_B\le\lim_{m\to\infty}\|G_{m,t}\|_B$. (Obviously the reverse inequality is also true.) These remarks, together with (10), show that $\{f=G_{*,t}+H_{*,t}\}_{t>0}$ is an optimal decomposition. So, since $y(t)$ is non-increasing, we have shown that (B,L^∞) is exactly monotone. $\square$

In the rest of this section we list some other couples which can readily be seen to be exactly monotone.

Example 2.3. The result of the previous theorem can be immediately generalized to show that the couple (B,L^∞_u) is exactly monotone for all choices of weight functions u, since this is equivalent to the exact monotonicity of $(B_{1/u},L^\infty)$. (Cf. Remark 1.5.)

Example 2.4. The couple (L^1_u,L^1_v) of weighted L^1 spaces on some arbitrary measure space is exactly monotone. This follows since for each element f we can choose $g_t=f\chi_{\{u\le tv\}}$.

Example 2.5. The couples of Lorentz spaces $(\Lambda(\phi_0),\Lambda(\phi_1))$ studied by Sharpley [19] are also all exactly monotone in view of the exact formula obtained in [19] for the K-functional.

It is interesting to note that the optimal decompositions of a function f for Sharpley's couples, obtained by dividing the graph of $|f|$ into two separate sequences of horizontal "slices" are of a radically different nature to the optimal decompositions obtained in the other examples mentioned here.

Example 2.6. The couple (L^1,L^p) for any $p\in(1,\infty]$ is exactly monotone in view of the exact formula for its K-functional which is given in [16]. (In fact some further small steps are needed to extend the formula given in [16] to the cases of more general functions f and more general measure spaces.) The papers [1] and [2] give more details and various generalizations of the results of [16]. With the help of the mapping of Stein–Weiss (see Remark 1.3), this result also extends to all weighted couples (L^1_u,L^p_v).

In Section 5 we shall prove a theorem which includes the exact monotonicity of (L^1,L^p) as a special case. In fact the couple (L^p_u,L^q_v) is also exactly monotone for all choices of $p,q\in[1,\infty]$ and all choices of weight functions u and v. For the proof of this in the remaining cases which are not covered by the preceding material of this section, we refer to Section 6.

3. Exactly monotone couples of finite dimensional lattices

Let (Y_0, Y_1) be a couple of Banach lattices of measurable functions on the measure space (Ω, Σ, μ) and suppose that $\dim Y_0$ and $\dim Y_1$ are both finite. Then of course $Y=Y_0+Y_1$ also satisfies $n=\dim Y<\infty$. Let $\{f_k\}_{k=1}^n$ be a basis of Y and let $\Omega^*=\{\omega\in\Omega:\sum_{k=1}^n |f_k(\omega)|>0\}$. Then, of course, each $f\in Y$ must vanish a.e. on $\Omega\backslash\Omega^*$. Furthermore, Ω^* must be the union of n atoms, $\Omega^*=\bigcup_{k=1}^n E_k$. Similar reasoning shows that there are also two subsets Ω_0^* and Ω_1^* of Ω^*, either or both of which may coincide with Ω^* or be empty, such that for every measurable function f on Ω, we have that $f\in Y_j$ if and only if $f=0$ a.e. on $\Omega\backslash\Omega_j^*$.

The map $\sum_{k=1}^n \alpha_k\chi_{E_k}\mapsto(\alpha_1, \alpha_2, \dots, \alpha_n)$ enables us to naturally identify the couple (Y_0, Y_1) with the couple of lattices (X_0, X_1) where

$$X_j=\{(\alpha_1, \alpha_2, \dots, \alpha_n)\in\mathbf{R}^n : \alpha_k=0 \text{ for all } k\notin I_j\} \tag{11}$$

and $I_j=\{k\in\{1,2,\dots,n\}:E_k\subset\Omega_j^*\}$. The lattice norm on X_j is naturally induced by $\|\cdot\|_{Y_j}$. That is, here we are considering $\mathbf{R}^n$ as the space of all real valued functions on a set of n points, and so the notation $x\le y$ means that $x=(x_1, x_2, \dots, x_n)$ and $y=(y_1, y_2, \dots, y_n)$ satisfy $x_k\le y_k$ for $k=1,2,\dots,n$.

Theorem 3.1. *Let $\overline{X}=(X_0, X_1)$ be a couple of Banach lattices on some measure space, such that* $\dim(X_0+X_1)\le 2$ *for* $j=0,1$. *Then $\overline{X}$ is exactly monotone.*

Remark 3.2. As we shall see in the next section, this result is false if we weaken the hypotheses to $\dim(X_0+X_1)\le 3$.

Proof. By the remarks preceding the statement of the theorem, we may suppose without loss of generality that the spaces X_j are each of the form (11) for $n=2$ and for index subsets I_j each containing at most two elements. We fix some element $f=(\alpha,\beta)\in X_0+X_1=\mathbf{R}^2$ and will show that it has a monotone optimal decomposition. It suffices to do this for the case when $\alpha\ge 0$ and $\beta\ge 0$ (cf. Remark 1.6). An obvious compactness argument guarantees the existence of an optimal decomposition $\{f=g_t+h_t\}_{t>0}$. We can assume (cf. again Remark 1.6) that

$$0\le g_t\le f \text{ and } 0\le h_t\le f \quad \text{for all } t>0. \tag{12}$$

If $\dim X_j=0$ for either $j=0$ or $j=1$ then the result is trivial and obvious. If $\dim X_0=1$ then I_0 is either $\{1\}$ or $\{2\}$ and g_t is of the form $g_t=\phi(t)e$ where $\phi\colon(0,\infty)\to[0,\infty)$ and the fixed element $e\in\mathbf{R}^2$ is either $(1,0)$ or $(0,1)$. Now $\phi(t)=\|g_t\|_{X_0}/\|e\|_{X_0}$ and, in view of (3), this must be a non-decreasing function of t and so the proof is complete. A slight variation of this argument takes care of the case $\dim X_1=1$. Thus we can suppose from here on that $\dim X_0=\dim X_1=2$, i.e. $X_0=X_1=\mathbf{R}^2$. Let us use the simpler notation $\|\cdot\|_0$ or $\|\cdot\|_1$ for $\|\cdot\|_{X_0}$ or $\|\cdot\|_{X_1}$, respectively. For $j=0,1$ and each $u\in\mathbf{R}^2$ and each $r>0$, let $B_j(u,r)$ denote the closed ball $\{v\in\mathbf{R}^2:\|u-v\|_j\leq r\}$. We denote its interior by $B_j^\circ(u,r)$ and its boundary by $\partial B_j(u,r)$. That is,

$$B_j^\circ(u,r)=\{v\in\mathbf{R}^2:\|u-v\|_j<r\} \quad \text{and} \quad \partial B_j(u,r)=\{v\in\mathbf{R}^2:\|u-v\|_j=r\}.$$

We shall make a temporary auxiliary assumption: (A) *For $j=0,1$, the boundary $\partial B_j(0,1)$ of the unit ball of X_j has a unique tangent at each point (x,y) and this tangent is not parallel to any other such tangent, except of course at the point $(-x,-y)$.* Since X_j is a lattice, $\partial B_j(0,1)$ is invariant under the maps $(x,y)\mapsto(-x,y)$ and $(x,y)\mapsto(x,-y)$, and the assumption (A) implies that the tangent is horizontal at the points of intersection with the y axis and vertical at the points of intersection with the x axis. For each fixed $t>0$ the balls $B_0^\circ(0,\|g_t\|_0)$ and $B_1(f,\|h_t\|_1)$ satisfy

$$B_0^\circ(0,\|g_t\|_0)\cap B_1(f,\|h_t\|_1)=\emptyset,$$

since any $g\in B_0^\circ(0,\|g_t\|_0)\cap B_1(f,\|h_t\|_1)$ would satisfy the impossible estimates

$$K(t,f;\overline{X})\leq\|g\|_0+t\|f-g\|_1<\|g_t\|_0+t\|h_t\|_1=K(t,f;\overline{X}).$$

The same argument shows that

$$B_0(0,\|g_t\|_0)\cap B_1^\circ(f,\|h_t\|_1)=\emptyset.$$

We deduce that the intersection of the corresponding closed balls, namely $J_t=B_0(0,\|g_t\|_0)\cap B_1(f,\|h_t\|_1)$, must be disjoint from each of the open balls $B_0^\circ(0,\|g_t\|_0)$ and $B_1^\circ(f,\|h_t\|_1)$, and therefore $J_t=\partial B_0(0,\|g_t\|_0)\cap\partial B_1(f,\|h_t\|_1)$. This set is non-empty since it contains the point g_t. It must also be convex. This means it cannot contain any point other than g_t, since our temporary assumption (A) precludes the possibility of either $\partial B_0(0,\|g_t\|_0)$ or $\partial B_1(f,\|h_t\|_1)$ containing any line segments. If g_t and h_t are both non-zero, then, since g_t lies on the boundaries of both of the non-empty disjoint open balls $B_0^\circ(0,\|g_t\|_0)$ and $B_1^\circ(f,\|h_t\|_1)$, it follows that the two uniquely determined tangents at g_t, to $\partial B_0(0,\|g_t\|_0)$ and to $\partial B_1(f,\|h_t\|_1)$, respectively, must both be the same line which we shall denote by L_t. We shall denote the slope of L_t by m_t. For $j=0$ and $j=1$ we can write the set $\partial B_j(0,1)\cap\{(x,y):x\geq 0,\ y\geq 0\}$ in the form $\{(x,\phi_j(x)):0\leq x\leq\delta_j\}$ where $\phi_j:[0,\delta_j]\to[0,\infty)$ is a strictly decreasing concave function with $\phi_j(\delta_j)=0$ and ϕ_j' exists and is strictly decreasing on $[0,\delta_j)$ with $\phi_j'(0)=0$ and $\lim_{x\to\delta_j}\phi_j'(x)=-\infty$. Thus, for our purposes here we can and shall unambiguously introduce the notation $\phi_j'(\delta_j)=-\infty$ so that now ϕ_j' is strictly decreasing on all of $[0,\delta_j]$. This representation of $\partial B_j(0,1)\cap\{(x,y):x\geq 0,\ y\geq 0\}$ immediately implies that, for each $r>0$,

$$\partial B_j(0,r)\cap\{(x,y):x\geq 0,\ y\geq 0\}=\left\{\left(x,r\phi_j\left(\frac{x}{r}\right)\right):0\leq x\leq r\delta_j\right\},$$

and also that the slope of the tangent line to $\partial B_j(0,r)$ at the point $(x,r\phi_j(x/r))$, equals $\phi_j'(x/r)$ for all $x\in[0,r\delta_j)$ and also for $x=r\delta_j$, in accordance with the convention adopted above. Let us write g_t in terms of its coordinates, i.e. $g_t=(x(t),y(t))$. By (12) we have $0\leq x(t)\leq\alpha$ and $0\leq y(t)\leq\beta$ for all $t>0$ and so both g_t and $h_t=(\alpha-x(t),\beta-y(t))$ are in the first quadrant $\{(x,y):x\geq 0,\ y\geq 0\}$. As a special case of the above formula for slopes of tangents, we obtain that

$$m_t=\phi_0'\left(\frac{x(t)}{\|g_t\|_0}\right),\quad \text{whenever } g_t\neq 0.$$

We will need a second formula in terms of ϕ_1' for m_t. This is easily obtained, e.g., with the help of the affine involution map J defined by

$$J(x,y)=(\alpha-x,\beta-y)=f-(x,y).$$

The map J maps each straight line in $\mathbf{R}^2$ onto another straight line with the same slope. Since L_t is also the tangent to $\partial B_1(f,\|h_t\|_1)$ at g_t, its slope m_t must equal the slope of the tangent $J(L_t)$ to $J(\partial B_1(f,\|h_t\|_1))=\partial B_1(0,\|h_t\|_1)$ at $J(g_t)=h_t$. This gives that

$$m_t=\phi_1'\left(\frac{\alpha-x(t)}{\|h_t\|_1}\right),\quad \text{whenever } h_t\neq 0.$$

We have to show that both $x(t)$ and $y(t)$ are non-decreasing functions of t. We shall now do this for $x(t)$. Thus we fix arbitrary numbers s and t with $0<s<t$ and have to show that $x(s)\leq x(t)$. This is obviously true if $g_s=0$ or $h_t=0$ since in these cases $x(s)=0$ or $x(t)=\alpha$, respectively. So from here on we can assume that both g_s and h_t are non-zero. We shall show that supposing $x(t)<x(s)$ leads to a contradiction. On the one hand it implies, since $0<\|g_s\|_0\leq\|g_t\|_0$ (by (3)), that $0\leq x(t)/\|g_t\|_0<x(s)/\|g_s\|_0\leq\delta_0$ and so we have

$$-\infty\leq m_s=\phi_0'\left(\frac{x(s)}{\|g_s\|_0}\right)<\phi_0'\left(\frac{x(t)}{\|g_t\|_0}\right)=m_t\leq 0. \tag{13}$$

On the other hand, $x(t)<x(s)$ also implies that $\alpha-x(s)<\alpha-x(t)$ and, since, again by (3), $0<\|h_t\|_1\leq\|h_s\|_1$, it then follows that $(\alpha-x(s))/\|h_s\|_1<(\alpha-x(t))/\|h_t\|_1$ and so

$$-\infty\leq m_t=\phi_1'\left(\frac{\alpha-x(t)}{\|h_t\|_1}\right)<\phi_1'\left(\frac{\alpha-x(s)}{\|h_s\|_1}\right)=m_s\leq 0.$$

This contradicts (13) and so proves that $x(t)$ must be non-decreasing. The proof that $y(t)$ is non-decreasing is exactly analogous and we leave it to the reader. The last step will be to extend our proof to the general case, i.e., where the unit balls of X_0 and X_1 do not necessarily satisfy the above mentioned temporary assumption (A). It is not difficult to show that, for $j=0,1$ and for each positive integer n, there exists a two-dimensional lattice $X_j(n)$ whose unit ball $B_{j,n}(0,1)$ satisfies assumption (A) and furthermore

$$B_j(0,1)\subset B_{j,n}(0,1)\subset B_j\left(0,1+\frac{1}{n}\right).$$

Then, by the preceding part of the argument, for each n there exists an optimal decomposition $\{f=g(t,n)+h(t,n)\}_{t>0}$ of f with respect to the couple $\overline{X}(n)=(X_0(n),X_1(n))$ such that, if $g(t,n)=(x(t,n),y(t,n))$, both $x(t,n)$ and $y(t,n)$ are non-decreasing functions of t. Furthermore, by (12), $0\le x(t,n)\le\alpha$ and $0\le y(t,n)\le\beta$ for all $t>0$ and $n\in\mathbf{N}$. By Helly's selection theorem (see e.g. [18], Exercise 13, p. 167) there exists a strictly increasing sequence of integers $\{n_k\}_{k\in\mathbf{N}}$ such that $x(t,n_k)$ and $y(t,n_k)$ converge for each t to non-decreasing functions $x(t)$ and $y(t)$. Let $g(t)=(x(t),y(t))$ and $h(t)=f-g(t)$. It is easy to check that $\{f=g(t)+h(t)\}_{t>0}$ is a monotone optimal decomposition of f for t with respect to the original couple $\overline{X}$. This completes the proof. $\square$

4. A counterexample in $\mathbf{R}^3$

Theorem 4.1. *Let X_0 be $\mathbf{R}^3$ equipped with the lattice norm*

$$\|(x,y,z)\|_0=\max\{|x|,|y|,\tfrac{1}{4}|y|+|z|,\tfrac{3}{7}|x|+\tfrac{5}{7}|y|+\tfrac{4}{7}|z|\}$$

and let X_1 be the subspace of $\mathbf{R}^3$ consisting of elements of the form $(x,y,0)$ equipped with the lattice norm

$$\|(x,y,0)\|_1=10^{-3}|x|+|y|.$$

Then the couple $\overline{X}=(X_0,X_1)$ is not exactly monotone.

Proof. We shall establish the result by determining the optimal decomposition $f=g_t+h_t$ for the element $f=(1,1,1)$ exactly when $t=10^{-6}$ and approximately when $t=10$. For the case $t=10^{-6}$ it is convenient to use the function

$$\phi(x,y)=\|(1-x,y,1)\|_0+10^{-6}\|(x,1-y,0)\|_1.$$

Obviously $K(10^{-6},f;\overline{X})=\inf\{\phi(x,y):(x,y)\in\mathbf{R}^2\}$. Note that $\phi(0,0)=1+10^{-6}$. We shall see that this is the infimum, and that it is not attained at any other point $(x,y)\neq(0,0)$. Now

$$\phi(x,y)\geq\tfrac{1}{4}|y|+1+10^{-6}|1-y|\geq\tfrac{1}{4}|y|+1+10^{-6}(1-|y|)=1+10^{-6}+\left(\tfrac{1}{4}-10^{-6}\right)|y|.$$

So if the infimum is attained at (x,y) we must have $y=0$. But then

$$\phi(x,0)=\|(1-x,0,1)\|_0+10^{-6}\|(x,1,0)\|_1\geq 1+10^{-9}|x|+10^{-6},$$

and so necessarily $x=0$. Consequently $f=g_t+h_t$, where $g_t=(1,0,1)$ and $h_t=(0,1,0)$ is the unique optimal decomposition of f for $t=10^{-6}$. Now to treat the case $t=10$ we shall use the function

$$\psi(x,y)=\|(x,1-y,1)\|_0+10\|(1-x,y,0)\|_1.$$

First observe that $\psi(0,0)=\frac{9}{7}+10\cdot10^{-3}=\frac{9}{7}+10^{-2}$. We shall not explicitly show that this is the infimum, but we shall see that the infimum can only be attained in a very small neighbourhood of $(0,0)$. Indeed, suppose that

$$\psi(x,y)\leq\psi(0,0). \tag{14}$$

Then it follows from the estimate

$$\psi(x,y)\geq\tfrac{5}{7}|1-y|+\tfrac{4}{7}+10|y|\geq\tfrac{5}{7}(1-|y|)+\tfrac{4}{7}+10|y|=\tfrac{9}{7}+\tfrac{65}{7}|y|$$

that

$$|y|\leq\tfrac{7}{6500}. \tag{15}$$

We then also have the estimate $\psi(x,y)\geq\frac{3}{7}|x|+\frac{5}{7}(1-|y|)+\frac{4}{7}$, which, combined with (14) and (15), yields that $\frac{3}{7}|x|\leq10^{-2}+\frac{5}{7}\frac{7}{6500}$ and so $|x|$ is considerably smaller than $\frac{1}{20}$. This shows that any optimal decomposition $f=g_t+h_t$ for $t=10$ must have g_t very close to $(0,1,1)$ and h_t very close to $(1,0,0)$. Thus the first coordinate of $|g_t|$ cannot be an increasing function of t which proves that $\overline{X}$ is not exactly monotone. □

Remark 4.2. There is nothing special about the fact that X_1 in the previous theorem has dimension 2. This choice was made only to simplify the calculations. To obtain an example of a couple (X_0,X_1) which is not exactly monotone and where both spaces have "full" dimension 3, we can simply use a small "perturbation" of the example of Theorem 4.1. For example, we can define X_0 as above and modify X_1 to now be $\mathbf{R}^3$ equipped with the lattice norm

$$\|(x,y,z)\|_1 = 10^{-3}|x| + |y| + 10^7|z|.$$

Then a straightforward variant of the above proof shows that here again the optimal decomposition of $f=(1,1,1)$ for $t=10^{-6}$ is exactly $(1,0,1)+(0,1,0)$ and for $t=10$ it is again very close to $(0,1,1)+(1,0,0)$. Thus we have the required counterexample.

Remark 4.3. It is easy to see that neither of the couples introduced in Theorem 4.1 and Remark 4.2 can be almost exactly monotone. Otherwise, for $f=(1,1,1)$ and each $n\in\mathbf{N}$ there would exist a decomposition into non-negative monotonic functions $\{f=g_{n,t}+h_{n,t}\}_{t>0}$ such that $\|g_{n,t}\|_{X_0}+t\|h_{n,t}\|_{X_1}\le(1+1/n)K(t,f;\overline{X})$. Then, as in the final step of the proof of Theorem 3.1, we could use Helly's selection theorem to pass to subsequences of $\{g_{n,t}\}$ and $\{h_{n,t}\}$ which, for each t, converge in $\mathbf{R}^3$, and therefore also in X_0 and X_1, to give a monotone optimal decomposition of f, contradicting what we have shown above.

5. The couple (L^1, X) for a large class of rearrangement invariant spaces X

The "large class" referred to in the title of this section consists of those spaces X which are exact interpolation spaces with respect to the couple $(L^1(\mu), L^\infty(\mu))$ on the same underlying measure space (Ω,Σ,μ). Characterizations of these spaces have been obtained by Calderón [6], Theorem 3, p. 280, and also by Mityagin [15]. Such spaces X are necessarily *rearrangement invariant.* That is, if $f\in X$ and g is a measurable function on Ω such that the non-increasing rearrangements of f and g satisfy $g^*(t)\le f^*(t)$ for all $t>0$, then $g\in X$ and $\|g\|_X\le\|f\|_X$. However, rearrangement invariance alone is not sufficient to imply exact interpolation with respect to (L^1, L^∞). Under appropriate conditions on (Ω,Σ,μ) it is sufficient to have any one of the additional conditions that X has the Fatou property, or it is separable, or it contains $L^1\cap L^\infty$ densely. We refer to [6], Theorem 4, p. 281, and Sections 4 and 5 of Chapter II of [13] for details of these matters.

Theorem 5.1. *Let (Ω,Σ,μ) be an arbitrary measure space. Let X be a Banach lattice of measurable functions on Ω which is an exact interpolation space with respect to $(L^1,L^\infty)=(L^1(\mu),L^\infty(\mu))$. Suppose also that X has the Fatou property. Let $f:\Omega\to[0,\infty)$ be an element of L^1+X and, for each $\lambda\in[0,\infty]$, define $f^\lambda=\min\{f,\lambda\}$ and $f_\lambda=f-f^\lambda$. Then, for each $t>0$, there exists $\lambda=\lambda(t)\in[0,\infty]$ such that*

$$K(t,f;L^1,X)=\|f_\lambda\|_{L^1}+t\|f^\lambda\|_X. \tag{16}$$

Furthermore, the couple (L^1, X) *is exactly monotone.*

Remark 5.2. Theorem 5.1 cannot be generalized to the case of all Banach lattices X on (Ω, Σ, μ). We can see this with the help of the couple (X_0, X_1) of Remark 4.2. Here $X_1 = L^1(\mu)$ for a suitable measure μ on $\Omega = \{1, 2, 3\}$, but neither this couple, nor (X_1, X_0) (cf. Remark 1.6) is exactly monotone.

Proof. Since $f \in L^1 + X$ and $X \subset L^1 + L^\infty$, we have $f = u + v + w$ where $u, v \in L^1$ and $w \in L^\infty$ and of course these three functions can all be taken to be non-negative. It then follows that $f_\lambda \in L^1$ for $\lambda = \|w\|_{L^\infty}$. Consequently $\lambda_* := \inf\{\lambda \in [0, \infty] : f_\lambda \in L^1\}$ satisfies $0 \le \lambda_* < \infty$. Let g be a measurable function which satisfies $0 \le g \le f$, $g \in L^1$ and $f - g \in X$. The main step of our proof will be to show that for a suitable choice of $\lambda \in [\lambda_*, \infty]$ the function $G = f - \min\{f, \lambda\}$ satisfies

$$G \in L^1 \quad \text{with } \|G\|_{L^1} \le \|g\|_{L^1} \tag{17}$$

and

$$f - G \in X \quad \text{with } \|f - G\|_X \le \|f - g\|_X. \tag{18}$$

Clearly, the function $\lambda \mapsto \|f_\lambda\|_{L^1}$ is non-increasing on $[0, \infty]$. By dominated convergence it is also continuous on (λ_*, ∞). By monotone convergence, we have

$$\lim_{\lambda \searrow \lambda_*} \|f_\lambda\|_{L^1} = \|f_{\lambda_*}\|_{L^1}$$

whether or not $\|f_{\lambda_*}\|_{L^1}$ is finite. Furthermore, by dominated convergence, we also have $\lim_{\lambda \nearrow \infty} \|f_\lambda\|_{L^1} = \|f_\infty\|_{L^1} = 0$. Using these properties we see that, if $\|g\|_{L^1} < \|f_{\lambda_*}\|_{L^1}$, then there exists some $\lambda \in (\lambda_*, \infty]$ such that $\|f_\lambda\|_{L^1} = \|g\|_{L^1}$. In the remaining case, when $\|g\|_{L^1} \ge \|f_{\lambda_*}\|_{L^1}$, which of course can only arise if $\|f_{\lambda_*}\|_{L^1} < \infty$, we set $\lambda = \lambda_*$. Obviously the function $G = f - \min\{\lambda, f\} = f_\lambda$, obtained by choosing λ as above, satisfies (17). To show that it also satisfies (18) it will suffice, in view of the interpolation properties of X, to show that for each $n \in \mathbf{N}$ there exists a linear operator S (depending on n) such that

$$S : L^p \longrightarrow L^p \quad \text{with norm not exceeding 1 for } p = 1, \infty, \tag{19}$$

and

$$\left(1 - \frac{1}{n}\right)(f - G) \le S(f - g). \tag{20}$$

Our construction of S will use a number of arguments similar to ones which appear in various papers, such as [6] and [7]. However it seems simpler to give a fairly self contained explanation rather than patching together miscellaneous components from those papers. Let us first construct S in the case where $\|g\|_{L^1} \geq \|f_{\lambda_*}\|_{L^1}$. We have that

$$\|f - G\|_{L^\infty} \leq \lambda = \lambda_*. \tag{21}$$

We can suppose that $\lambda_* > 0$ since if $\lambda_* = 0$ we can of course simply take $S = 0$. Thus it follows from the definition of λ_* that $F_m = \{\omega \in \Omega : (1 - 1/m)\lambda_* \leq f(\omega) \leq (1 + 1/m)\lambda_*\}$ satisfies $\mu(F_m) = \infty$ for each $m \in \mathbf{N}$. We now construct a bounded linear functional ϕ_m on $L^1 + L^\infty$ for each m. We do this in one of two different ways, corresponding to two separate subcases.

Subcase 1. This occurs if F_m has a measurable subset F_m^* with the property that $\mu(F_m^*) = \infty$ and every measurable subset of F_m^* has measure which is either 0 or ∞. Then we have $h\chi_{F_m^*} = 0$ a.e. for each $h \in L^1$. In this case we define ϕ_m by setting $\phi_m(h) = \psi_m(h\chi_{F_m^*})$, where ψ_m is a norm one linear functional on L^∞ such that $\psi_m(f\chi_{F_m^*}) = \|f\chi_{F_m^*}\|_{L^\infty}$.

Subcase 2. If Subcase 1 is not applicable then F_m must contain a measurable subset of finite positive measure. We claim that this implies that the quantity

$$M := \sup\{\mu(F) : F \in \Sigma,\ F \subset F_m,\ \mu(F) < \infty\}$$

must be infinite, since if not there exists a sequence $\{E_k\}_{k \in \mathbf{N}}$ of measurable subsets of F_m with $M - 1/k \leq \mu(E_k) < \infty$, and also, necessarily $\mu(E_1 \cup E_2 \cup \ldots \cup E_k) \leq M$ for all $k \in \mathbf{N}$. Then $\mu(\bigcup_{k \in \mathbf{N}} E_k) = M$ and it is easy to check that the set $F_m^* := F_m \setminus \bigcup_{k \in \mathbf{N}} E_k$ has the property dealt with in Subcase 1. Since $M = \infty$, there exists a sequence of measurable sets $\{E_k\}_{k \in \mathbf{N}}$ such that $E_k \subset F_m$ and $k < \mu(E_k) < \infty$. By passing if necessary to a subsequence, we can suppose that $\{E_k\}_{k \in \mathbf{N}}$ has the further property that the bounded sequence $\{(1/\mu(E_k)) \int_{E_k} f\, d\mu\}_{k \in \mathbf{N}}$ converges as k tends to ∞. We can now define ϕ_m by setting $\phi_m(h) = \text{B-}\lim_{k\to\infty}(1/\mu(E_k)) \int_{E_k} h\, d\mu$, where $\text{B-}\lim_{k\to\infty}$ denotes a Banach limit on l^∞, i.e. a norm one linear functional which extends the functional $\psi(\{\alpha_k\}) = \lim_{k\to\infty} \alpha_k$, defined on the subspace of convergent sequences in l^∞.

Note that in both of these subcases we have

$$\phi_m(h) = 0 \qquad \text{for all } h \in L^1, \tag{22}$$

$$|\phi_m(h)| \leq \|h\|_{L^\infty} \qquad \text{for all } h \in L^\infty, \tag{23}$$

and so, since $g \in L^1$, we also have

$$\left(1 - \frac{1}{m}\right)\lambda_* \leq \phi_m(f) = \phi_m(f - g) \leq \left(1 + \frac{1}{m}\right)\lambda_*. \tag{24}$$

It follows from (22) and (23) that the operator S defined by

$$Sh=\phi_m(h)\frac{f-G}{\|f-G\|_{L^\infty}}$$

has the required boundedness property (19) for all choices of $n\in\mathbf{N}$. Furthermore, at all points $\omega\in\Omega$, we have, using (24) and (21), that

$$S(f-g)\geq\left(1-\frac{1}{m}\right)\lambda_*\frac{f-G}{\lambda_*},$$

i.e., we can obtain the second required property (20) for any given $n\in\mathbf{N}$ by choosing $m=n$. As a preliminary to the next step, we consider another similar operator which can be constructed, whenever $\lambda_*>0$, using the same functional ϕ_m, and the set $F:=\{\omega\in\Omega:f(\omega)\leq\lambda_*\}$. This is the operator U_m which is defined by

$$U_m h=\left(1-\frac{1}{m}\right)\phi_m(h\chi_F)\frac{f\chi_F}{\|f\chi_{F_m}\|_{L^\infty}} \tag{25}$$

and which clearly has norm not exceeding 1 on L^∞ and maps L^1 to $\{0\}$. Furthermore

$$U_m(f-g)=U_m(f)\geq\left(1-\frac{1}{m}\right)f\chi_F \quad \text{at all points of } \Omega. \tag{26}$$

We now turn to constructing S in the remaining case where $\|g\|_{L^1}<\|f_{\lambda_*}\|_{L^1}$. Let r be a constant in $(0,1)$ whose precise value will be specified later. Since in this case we have $\lambda>\lambda_*$, the sequence $\{\lambda_k\}_{k=0}^\infty$ defined by

$$\lambda_k=\lambda_*+r^k(\lambda-\lambda_*)$$

is strictly decreasing. Note also that

$$\frac{\lambda_k}{\lambda_{k+1}}\leq\frac{1}{r}. \tag{27}$$

We define a pairwise disjoint sequence of measurable sets $\{\Lambda_k\}_{k=0}^\infty$ by setting

$$\Lambda_0:=\{\omega\in\Omega:\lambda_0\leq f(\omega)\}$$

and

$$\Lambda_k:=\{\omega\in\Omega:\lambda_k\leq f(\omega)<\lambda_{k-1}\}\quad \text{for } k=1,2,\dots\,.$$

For each $\alpha>\lambda_*$ it follows easily (e.g. by applying Chebyshev's inequality to the function $f_\beta\in L^1$, where β is some number in (λ_*,α)) that the set $\{\omega\in\Omega: f(\omega)>\alpha\}$ has finite measure. Thus, for each $k=0,1,\dots$, we have $\mu(\Lambda_k)<\infty$. We define a sequence of disjoint intervals $I_k=[\alpha_k,\alpha_{k+1})$ by setting $\alpha_0=0$ and $\alpha_k=\sum_{j=0}^{k-1}\mu(\Lambda_j)$, i.e. we have $|I_k|=\mu(\Lambda_k)$ for $k=0,1,\dots$. Let $\tilde{u}:[0,\infty)\to[0,\infty)$ be the function

$$\tilde{u}=\sum_{k\in K}\left(\frac{1}{\mu(\Lambda_k)}\int_{\Lambda_k}(f-G)\,d\mu\right)\chi_{I_k},$$

where K is the set of non-negative integers k such that $\mu(\Lambda_k)>0$. It is clear that $\tilde{u}$ is non-increasing. Let $U: L^1([0,\infty),dx)+L^\infty([0,\infty),dx)\to L^1(\mu)+L^\infty(\mu)$ be the operator defined by

$$Uh=\sum_{k\in K}\left(\frac{1}{\mu(\Lambda_k)}\int_{I_k}h\,dx\right)\chi_{\Lambda_k}.$$

Of course $U: L^p([0,\infty),dx)\to L^p(\mu)$ with norm 1 for $p=1,\infty$. Furthermore,

$$U\tilde{u}=\sum_{k\in K}\left(\frac{1}{\mu(\Lambda_k)}\int_{\Lambda_k}(f-G)\,d\mu\right)\chi_{\Lambda_k},$$

and $f-G=\min\{f,\lambda\}=\min\{f,\lambda_0\}$. Consequently, $f-G=\lambda=\lambda_0$ on Λ_0 and for each $k=1,2,\dots$, $f(\omega)-G(\omega)=f(\omega)\in[\lambda_k,\lambda_{k-1})$ for all $\omega\in\Lambda_k$. So, using also (27), we obtain that

$$r(f-G)\le U\tilde{u}\quad\text{at almost every point of }\bigcup_{k=0}^{\infty}\Lambda_k.$$

We observe that, by our definition of G,

$$\int_{\Lambda_0}g\,d\mu\le\int_\Omega g\,d\mu=\int_\Omega G\,d\mu=\int_{\Lambda_0}G\,d\mu \tag{28}$$

and consequently,

$$\int_{\Lambda_0}(f-G)\,d\mu\le\int_{\Lambda_0}(f-g)\,d\mu. \tag{29}$$

We now define the operator $V: L^1(\mu)+L^\infty(\mu)\to L^1([0,\infty),dx)+L^\infty([0,\infty),dx)$ by

$$Vh=\sum_{k\in K}\left(\frac{1}{|I_k|}\int_{\Lambda_k}h\,d\mu\right)\chi_{I_k}.$$

It is clear that $V: L^p(\mu) \to L^p([0,\infty), dx)$ with norm 1 for $p=1,\infty$. We claim that for every $t>0$

$$\int_0^t V(f-g)\,dx \geq \int_0^t \tilde{u}\,dx. \tag{30}$$

By (29), $V(f-g)$ assumes a constant value greater than or equal to λ on I_0. Since $\tilde{u}(x) \leq \lambda$ for all x, we see that (30) holds for all $t \in I_0 = [0, \alpha_1)$. Using once more the fact that $f-G=f$ on each Λ_k for $k \geq 1$ we see that $V(f)$ and $\tilde{u}$ assume the same constant value on I_k for each $k \geq 1$. In other words,

$$Vf(x) = \tilde{u}(x) \quad \text{for all } x \geq \alpha_1.$$

So, for all $t \geq \alpha_1$,

$$\begin{aligned}
\int_0^t V(f-g)\,dx &= \int_0^{\alpha_1} V(f-g)\,dx + \int_{\alpha_1}^t V(f-g)\,dx \\
&= \int_{\Lambda_0} (f-g)\,d\mu + \int_{\alpha_1}^t Vf\,dx - \int_{\alpha_1}^t Vg\,dx \\
&\geq \int_{\Lambda_0} (f-g)\,d\mu + \int_{\alpha_1}^t \tilde{u}\,dx - \int_{\alpha_1}^\infty Vg\,dx \\
&= \int_{\Lambda_0} (f-g)\,d\mu + \int_{\alpha_1}^t \tilde{u}\,dx - \sum_{k=1}^\infty \int_{\Lambda_k} g\,d\mu \\
&\geq \int_{\Lambda_0} f\,d\mu + \int_{\alpha_1}^t \tilde{u}\,dx - \int_\Omega g\,d\mu.
\end{aligned}$$

By (28) this last expression equals

$$\begin{aligned}
\int_{\Lambda_0} f\,d\mu + \int_{\alpha_1}^t \tilde{u}\,dx - \int_{\Lambda_0} G\,d\mu &= \int_{\Lambda_0} (f-G)\,d\mu + \int_{\alpha_1}^t \tilde{u}\,dx \\
&= \int_0^{\alpha_1} \tilde{u}\,dx + \int_{\alpha_1}^t \tilde{u}\,dx = \int_0^t \tilde{u}\,dx,
\end{aligned}$$

and so we have established (30) for all $t>0$. Let h^* denote the non-increasing rearrangement of $h \in L^1([0,\infty), dx) + L^\infty([0,\infty), dx)$. Then, for each $t>0$,

$$\int_0^t h^*\,dx = \sup\left\{ \int_E |h|\,dx : E \subset [0,\infty),\ E \text{ measurable},\ |E| = t \right\} \geq \int_0^t h\,dx. \tag{31}$$

(Cf., e.g., Proposition 3.3 on p. 53 of [3] or Assertion 8 on p. 64 of [13].) Since $\tilde{u}=(\tilde{u})^*$, we obtain from (31) and (30) that

$$\int_0^t (V(f-g))^*\,dx \geq \int_0^t (\tilde{u})^*\,dx \quad \text{for all } t>0.$$

This is a sufficient condition, by Theorem 1 of [6], p. 278, (and also a necessary one) for the existence of an operator $T\colon L^1([0,\infty),dx)+L^\infty([0,\infty),dx)\to L^1([0,\infty),dx)+L^\infty([0,\infty),dx)$ such that $T(V(f-g))=\tilde{u}$ and $T\colon L^p([0,\infty),dx)\to L^p([0,\infty),dx)$ with norm not exceeding 1 for $p=1,\infty$. Combining the previous steps, and writing $\Lambda_*=\bigcup_{k=0}^\infty \Lambda_k$ we see that the operator S_0 defined by

$$S_0h=UTVh$$

satisfies $S_0\colon L^p(\mu)\to L^p(\mu)$ with norm not exceeding 1 for $p=1,\infty$ and also that $S_0((f-g)\chi_{\Lambda_*})\geq r(f-G)\chi_{\Lambda_*}$ at almost every point of Ω. To complete the construction of S for any given $n\in\mathbf{N}$ we need to choose $r=1-1/n$ and to find a second operator $S_1\colon L^\infty(\mu)\to L^\infty(\mu)$ with norm not exceeding 1 such that $S_1(L^1(\mu))=\{0\}$ and

$$S_1(f-g)\geq\left(1-\frac{1}{n}\right)(f-G)\chi_{\Omega\setminus\Lambda_*} \quad \text{at almost every point of } \Omega. \tag{32}$$

Then, of course,

$$Sh:=\chi_{\Lambda_*}S_0(h\chi_{\Lambda_*})+\chi_{\Omega\setminus\Lambda_*}S_1h$$

will have the required properties (19) and (20). Now $\Omega\setminus\Lambda_*=\{\omega\in\Omega : f(\omega)\leq\lambda_*\}$ and so, if $\lambda_*=0$, then both of the functions $(f-g)\chi_{\Omega\setminus\Lambda_*}$ and $(f-G)\chi_{\Omega\setminus\Lambda_*}$ vanish identically, i.e., we can simply take $S_1=0$. If, on the other hand, $\lambda_*>0$, we can use the operator U_m defined above by (25). We have $F=\Omega\setminus\Lambda_*$ in that definition, and furthermore, $f\chi_F=(f-G)\chi_F$ and $U_m(h)=0$ for all $h\in L^1(\mu)$. Thus, if we choose $m=n$ and $S_1=U_n$, then (26) immediately gives us (32). Having constructed the operator S we can now easily finish the proof of the theorem: Given any fixed $t>0$, there exists a sequence of functions $\{g_n\}_{n\in\mathbf{N}}$ such that (i) $0\leq g_n(\omega)\leq f(\omega)$ for a.e. $\omega\in\Omega$, (ii) $g_n\in L^1$, (iii) $f-g_n\in X$ and

$$\|g_n\|_{L^1}+t\|f-g_n\|_X\leq\frac{1}{n}+K(t,f;L^1,X). \tag{33}$$

We shall now define a new sequence $\{G_n\}_{n\in\mathbf{N}}$ by choosing $G_n=f-\min\{\lambda_n,f\}=f_{\lambda_n}$, where $\lambda_n\in[\lambda_*,\infty]$ is chosen to satisfy $\|f_{\lambda_n}\|_{L^1}=\|g_n\|_{L^1}$, if $\|g_n\|_{L^1}<\|f_{\lambda_*}\|_{L^1}$, and otherwise $\lambda_n=\lambda_*$. Applying our main step for each n, we see that conditions (i), (ii), (iii) and (33) all hold when g_n is replaced by G_n. Thus $\{\|G_n\|_{L^1}\}_{n\in\mathbf{N}}$ and $\{\|f-G_n\|_X\}_{n\in\mathbf{N}}$ are both bounded sequences and

$$\limsup_{n\to\infty}\,(\|G_n\|_{L^1}+t\|f-G_n\|_X)\le K(t,f;L^1,X). \tag{34}$$

By passing, if necessary, to a subsequence, we can suppose furthermore that there exists $\lambda_{**}=\lambda_{**}(t)\in[\lambda_*,\infty]$ such that either

$$\lambda_n\nearrow\lambda_{**} \tag{35}$$

or

$$\lambda_n\searrow\lambda_{**}. \tag{36}$$

If (35) holds, then, using the Fatou property of X, we obtain that the pointwise limit $\min\{f,\lambda_{**}\}$ of the norm bounded monotone increasing sequence $f-G_n=\min\{f,\lambda_n\}$ is an element of X with norm

$$\|\min\{f,\lambda_{**}\}\|_X=\lim_{n\to\infty}\|\min\{f,\lambda_n\}\|_X. \tag{37}$$

Since $G_1=f-\min\{f,\lambda_1\}\in L^1$, we can apply dominated convergence to the monotone decreasing sequence $\{G_n\}$ of non-negative functions to show that

$$f-\min\{f,\lambda_{**}\}\in L^1\quad\text{and}\quad\|f-\min\{f,\lambda_{**}\}\|_{L^1}=\lim_{n\to\infty}\|f-\min\{f,\lambda_n\}\|_{L^1}. \tag{38}$$

If, on the other hand, (36) holds, then we still obtain (38) by monotone convergence, and, instead of (37), we have simply that

$$\min\{f,\lambda_{**}\}\in X\quad\text{and}\quad\|\min\{f,\lambda_{**}\}\|_X\le\|\min\{f,\lambda_n\}\|_X\quad\text{for all } n\in\mathbf{N}.$$

Thus in both cases we can substitute in (34) to obtain that (16) holds for $\lambda=\lambda_{**}(t)$. Finally, to show that (L^1,X) is exactly monotone, it suffices to show that (16) also holds for $\lambda=\lambda(t)$, where $\lambda(t)$ is a non-increasing function of t. Let us define the function $\psi\colon[\lambda_*,\infty]\to[\lambda_*,\infty]$ by setting

$$\psi(\lambda)=\inf\{\alpha\in[\lambda_*,\lambda]:\|f-\min\{f,\alpha\}\|_{L^1}=\|f-\min\{f,\lambda\}\|_{L^1}\}.$$

We observe that this infimum is always attained: This is obviously the case when $\lambda=\lambda_*$ and so $\psi(\lambda_*)=\lambda_*$. Furthermore, for each $\lambda>\lambda_*$, we have, by monotone convergence, that

$$\|f-\min\{f,\psi(\lambda)\}\|_{L^1}=\|f-\min\{f,\lambda\}\|_{L^1}<\infty. \tag{39}$$

From (39) we also obtain that $\int_\Omega \min\{f,\lambda\}-\min\{f,\psi(\lambda)\}\,d\mu=0$ for each $\lambda>\lambda_*$ and so the non-negative function integrand satisfies

$$\min\{f,\lambda\}-\min\{f,\psi(\lambda)\}=0 \quad \text{for a.e. } \omega\in\Omega. \tag{40}$$

Obviously (40) also holds when $\lambda=\lambda_*$. We deduce that

$$\|\min\{f,\lambda\}\|_X=\|\min\{f,\psi(\lambda)\}\|_X \quad \text{for all } \lambda\in[\lambda_*,\infty].$$

We can now define the function $\lambda(t)$ by setting $\lambda(t)=\psi(\lambda_{**}(t))$. The preceding remarks show that, for each fixed $t>0$, (16) holds also for $\lambda=\lambda(t)$. Suppose that $0<s<t$. Then, by (3),

$$0\le\|f-\min\{f,\lambda(t)\}\|_{L^1}-\|f-\min\{f,\lambda(s)\}\|_{L^1}=\int_\Omega \min\{f,\lambda(s)\}-\min\{f,\lambda(t)\}\,d\mu.$$

On the one hand, if this integral is strictly positive, then we must have $\lambda(s)>\lambda(t)$. On the other hand, if it is zero, then, necessarily,

$$\psi(\lambda(s))=\psi(\lambda(t)). \tag{41}$$

But, since the infimum in the definition of ψ is attained, we have that $\psi(\psi(\lambda))=\psi(\lambda)$ for each $\lambda\in[\lambda_*,\infty]$. Consequently (41) implies that $\lambda(s)=\lambda(t)$. Thus we see that $\lambda(t)$ is a non-increasing function, which shows that (L^1,X) is exactly monotone and so completes the proof of the theorem. □

6. The couple (L^p_u, L^q_v) for arbitrary p and q in $[1,\infty]$ and arbitrary weight functions u and v

In this section we complement the remarks of Section 2 and show that the couple (L^p_u,L^q_v) on an arbitrary measure space (Ω,Σ,μ) is exactly monotone for all $p,q\in[1,\infty]$ and all weight functions u and v on Ω.

The case $\max\{p,q\}=\infty$ is covered by Theorem 2.1 and Example 2.3. The case $\min\{p,q\}=1$ is covered by Example 2.4 when $p=q=1$ and by Example 2.6 or Theorem 5.1 when $\max\{p,q\}>1$. (As already mentioned, the case of general weight functions here can be deduced from the case where both u and v are identically 1, via the mapping of Stein–Weiss referred to in Remark 1.3.)

Thus the remaining case which we have to treat is when both p and q are in $(1,\infty)$. Although the K-functional for (L^p_u,L^q_v) looks quite different when $p=q$ as compared to when $p\neq q$, and although for $p\neq q$, its formula is very much simpler when u and v are identically 1, it turns out that we can just as easily treat all these cases simultaneously by the same "calculus of variations" approach similar to that used by Bastero, Raynaud and Rezola in [2] to obtain various exact K-functional formulæ. We shall use some rather straightforward modifications or generalizations of some of the proofs and results of [2].

Let f be an arbitrary non-negative function in $L_u^p+L_v^q$. For some fixed $t>0$, let $\{G_k\}_{k\in\mathbf{N}}$ be a sequence of functions in L_u^p such that $f-G_k\in L_v^q$ and

$$\lim_{k\to\infty}\|G_k\|_{L_u^p}+t\|f-G_k\|_{L_v^q}=K(t,f;L_u^p,L_v^q).$$

We can of course (Remark 1.6) choose the functions G_k so that they satisfy $0\le G_k\le f$. Furthermore, since $p,q\in(1,\infty)$, we can suppose, by passing if necessary to a subsequence, that G_k converges weakly in L_u^p to a function $g=g_t\in L_u^p$ and $f-G_k$ converges weakly in L_v^q to a function $h=h_t\in L_v^q$. These functions satisfy $\|g\|_{L_u^p}+t\|h\|_{L_v^q}\le K(t,f;L_u^p,L_v^q)$. Furthermore, since $\int_\Omega(g+h)\phi\,d\mu=\int_\Omega f\phi\,d\mu$ for all $\phi\in L_{1/u}^{p'}\cap L_{1/v}^{q'}$, we have that $f=g+h$ and so

$$\|g\|_{L_u^p}+t\|f-g\|_{L_v^q}=K(t,f;L_u^p,L_v^q) \tag{42}$$

for the particular $t>0$ chosen above. Since $\int_\Omega g\phi\,d\mu$ and $\int_\Omega(f-g)\phi\,d\mu$ are non-negative for every non-negative $\phi\in L_{1/u}^{p'}\cap L_{1/v}^{q'}$ we also have that g and $h=f-g$ are non-negative almost everywhere.

Let $F=\{\omega\in\Omega:f(\omega)>0\}$. Our next step will be to show (cf. [2]) that the function g obtained as above must satisfy

$$\text{(i) } g(\omega)<f(\omega) \text{ for a.e. } \omega\in F \quad\text{or}\quad \text{(ii) } g(\omega)=f(\omega) \text{ for a.e. } \omega\in\Omega. \tag{43}$$

Suppose that (43) is false, i.e. that the sets $B=\{\omega\in F:g(\omega)=f(\omega)\}$ and $F\setminus B$ both have positive measure. Then, since F is σ-finite, B has a subset B' with positive and finite measure, and furthermore, for some $n\in\mathbf{N}$, the subset

$$B_n'=\left\{\omega\in B':\frac{1}{n}<f(\omega)<n,\ \frac{1}{n}<u(\omega)<n,\ \frac{1}{n}<v(\omega)<n\right\}$$

also has finite positive measure. We define the function $\phi:\mathbf{R}\to[0,\infty)$ by

$$\phi(\delta)=\|g+\delta\chi_{B_n'}\|_{L_u^p}+t\|f-g-\delta\chi_{B_n'}\|_{L_v^q}.$$

We claim that ϕ is differentiable at every point $\delta\in(-1/n,0)$ and, for these δ,

$$\phi'(\delta)=\frac{\int_{B_n'}(g+\delta)^{p-1}u^p\,d\mu}{(\int_\Omega(g+\delta\chi_{B_n'})^pu^p\,d\mu)^{1-1/p}}-\frac{t\int_{B_n'}(f-g-\delta)^{q-1}v^q\,d\mu}{(\int_\Omega(f-g-\delta\chi_{B_n'})^qv^q\,d\mu)^{1-1/q}}.$$

This follows of course from a standard theorem for differentiating under the integral sign. But note that various conditions appearing in the definition of B_n' have been imposed to ensure the validity of this theorem. To be more specific, what we need and have used here (and will also use again later) is the following simple fact, which follows immediately from Lagrange's theorem and Lebesgue's dominated convergence theorem.

Fact 6.1. Suppose that ϱ and w are non-negative measurable functions on (Ω,Σ,μ) such that $\varrho^r w^r \in L^1(\mu)$ for some $r\in(1,\infty)$. Let $A\in\Sigma$ be such that the functions $\varrho^{r-1}w^r\chi_A$ and $w^r\chi_A$ are also in $L^1(\mu)$. Let $\psi(\delta)=\int_\Omega |\varrho+\delta\chi_A|^r w^r\,d\mu$. Then, for each $\delta>-\inf_{\omega\in A}\varrho(\omega)$, ψ is differentiable at δ and

$$\psi'(\delta)=\int_A r(\varrho+\delta)^{r-1}w^r\,d\mu.$$

From our assumptions about B, $F\backslash B$ and B_n' it follows that the one sided limit $\lim_{\delta\nearrow 0}\phi'(\delta)$ exists and is strictly positive. But this is impossible, since, by (42), we have $\phi(0)\le\phi(\delta)$ for all $\delta\neq 0$. This contradiction proves (43).

We next claim that

$$\text{(44)}\qquad \text{(i) } g(\omega)>0 \text{ for a.e. } \omega\in F \quad\text{or}\quad \text{(ii) } g(\omega)=0 \text{ for a.e. } \omega\in\Omega.$$

This is proved by an exactly analogous argument to the one we have just presented for (43). That is, one has only to permute the rôles of L_u^p and L_v^q, and also the rôles of the functions g and $h=f-g$.

We now establish another property of the function $g=g_t$ in the case when it satisfies

$$\text{(45)}\qquad 0<g_t(\omega)<f(\omega) \quad \text{for a.e. } \omega\in F.$$

For each $n\in\mathbf{N}$, let F_n be the subset of F consisting of all points ω at which the values of the functions $g(\omega)$, $f(\omega)-g(\omega)$, $u(\omega)$ and $v(\omega)$ are all in the range $(1/n,n)$. Let B be any measurable subset of F_n and consider the function

$$\phi(\delta)=\|g+\delta\chi_B\|_{L_u^p}+t\|f-g-\delta\chi_B\|_{L_v^q}.$$

Since of course $\mu(B)<\infty$ we can use Fact 6.1 to show that, for all $\delta\in(-1/n,1/n)$,

$$\phi'(\delta)=\frac{\int_B(g+\delta)^{p-1}u^p\,d\mu}{\left(\int_\Omega(g+\delta\chi_B)^p u^p\,d\mu\right)^{1-1/p}}-\frac{t\int_B(f-g-\delta)^{q-1}v^q\,d\mu}{\left(\int_\Omega(f-g-\delta\chi_B)^q v^q\,d\mu\right)^{1-1/q}}.$$

Since ϕ assumes a minimum value at $\delta=0$, it follows that

$$\phi'(0)=\int_B\left(\frac{g^{p-1}u^p}{\|g\|_{L_u^p}^{p-1}}-\frac{t(f-g)^{q-1}v^q}{\|f-g\|_{L_v^q}^{q-1}}\right)d\mu=0$$

for all sets B as above. This implies that

$$\frac{g^{p-1}u^p}{\|g\|_{L_u^p}^{p-1}}=\frac{t(f-g)^{q-1}v^q}{\|f-g\|_{L_v^q}^{q-1}} \tag{46}$$

at almost every point of F_n and so also at almost every point of $\bigcup_{n\in\mathbf{N}}F_n=F$.

We are now ready to consider the behaviour of the above functions $g=g_t$ and $h=h_t=f-g_t$ as t ranges over all possible values in $(0,\infty)$.

Let E_* denote the set of all numbers $t>0$ for which the function $g=g_t$ satisfies (45). In view of (43) and (44) the set $(0,\infty)\backslash E_*$ is the union of the two disjoint sets $E_0=\{t>0: g_t(\omega)=0$ for a.e. $\omega\in\Omega\}$ and $E_f=\{t>0: g_t(\omega)=f(\omega)$ for a.e. $\omega\in\Omega\}$. Since $t\mapsto\|g_t\|_{L_u^p}$ has to be a non-decreasing function on $(0,\infty)$ (cf. (3)) we see that either E_0 is empty, or it is an interval whose left endpoint is 0. Similarly, either E_f is empty, or it is an interval whose right endpoint is ∞.

Suppose that $0<s<t$. We claim that

$$g_s(\omega)\le g_t(\omega)\quad\text{for a.e. }\omega\in\Omega. \tag{47}$$

This is obvious if $s\in E_0$ or if $t\in E_f$. It is also obvious if $s\in E_f$ or, alternatively, if $t\in E_0$, since then of course $t\in E_f$ or $s\in E_0$, respectively. Thus it remains only to consider the case when both s and t are in E_*. Here we can apply (46) to obtain that

$$\frac{g_s^{p-1}u^p}{(f-g_s)^{q-1}v^q}=\frac{s\|g_s\|_{L_u^p}^{p-1}}{\|f-g_s\|_{L_v^q}^{q-1}}\quad\text{and}\quad\frac{g_t^{p-1}u^p}{(f-g_t)^{q-1}v^q}=\frac{t\|g_t\|_{L_u^p}^{p-1}}{\|f-g_t\|_{L_v^q}^{q-1}} \tag{48}$$

at almost every point of F. Now, using (3) once more (i.e. that $t\mapsto\|g_t\|_{L_u^p}$ is non-decreasing and $t\mapsto\|f-g_t\|_{L_v^q}$ is non-increasing), we deduce from (48) that

$$\frac{g_s(\omega)^{p-1}}{(f(\omega)-g_s(\omega))^{q-1}}<\frac{g_t(\omega)^{p-1}}{(f(\omega)-g_t(\omega))^{q-1}}\quad\text{for almost every }\omega\in F. \tag{49}$$

For each fixed $\omega\in F$ we have $f(\omega)>0$ and therefore the continuous function $x\mapsto x^{p-1}/(f(\omega)-x)^{q-1}$ is a strictly increasing map of the interval $(0,f(\omega))$ onto $(0,\infty)$. Hence this function has a strictly increasing inverse on $(0,\infty)$ which can be applied to (49) to yield that $g_s(\omega)<g_t(\omega)$ for a.e. $\omega\in F$. Since $g_s(\omega)=g_t(\omega)=f(\omega)=0$ for all $\omega\notin F$ this establishes (47).

We have still not quite established that (L^p_u, L^q_v) is exactly monotone, since it could happen that the exceptional subset of measure zero $N_{s,t}\subset F$, which contains all points ω where (47) does not hold, depends on s and t in such a way that $\bigcup\{N_{s,t}:0<s<t\}$ might not be contained in a set of zero measure. To overcome this (small) problem we first consider the set $N_*=\bigcup\{N_{s,t}:0<s<t,\ s\in\mathbf{Q},\ t\in\mathbf{Q}\}$. This is of course measurable and $\mu(N_*)=0$, and for each $\omega\in F\setminus N_*$ we have that the function $t\mapsto g_t(\omega)$ restricted to $(0,\infty)\cap\mathbf{Q}$ is non-decreasing. Now let us define $G_t(\omega)$ for each $t>0$ and each $\omega\in F\setminus N_*$ by $G_t(\omega)=\sup\{g_s(\omega):0<s\le t,\ s\in\mathbf{Q}\}$. It is then easy to check that $\|G_t\|_{L^p_u}+t\|f-G_t\|_{L^q_v}=K(t,f;L^p_u,L^q_v)$ for all rational *and* irrational points $t\in(0,\infty)$ and to use the decomposition $\{f=G_t+(f-G_t)\}_{t>0}$ to show that (L^p_u, L^q_v) is exactly monotone.

7. The K-divisibility constant and λ-monotone couples

According to the Brudnyi–Krugljak K-divisibility theorem ([5], p. 325), for any given Banach couple $\overline{X}$, there exists a constant C having the following property.

(**) *If x is an arbitrary element of X_0+X_1 for which $K(t,x;\overline{X})\le\sum_{n=1}^\infty\phi_n(t)$ for all $t>0$, where the functions ϕ_n are all positive and concave and $\sum_{n=1}^\infty\phi_n(1)<\infty$, then there exist elements $x_n\in X_0+X_1$ such that $x=\sum_{n=1}^\infty x_n$ and $K(t,x_n;\overline{X})\le C\phi_n(t)$ for all $t>0$.*

We shall let $\gamma(\overline{X})$ denote the K-divisibility constant for $\overline{X}$, i.e. the infimum of all numbers C for which (**) holds. We recall (cf. [9]) that

$$1\le\gamma(\overline{X})\le 3+2\sqrt{2} \tag{50}$$

for every Banach couple $\overline{X}$.

In this section we shall investigate certain connections between the condition of exact monotonicity for couples of lattices $\overline{X}$ and the value of $\gamma(\overline{X})$. On the one hand, when $\overline{X}$ is exactly monotone, or "close" to being exactly monotone, we shall obtain an estimate for $\gamma(\overline{X})$ which is sharper than (50). On the other hand we shall see that if $\gamma(\overline{X})$ is "small" then this implies that $\overline{X}$ has a property similar to exact monotonicity. In particular (see Corollary 7.2) every couple of lattices $\overline{X}$ satisfying $\gamma(\overline{X})=1$ must necessarily be "extremely close" to being exactly monotone.

The precise formulations of these results are in terms of the notion of λ-monotone couples and almost exactly monotone couples (see Definition 1.4).

In fact every couple of Banach lattices is λ-monotone for some λ. More precisely we have the following result.

Theorem 7.1. *Each couple $\overline{X}$ of Banach lattices of measurable functions is λ-monotone for every $\lambda > \gamma(\overline{X})$.*

Proof. As already observed in Remark 1.6, it suffices to obtain the decompositions $f = g_t + h_t$ for the case where $f \geq 0$. This can be done exactly as in the proof of Theorem 4.1 of [10], i.e. we can set $g_t = \xi_0(t)$ and $h_t = \xi_1(t)$ in the notation of [10]. Note that the estimate (ii) at the beginning of the proof in [10] corresponds exactly to (5) above with $\lambda = C_p(1+\varepsilon)$. In our case $p=1$ and it is clear that we can take $C_p = C'_p = \gamma(\overline{X})$ and $\varepsilon > 0$ arbitrarily small. □

Corollary 7.2. *If $\gamma(\overline{X}) = 1$ then $\overline{X}$ is almost exactly monotone.*

Corollary 7.3. *If $\overline{X}$ is either of the couples introduced in Theorem* 4.1 *and Remark* 4.2 *then $\gamma(\overline{X}) > 1$.*

Proof. This is an immediate consequence of Corollary 7.2 and Remark 4.3. □

Remark 7.4. We can rewrite the result of Theorem 7.1 as $\lambda(\overline{X}) \leq \gamma(\overline{X})$ if we define $\lambda(\overline{X})$ to be the infimum of all $\lambda \geq 1$ such that $\overline{X}$ is λ-monotone. In fact, Theorem 7.7 below will enable us to obtain an approximate reverse of this inequality so that altogether we will have

$$\lambda(\overline{X}) \leq \gamma(\overline{X}) \leq 4\lambda(\overline{X}).$$

The rôle played by the proof of Theorem 4.1 of [10] in the proof of the preceding theorem, points to the fact that the λ-monotonicity of each couple of lattices $\overline{X}$ is also related to the formula to within equivalence for $K(t, f; \overline{X})$ stated above as (6) (i.e. Property $(*)$). The proof of Theorem 4.1 of [10] shows that the constant C in (6) can be chosen to be any number greater than $2\gamma(\overline{X})$. Our next (very simple) result provides an alternative estimate for this constant C. Since Theorem 7.1 does not exclude the possibility that a given couple of lattices $\overline{X}$ may be λ-monotone also for some $\lambda \leq \gamma(\overline{X})$, it is possible that this alternative estimate for C may sometimes be sharper than the one provided by Theorem 4.1 of [10].

Proposition 7.5. *Let $\overline{X}$ be a λ-monotone couple of Banach lattices. Then for each $f \in X_0 + X_1$ there exists an increasing family $\{E_t\}_{t>0}$ of measurable subsets of the underlying space such that*

$$\|f\chi_{E_t}\|_{X_0} + t\|f(1-\chi_{E_t})\|_{X_1} \leq 2\lambda K(t, f; \overline{X})$$

for each $t > 0$.

Proof. This is similar to a different (and quite simple) part of the proof of Theorem 4.1 in [10]. Let $f = g_t + h_t$ be the decomposition which exists according to

Definition 1.4 and, for each $t>0$, let E_t be the set where $|g_t|\geq|h_t|$. It is easy to check that these sets have all the required properties. □

We next present a simple lemma which will be needed for the proof of the last main result of this section.

Lemma 7.6. *Suppose that $\overline{X}$ is a λ-monotone couple of Banach lattices. Then, for each $\varepsilon>0$ and each non-negative function $f\in X_0+X_1$, there exists a decomposition $\{f=G_t+H_t\}_{t>0}$ such that*

$$\|G_t\|_{X_0}+t\|H_t\|_{X_1}\leq\lambda(1+\varepsilon)K(t,f;\overline{X})\quad \textit{for all } t>0, \tag{51}$$

and for a.e. ω in the underlying measure space $G_t(\omega)$ is a non-decreasing non-negative function of t and $H_t(\omega)$ is a non-increasing non-negative function of t. Furthermore, we can suppose that the functions $t\mapsto\|G_t\|_{X_0}$ and $t\mapsto\|H_t\|_{X_1}$ are continuous on $(0,\infty)$.

Proof. Fix a non-negative function $f\in X_0+X_1$ and let $f=g_t+h_t$ be a decomposition having all the properties specified in Definition 1.4 and Remark 1.6. We first set $G_t=g_t$ and $H_t=h_t$ for each t of the form $t=(1+\varepsilon)^n$ for each $n\in\mathbf{Z}$. Then we extend G_t and H_t to all of $(0,\infty)$, so that they are affine functions of t on each interval $[(1+\varepsilon)^n,(1+\varepsilon)^{n+1}]$. To show that (51) holds, given any fixed $t>0$, we choose $n\in\mathbf{Z}$ and $\theta\in[0,1]$ so that $t=(1-\theta)(1+\varepsilon)^n+\theta(1+\varepsilon)^{n+1}$. Then $G_t=(1-\theta)g_{(1+\varepsilon)^n}+\theta g_{(1+\varepsilon)^{n+1}}$ and $H_t=(1-\theta)h_{(1+\varepsilon)^n}+\theta h_{(1+\varepsilon)^{n+1}}$. By (5) we have that

$$\|G_s\|_{X_0}+s\|H_s\|_{X_1}\leq\lambda K(s,f)\quad\text{for } s=(1+\varepsilon)^n \text{ and } s=(1+\varepsilon)^{n+1}.$$

Consequently,

$$\begin{aligned}\|G_t\|_{X_0}+(1+\varepsilon)^n\|H_t\|_{X_1}&\leq(1-\theta)(\|G_{(1+\varepsilon)^n}\|_{X_0}+(1+\varepsilon)^n\|H_{(1+\varepsilon)^n}\|_{X_1})\\&\quad+\theta(\|G_{(1+\varepsilon)^{n+1}}\|_{X_0}+(1+\varepsilon)^n\|H_{(1+\varepsilon)^{n+1}}\|_{X_1})\\&\leq(1-\theta)\lambda K((1+\varepsilon)^n,f)+\theta\lambda K((1+\varepsilon)^{n+1},f).\end{aligned}$$

The concavity of the function $t\mapsto K(t,f)$ implies that this last expression does not exceed $\lambda K(t,f)$. We deduce (51) immediately, since

$$\|G_t\|_{X_0}+t\|H_t\|_{X_1}\leq(1+\varepsilon)(\|G_t\|_{X_0}+(1+\varepsilon)^n\|H_t\|_{X_1}).$$

It is very easy to check that G_t and H_t also have the other properties stated in the lemma. □

Our final main result in this section is the inequality

$$\gamma(\overline{X}) \le 4\lambda(\overline{X}) \tag{52}$$

which has already been alluded to above. That is we must show that $\gamma(\overline{X}) \le 4\lambda$ for each λ such that the couple of lattices $\overline{X}$ is λ-monotone. We can deduce this easily from Theorem 7.7 which we shall state immediately after this paragraph. This theorem is an analogue of Theorem 4 of [8], pp. 49–50, and of Theorem 1.7 of [9], pp. 71–72, i.e. it is a variant of the so-called "strong fundamental lemma" of [9]. The estimate (52) will follow from the fact that $\gamma(\overline{X}) \le 4\lambda(1+\varepsilon)$ for each λ and ε as in the statement of Theorem 7.7, and this in turn can be deduced from Theorem 7.7 in exactly the same way as Theorem 1 of [8] is deduced from Theorem 4 of [8] on pp. 54–55 of [8], except, of course, that the constant 8 appearing there has to be replaced here by 4λ. (Cf. also Remarks 1.34 and 1.36 and Proposition 1.40 of [10].)

Theorem 7.7. *Let $\overline{X}=(X_0,X_1)$ be a λ-monotone Banach couple of lattices of measurable functions and let $\overline{X^\sim}$ denote the couple $(X_0^\sim, X_1^\sim)$, where $X_j^\sim$ is the Gagliardo completion of X_j in X_0+X_1, $j=0,1$. Let $f \in X_0+X_1$. Then for each $\varepsilon>0$ there exists a sequence of elements $\{u_{n,\varepsilon}\}_{n\in\mathbf{Z}}=\{u_n\}_{n\in\mathbf{Z}}$ in X_0+X_1 such that $u_n \in X_0 \cap X_1$ for all but at most two values of n, $\sum_{n=-\infty}^{\infty} u_n = f$, with convergence in the X_0+X_1 norm, and*

$$\sum_{n=-\infty}^{\infty} \min\{\|u_n\|_{X_0^\sim}, t\|u_n\|_{X_1^\sim}\} \le 4\lambda(1+\varepsilon)K(t,f;\overline{X}) \quad \textit{for all } t>0. \tag{53}$$

(In the preceding estimate we set $\|u_n\|_{X_j^\sim}=\infty$ if $u_n \notin X_j^\sim$.)

Proof. Clearly it suffices to consider the case where $f \ne 0$ is a non-negative function. Many, but not all steps of this proof are modelled on the proof of Theorem 1.7 in Section 2 of [9]. For the benefit of the reader who may wish to refine either of these theorems, we shall draw attention at various stages to some of the similarities and differences between the two proofs. We first need to choose a constant $r>1$. We can of course suppose, without loss of generality, that the number ε appearing in the statement of the theorem satisfies

$$1+\varepsilon \le r. \tag{54}$$

In fact, we shall see later that the optimal value for r is 2. But we shall present most of the steps of the proof for general r, again with a view to facilitating future improvements. We introduce the set $D(f)=\{(\|G_t\|_{X_0}, \|H_t\|_{X_1}) : t \in (0,\infty)\}$, where $f=G_t+H_t$ is the continuous decomposition of f constructed in Lemma 7.6. This set will play a rôle more or less analogous to that of the set $D(a)$ introduced in [9]. (Let us note in passing that $D(f)$ is always non-empty, whereas the set $D(a)$ of [9] may be empty. This case is not dealt with explicitly in [9], but it can be immediately disposed of, since $D(a)=\emptyset$ if and only if $a=0$.) Let us set

$$x_{-\infty}=\lim_{t\to 0}\|G_t\|_{X_0},\quad x_\infty=\lim_{t\to\infty}\|G_t\|_{X_0},\quad y_{-\infty}=\lim_{t\to 0}\|H_t\|_{X_1},\quad y_\infty=\lim_{t\to\infty}\|H_t\|_{X_1}.$$

These are approximate counterparts of the quantities defined by the formulæ (2.1) on p. 74 of [9]. However, they do not necessarily satisfy the formulæ (2.2) of [9]. (Note also that here we have permuted part of the notation adopted in [9] so that $y_{-\infty}$ is now the "largest" and y_∞ is now the "smallest" value of y for $(x,y)\in D(f)$.) The next step of the corresponding proof in [9] is to construct a certain finite or infinite sequence of points $\{(x_n,y_n)\}_{\nu_{-\infty}<n<\nu_\infty}$ in $D(a)$. (We have taken this opportunity to correct a minor misprint in [9], where the range of n for this sequence is incorrectly stated to be $\nu_{-\infty}-1<n<\nu_\infty+1$.) Here, analogously, we shall now construct a special sequence of points lying on $D(f)$. This is done in a way which is quite similar to the construction of the sequence $\{(x_n,y_n)\}_{\nu_{-\infty}<n<\nu_\infty}$ in [9], except that in some cases we have to make some modifications when n is at one of the "endpoints" $\nu_{-\infty}$ and ν_∞ if these are finite. Here the index n will range over a possibly larger set which we will denote by $\varrho_{-\infty}<n<\varrho_\infty$. (These modifications are needed because of the above mentioned possible failure of the quantities $x_{\pm\infty}$ and $y_{\pm\infty}$ to satisfy (2.2) of [9].) The actual values of the four parameters $\varrho_{-\infty}$, ϱ_∞, $\nu_{-\infty}$ and ν_∞ will be determined in the course of the construction. They can either be integers, or $\pm\infty$. More specifically, they will satisfy $-\infty\le\varrho_{-\infty}\le\nu_{-\infty}<0<\nu_\infty\le\varrho_\infty\le+\infty$. Our sequence $\{(x_n,y_n)\}_{\varrho_{-\infty}<n<\varrho_\infty}$ of points of $D(f)$ will correspond to an *increasing* sequence of points $\{t_n\}_{\varrho_{-\infty}<n<\varrho_\infty}$ in $(0,\infty)$, where we set $x_n=\|G_{t_n}\|_{X_0}$ and $y_n=\|H_{t_n}\|_{X_1}$. In the two trivial cases, where $G_t=0$ for all t, or $H_t=0$ for all t, we can prove the theorem by simply choosing $u_0=f$ and $u_n=0$ for all $n\neq 0$. So we can assume that the sets $\{t>0:G_t\neq 0\}$ and $\{t>0:H_t\neq 0\}$ are both non-empty. These sets are necessarily intervals of the form (α,∞) and $(0,\beta)$, respectively. Furthermore, since $f\neq 0$, we have $\beta>\alpha$. We begin the construction of $\{t_n\}$ by choosing some arbitrary $t_0\in(\alpha,\beta)$ and, correspondingly, $(x_0,y_0)=(\|G_{t_0}\|_{X_0},\|H_{t_0}\|_{X_1})$. Then, for each $n>0$, we construct $(x_n,y_n)=(\|G_{t_n}\|_{X_0},\|H_{t_n}\|_{X_1})\in D(f)$ inductively such that $t_n>t_{n-1}$ and

$$\begin{cases} x_n=rx_{n-1}, \\ y_n\le \dfrac{1}{r}y_{n-1} \end{cases} \quad\text{or}\quad \begin{cases} x_n\ge rx_{n-1}, \\ y_n=\dfrac{1}{r}y_{n-1} \end{cases}$$

holds. Because of the continuity and monotonicity of the functions $t\mapsto\|G_t\|_{X_0}$ and $t\mapsto\|H_t\|_{X_1}$, such (x_n,y_n) and t_n will always exist whenever the integer n satisfies

$$rx_{n-1}<x_\infty \quad\text{and}\quad \frac{1}{r}y_{n-1}>y_\infty. \tag{55}$$

If (55) holds for every positive n then we obtain an infinite sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ and, accordingly, we set $\varrho_\infty = \nu_\infty = \infty$. In this case, since $x_0 > 0$, it follows that $\lim_{n\to\infty} x_n = \infty$ and therefore $\lim_{n\to\infty} t_n = \infty$. On the other hand, if at some stage of the construction we encounter an integer $n > 0$ which satisfies

$$r x_{n-1} \ge x_\infty \quad \text{or} \quad \frac{1}{r} y_{n-1} \le y_\infty,$$

then we set $\nu_\infty = n$. In such a situation there are two possibilities which must be dealt with separately. First, if

$$\frac{1}{r} y_{n-1} \le y_\infty \quad \text{or} \quad y_\infty = 0, \tag{56}$$

then, as in [9], the construction stops at this stage, i.e. we also set $\varrho_\infty = n$ and do not define (x_n, y_n) and t_n. The remaining possibility is that

$$r x_{n-1} \ge x_\infty \quad \text{and} \quad y_\infty > 0. \tag{57}$$

In this case we set $\varrho_\infty = \nu_\infty + 1$ and (in contrast to [9]) the construction has one more step, i.e. we choose t_{ν_∞} sufficiently large so that the additional point $(x_{\nu_\infty}, y_{\nu_\infty}) = (\|G_{t_{\nu_\infty}}\|_{X_0}, \|H_{t_{\nu_\infty}}\|_{X_1})$ satisfies $y_{\nu_\infty} \le (1+\varepsilon) y_\infty$.

Now, in a similar way, for $n < 0$ we go "backwards" and inductively construct $(x_n, y_n) = (\|G_{t_n}\|_{X_0}, \|H_{t_n}\|_{X_1}) \in D(f)$ such that $t_n < t_{n+1}$ and

$$\begin{cases} x_n = \dfrac{1}{r} x_{n+1}, \\ y_n \ge r y_{n+1} \end{cases} \quad \text{or} \quad \begin{cases} x_n \le \dfrac{1}{r} x_{n+1}, \\ y_n = r y_{n+1} \end{cases}$$

holds. Again the existence of these points is guaranteed by the properties of $t \mapsto \|G_t\|_{X_0}$ and $t \mapsto \|H_t\|_{X_1}$ whenever the negative integer n satisfies

$$\frac{1}{r} x_{n+1} > x_{-\infty} \quad \text{and} \quad r y_{n+1} < y_{-\infty}. \tag{58}$$

If (58) holds for all negative n then we obtain an infinite sequence $\{(x_n, y_n)\}_{n=-\infty}^{-1}$ and we set $\nu_{-\infty} = \varrho_{-\infty} = -\infty$. In this case, since $y_0 > 0$, we have $\lim_{n\to-\infty} y_n = \infty$ and therefore $\lim_{n\to-\infty} t_n = 0$. If, however, we encounter an integer $n < 0$ for which

$$\frac{1}{r} x_{n+1} \le x_{-\infty} \quad \text{or} \quad r y_{n+1} \ge y_{-\infty},$$

then we set $\nu_{-\infty} = n$. Here again there are two possibilities which need to be treated separately. The first occurs when

$$\frac{1}{r} x_{n+1} \le x_{-\infty} \quad \text{or} \quad x_{-\infty} = 0,$$

and if this happens we proceed as in [9], setting $\varrho_{-\infty}=\nu_{-\infty}=n$ and not defining (x_n, y_n) and t_n. On the other hand, if

$$ry_{n+1} \geq y_{-\infty} \quad \text{and} \quad x_{-\infty} > 0,$$

then we set $\varrho_{-\infty}=\nu_{-\infty}-1$ and, as an additional step, choose $t_{\nu_{-\infty}}>0$ sufficiently small so that $(x_{\nu_{-\infty}}, y_{\nu_{-\infty}})=(\|G_{t_{\nu_{-\infty}}}\|_{X_0}, \|H_{t_{\nu_{-\infty}}}\|_{X_1})$ satisfies $x_{\nu_{-\infty}} \leq (1+\varepsilon)x_{-\infty}$. Note that in *all* cases, whether or not $\nu_{\pm\infty}$ and $\varrho_{\pm\infty}$ are finite, we have defined (x_n, y_n) and t_n for all integers n which satisfy $\varrho_{-\infty}<n<\varrho_\infty$ and for no integer n outside this range. We can now define the sequence $\{u_n\}_{n\in\mathbf{Z}}$ by

$$u_n = \begin{cases} G_{t_n} - G_{t_{n-1}} = H_{t_{n-1}} - H_{t_n}, & \text{if } \varrho_{-\infty}+1 < n < \varrho_\infty, \\ f - G_{t_{\varrho_\infty - 1}} = H_{t_{\varrho_\infty - 1}}, & \text{if } n = \varrho_\infty < \infty, \\ f - H_{t_{\varrho_{-\infty}+1}} = G_{t_{\varrho_{-\infty}+1}}, & \text{if } n = \varrho_{-\infty}+1 > -\infty, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\sum_{n=-\infty}^{\infty} u_n = f$, where the series converges in the X_0+X_1 norm. In fact, if $\varrho_{-\infty}>-\infty$, then $\sum_{n=\varrho_{-\infty}+1}^{0} u_n = G_{t_0}$, and if $\varrho_{-\infty}=-\infty$ then

$$\left\| \sum_{n=-\infty}^{0} u_n - G_{t_0} \right\|_{X_0} = \lim_{n\to-\infty} \|G_{t_n}\|_{X_0} \leq \lim_{n\to\infty} \|G_{t_0}\|_{X_0} r^{-n} = 0.$$

Similarly, $\sum_{n=1}^{\infty} u_n = H_{t_0}$ with convergence in the X_1 norm, whether or not ϱ_∞ is finite.

As a first step towards proving (53) we need some preliminary estimates for $\|u_n\|_{X_0}$ and $\|u_n\|_{X_1}$. This is exactly the place where the monotonicity of G_t and H_t enables us to obtain better bounds than those which hold in the analogous proof for a general Banach couple (cf. (2.9) and (2.10) on p. 75 of [9]). If $\varrho_{-\infty}+1<n<\varrho_\infty$ then $\|u_n\|_{X_0} = \|G_{t_n} - G_{t_{n-1}}\|_{X_0} \leq \|G_{t_n}\|_{X_0}$, so we have

$$\|u_n\|_{X_0} \leq \|G_{t_n}\|_{X_0} = x_n \quad \text{for } \varrho_{-\infty} < n < \varrho_\infty \tag{59}$$

(i.e. we have also observed that obviously (59) holds also for $n=\varrho_{-\infty}+1$, if $\varrho_{-\infty}$ is finite). Similarly $\|u_n\|_{X_1} = \|H_{t_{n-1}} - H_{t_n}\|_{X_1} \leq \|H_{t_{n-1}}\|_{X_1}$ for $\varrho_{-\infty}+1<n<\varrho_\infty$ and so

$$\|u_n\|_{X_1} \leq \|H_{t_{n-1}}\|_{X_1} = y_{n-1} \quad \text{for } \varrho_{-\infty}+1 < n < \varrho_\infty + 1 \tag{60}$$

(where again the additional case where $n=\varrho_\infty$ is obvious, when ϱ_∞ is finite).

Now let us fix an arbitrary $t>0$ and show that (53) holds for this t. There are three cases which must be considered. Case 1 is when there exists an integer n^* in the range $\nu_{-\infty}+1<n^*<\nu_\infty$ such that $t_{n^*-1}\le t\le t_{n^*}$. Case 2 is when $\nu_\infty<\infty$ and $t\ge t_{\nu_\infty-1}$. The remaining possibility, Case 3, is when $\nu_{-\infty}>-\infty$ and $t\le t_{\nu_{-\infty}+1}$. Let us first deal with Case 1. We use the notation $m_n=\min\{\|u_n\|_{X_0},t\|u_n\|_{X_1}\}$ and write the sum

$$\sum_{n=-\infty}^{\infty}\min\{\|u_n\|_{X_0},t\|u_n\|_{X_1}\}=\sum_{n=-\infty}^{n^*-1}m_n+m_{n^*}+\sum_{n=n^*+1}^{\infty}m_n=I_-+m_{n^*}+I_+.$$

We note that, by (59),

$$I_-=\sum_{n=\varrho_{-\infty}+1}^{n^*-1}m_n\le\sum_{n=\varrho_{-\infty}+1}^{n^*-1}x_n=x'_{\varrho_{-\infty}+1}+\sum_{n=\nu_{-\infty}+1}^{n^*-1}x_n. \tag{61}$$

Here we are using the notation

$$x'_{\varrho_{-\infty}+1}=\begin{cases}0, & \text{if } \varrho_{-\infty}=\nu_{-\infty},\\ x_{\varrho_{-\infty}+1}, & \text{if } \varrho_{-\infty}=\nu_{-\infty}-1>-\infty.\end{cases}$$

Our construction of $\{(x_n,y_n)\}$ ensures that

$$x_n\le\frac{1}{r}x_{n+1},\quad \text{whenever } \nu_{-\infty}<n<\nu_\infty-1. \tag{62}$$

Consequently $x_n\le(1/r)^{n^*-n-1}x_{n^*-1}$ for $\nu_{-\infty}<n\le n^*-1$ and so

$$\sum_{n=\nu_{-\infty}+1}^{n^*-1}x_n\le\sum_{n=\nu_{-\infty}+1}^{n^*-1}\left(\frac{1}{r}\right)^{n^*-n-1}x_{n^*-1}=x_{n^*-1}\frac{1-r^{-n^*+\nu_{-\infty}+1}}{1-r^{-1}}, \tag{63}$$

where we are adopting the convention that $r^{-n^*+\nu_{-\infty}+1}=0$ if $\nu_{-\infty}=-\infty$. If $\varrho_{-\infty}$ is finite and equal to $\nu_{-\infty}-1$, then

$$x_{\varrho_{-\infty}+1}=x_{\nu_{-\infty}}\le(1+\varepsilon)x_{-\infty}\le(1+\varepsilon)x_{\nu_{-\infty}+1}\le(1+\varepsilon)r^{-n^*+\nu_{-\infty}+2}x_{n^*-1}. \tag{64}$$

Combining (61), (63) and (64) gives us that

$$\begin{aligned}I_-&\le x_{n^*-1}\left(\frac{1-r^{-n^*+\nu_{-\infty}+1}}{1-r^{-1}}+(1+\varepsilon)r^{-n^*+\nu_{-\infty}+2}\right)\\ &=x_{n^*-1}\left(\frac{1}{1-r^{-1}}+r^{-n^*+\nu_{-\infty}+2}\left(1+\varepsilon-\frac{r^{-1}}{1-r^{-1}}\right)\right)\\ &\le x_{n^*-1}\left(\frac{1}{1-r^{-1}}+\left|1+\varepsilon-\frac{r^{-1}}{1-r^{-1}}\right|\right).\end{aligned} \tag{65}$$

On the other hand, if $\varrho_{-\infty}=\nu_{-\infty}$, whether or not this quantity is finite, a simpler version of the preceding estimates gives us that

$$I_{-}\le x_{n^*-1}\frac{1}{1-r^{-1}}. \tag{66}$$

We next apply very similar arguments to estimate I_{+}. By (60) we have

$$I_{+}=\sum_{n=n^*+1}^{\varrho_\infty} m_n \le t\sum_{n=n^*+1}^{\varrho_\infty} y_{n-1}=t\sum_{n=n^*+1}^{\nu_\infty} y_{n-1}+ty'_{\varrho_\infty-1}, \tag{67}$$

where

$$y'_{\varrho_\infty-1}=\begin{cases} 0, & \text{if } \varrho_\infty=\nu_\infty,\\ y_{\varrho_\infty-1}, & \text{if } \varrho_\infty=\nu_\infty+1<\infty.\end{cases}$$

Our construction of $\{(x_n,y_n)\}$ ensures that $y_n\le(1/r)y_{n-1}$, whenever $\nu_{-\infty}+1<n<\nu_\infty$. Consequently $y_{n-1}\le(1/r)^{-n^*-1+n}y_{n^*}$ whenever $n^*+1\le n<\nu_\infty+1$. So

$$\sum_{n=n^*+1}^{\nu_\infty} y_{n-1}\le\sum_{n=n^*+1}^{\nu_\infty}\left(\frac{1}{r}\right)^{-n^*-1+n} y_{n^*}=y_{n^*}\frac{1-r^{n^*-\nu_\infty}}{1-r^{-1}}, \tag{68}$$

where we are adopting the convention that $r^{n^*-\nu_\infty}=0$ if $\nu_\infty=\infty$. If ϱ_∞ is finite and equal to $\nu_\infty+1$, then

$$y_{\varrho_\infty-1}=y_{\nu_\infty}\le(1+\varepsilon)y_\infty\le(1+\varepsilon)y_{\nu_\infty-1}\le(1+\varepsilon)r^{n^*-\nu_\infty+1}y_{n^*} \tag{69}$$

and we can combine (67), (68) and (69) to obtain that

$$\begin{aligned} I_{+}&\le ty_{n^*}\left(\frac{1-r^{n^*-\nu_\infty}}{1-r^{-1}}+(1+\varepsilon)r^{n^*-\nu_\infty+1}\right)\\ &=ty_{n^*}\left(\frac{1}{1-r^{-1}}+r^{n^*-\nu_\infty+1}\left(1+\varepsilon-\frac{r^{-1}}{1-r^{-1}}\right)\right)\\ &\le ty_{n^*}\left(\frac{1}{1-r^{-1}}+\left|1+\varepsilon-\frac{r^{-1}}{1-r^{-1}}\right|\right).\end{aligned}$$

On the other hand, when $\varrho_\infty=\nu_\infty$, whether or not ϱ_∞ is finite, we obtain similarly that

$$I_{+}\le ty_{n^*}\frac{1}{1-r^{-1}}.$$

Summarizing the preceding estimates, we see that in all subcases of Case 1, i.e. whether or not the quantities $\varrho_{\pm\infty}$ and $\nu_{\pm\infty}$ are equal to each other or finite,

$$I_{-}+I_{+}\le\left(\frac{1}{1-r^{-1}}+\left|1+\varepsilon-\frac{r^{-1}}{1-r^{-1}}\right|\right)(x_{n^*-1}+ty_{n^*}). \tag{70}$$

We observe that

$$x_{n^*-1}+sy_{n^*}\le\lambda(1+\varepsilon)K(s,f)\quad\text{for } s=t_{n^*-1} \text{ and } s=t_{n^*}\tag{71}$$

and, since the left-hand side of (71) is an affine function of s and the right-hand side is a concave function of s, we obtain the same inequality for $s=t$. This, combined with (70), gives

$$I_-+I_+\le\lambda(1+\varepsilon)\biggl(\frac{r}{r-1}+\biggl|\varepsilon+\frac{r-2}{r-1}\biggr|\biggr)K(t,f).\tag{72}$$

We can also see that $m_{n^*}\le\lambda(1+\varepsilon)rK(t,f)$ since either $rx_{n^*-1}=x_{n^*}$ holds, in which case $m_{n^*}\le\|u_{n^*}\|_{X_0}\le x_{n^*}=rx_{n^*-1}\le\lambda(1+\varepsilon)rK(t,f)$, or otherwise $ry_{n^*}=y_{n^*-1}$ must hold and then $m_{n^*}\le t\|u_{n^*}\|_{X_1}\le try_{n^*}\le tr\|H_t\|_{X_1}\le\lambda(1+\varepsilon)rK(t,f)$. Combining the estimate for m_{n^*} with (72) we obtain that, in Case 1,

$$\sum_{n=-\infty}^{\infty}\min\{\|u_n\|_{X_0},t\|u_n\|_{X_1}\}\le\lambda(1+\varepsilon)\biggl(r+\frac{r}{r-1}+\biggl|\varepsilon+\frac{r-2}{r-1}\biggr|\biggr)K(t,f).\tag{73}$$

We now turn to Case 2, i.e. when $\nu_\infty<\infty$ and $t\ge t_{\nu_\infty-1}$. Now we write $\widetilde{m}_n=\min\{\|u_n\|_{X_0^\sim},t\|u_n\|_{X_1^\sim}\}$ and we shall estimate $\sum_{n=-\infty}^{\infty}\widetilde{m}_n$. We first observe that, quite similarly to before, using (59) and (62), we have

$$\begin{aligned}\tilde{I}_-:=\sum_{n=-\infty}^{\nu_\infty-1}\widetilde{m}_n&\le\sum_{n=-\infty}^{\nu_\infty-1}\|u_n\|_{X_0^\sim}\le\sum_{n=\varrho_{-\infty}+1}^{\nu_\infty-1}\|u_n\|_{X_0}\\&\le x'_{\varrho_{-\infty}+1}+\sum_{n=\nu_{-\infty}+1}^{\nu_\infty-1}x_n\le x'_{\varrho_{-\infty}+1}+\sum_{n=\nu_{-\infty}+1}^{\nu_\infty-1}\Bigl(\frac{1}{r}\Bigr)^{\nu_\infty-1-n}x_{\nu_\infty-1}.\end{aligned}$$

By substituting $n^*=\nu_\infty$ in (63), (64), (65) and (66) we obtain that, whatever the value of $\varrho_{-\infty}$, whether or not it is finite or equal to $\nu_{-\infty}$,

$$\tilde{I}_-\le x_{\nu_\infty-1}\biggl(\frac{r}{r-1}+\biggl|\varepsilon+\frac{r-2}{r-1}\biggr|\biggr)\le\lambda(1+\varepsilon)\biggl(\frac{r}{r-1}+\biggl|\varepsilon+\frac{r-2}{r-1}\biggr|\biggr)K(t,f).\tag{74}$$

Since now $\nu_\infty<\infty$, there are only two possibly non-zero terms in $\sum_{n=\nu_\infty}^{\infty}\widetilde{m}_n$, namely $\widetilde{m}_{\nu_\infty}$ and $\widetilde{m}_{\nu_\infty+1}$, and we have to estimate these terms in the two possible "subcases" (56) and (57). Let us first suppose that (56) holds and so $\varrho_\infty=\nu_\infty$ and $\widetilde{m}_{\nu_\infty+1}=0$. One possibility here is that $y_{\nu_\infty-1}/r\le y_\infty$ and so

$$\widetilde{m}_{\nu_\infty}\le t\|H_{t_{\nu_\infty-1}}\|_{X_1}=ty_{\nu_\infty-1}\le try_\infty\le\lambda(1+\varepsilon)rK(t,f).$$

Alternatively, we must have $y_\infty=0$ and also $y_{\nu_\infty-1}/r>y_\infty$. Since (55) does not hold, this also implies that $rx_{\nu_\infty-1}\ge x_\infty$. It follows that

$$K(s,f)\le \lim_{\tau\to\infty}(\|G_\tau\|_{X_0}+s\|H_\tau\|_{X_1})=x_\infty<\infty \quad \text{for each } s>0.$$

Consequently, $\|f\|_{X_0^\sim}=\lim_{s\to\infty}K(s,f)\le x_\infty$ (cf. [9], (2.2)) and

$$\widetilde{m}_{\nu_\infty}\le\|u_{\nu_\infty}\|_{X_0^\sim}=\|H_{t_{\nu_\infty-1}}\|_{X_0^\sim}\le\|f\|_{X_0^\sim}\le x_\infty\le rx_{\nu_\infty-1}\le\lambda(1+\varepsilon)rK(t,f).$$

It remains to deal with the second "subcase" i.e. when (57) holds. Then $\varrho_\infty=\nu_\infty+1$ and so, by (59), $\widetilde{m}_{\nu_\infty}\le\|u_{\nu_\infty}\|_{X_0}\le x_{\nu_\infty}\le x_\infty\le rx_{\nu_\infty-1}$. Since $t\ge t_{\nu_\infty-1}$, this last term is dominated by $r\|G_t\|_{X_0}$. We also have that $\widetilde{m}_{\nu_\infty+1}\le t\|u_{\nu_\infty+1}\|_{X_1}=t\|H_{t_{\nu_\infty}}\|_{X_1}\le t(1+\varepsilon)y_\infty\le t(1+\varepsilon)\|H_t\|_{X_1}$. Combining these estimates and also using (54), we obtain that

$$\widetilde{m}_{\nu_\infty}+\widetilde{m}_{\nu_\infty+1}\le r\|G_t\|_{X_0}+tr\|H_t\|_{X_1}\le\lambda(1+\varepsilon)rK(t,f).$$

These estimates combined with (74) show that, in all possible subcases of Case 2,

$$\sum_{n=-\infty}^{\infty}\widetilde{m}_n\le\lambda(1+\varepsilon)\left(r+\frac{r}{r-1}+\left|\varepsilon+\frac{r-2}{r-1}\right|\right)K(t,f). \tag{75}$$

An analogous argument, whose details we leave to the reader, shows that (75) also holds in the remaining case, namely Case 3. Thus, (cf. (73)) it holds for all cases. We now substitute $r=2$ to obtain

$$\sum_{n=-\infty}^{\infty}\widetilde{m}_n\le\lambda(1+\varepsilon)(4+\varepsilon)K(t,f)\quad\text{for all } t>0.$$

This immediately gives (53), since we can of course carry out all preceding steps of the proof with ε replaced by any smaller positive number. This completes the proof of the theorem, and consequently, as already explained above, also establishes (52). □

Acknowledgments. We are grateful to Mieczysław Mastyło for some helpful comments. We also thank Yuri Brudnyi and Pavel Shvartsman for drawing our attention to a number of useful references.

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Received September 2, 1999

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