Generalised parabolic bundles and applications to torsionfree sheaves on nodal curves

Usha Bhosle

Introduction

Let X be an irreducible nonsingular projective curve over an algebraically closed field. Let E be a vector bundle of rank k and degree d on X. We define generalised parabolic vector bundles (or *GPB's*) by extending the notion of a parabolic structure at a point of X to a parabolic structure over a divisor on X as follows.

Definition 1. A parabolic structure on E over a divisor D consists of 1) a flag $\mathcal{F}$ of vector subspaces of the vector space $E_{|D}=E\otimes O_D$:

$$\mathcal{F}\colon F_0(E) = E_{|D} \supset F_1(E) \supset \dots \supset F_r(E) = 0$$

2) real numbers $\alpha_1, \dots, \alpha_r$ (with $0\leqq\alpha_1<\alpha_2<\dots<\alpha_r<1$) called weights associated to the flag.

Definition 2. A *GPB* is a vector bundle E together with parabolic structures over finitely many divisors D_i.

We define semistability, stability of *GPB's*, study their properties and construct moduli spaces in some important cases. The main results are the following:

Result 1. (Proposition 2.2.) The moduli space P of generalised parabolic line bundles L with $\mathcal{F}$ given by $F_0(L)=L_{x_1}\oplus L_{x_2}\supset F_1(L)\supset o$; $x_1, x_2\in X$, $\dim f_1(L)=1$, is a nonsingular projective variety, it is in fact a $\mathbf{P}^1$-bundle over $\operatorname{Pic} X$.

Result 2. (Theorem 1.) There exists a coarse moduli space $M(k, d, a)$ of equivalence classes of semistable *GPB's* of rank k, degree d and with a parabolic structure over a divisor D of degree 2 given by $\mathcal{F}$: $F_0(E)=E_{|D}\supset F_1(E)\supset o$, $a=\dim F_1E$, weights $(\alpha_1, \alpha_2)=(0, \alpha)$. This space is a normal projective variety of dimension $k^2(g-1)+1+\dim F$, F being the flag variety of flags of type $\mathcal{F}$. If k and d are mutually coprime, α near 1 and $a=k$, then $M(k, d, k)$ is nonsingular and is a fine moduli space.

We have an interesting application of *GPB's* to the study of the moduli space $U(k, d)$ of torsionfree coherent sheaves of rank k and degree d on a nodal curve X_0. Let $\pi\colon X \to X_0$ be the normalisation map. For simplicity of exposition, let us assume that X_0 has a unique node x_0 and let x_1, x_2 be two points in X lying over x_0, $D=x_1+x_2$.

Result 3. The moduli space P (result 1) is a desingularisation of the compactified Jacobian $\bar{J}$ of X_0.

Result 4. (Theorem 3.) There is a birational surjective morphism $f\colon M(k, d, k) \to U(k, d)$. If $U_k \subset U(k, d)$ is the open subset corresponding to locally free sheaves, then the restriction of f induces an isomorphism of $f^{-1}(U_k)$ onto U_k.

In particular from results 2 and 4 it follows that if $(k, d)=1$, then $M=M(k, d, k)$ is a desingularization of $U(k, d)$. The moduli space $U=U(k, d)$ has a stratification. $U=\bigcup_{r=0}^{k} U_r$ where $U_a=\{F|\text{stalk } F_{x_0} \approx a\mathcal{O}_{x_0} \oplus (k-a)m_0\}$, $\mathcal{O}_{x_0}$ and m_0 being the local ring and maximum ideal at x_0. The space M also has a stratification $M=\bigcup_{r=0}^{k} M_r$ such that $f(M_r) \subsetneqq U_r$, for all $r>0$ (proposition 4.3). We have a morphism det: $U_k \to J$ defined by $\det F=\Lambda^k F$. An interesting question to ask is: Does this morphism extend to U?

Result 5. (Proposition 4.7.)

(1) The morphism det: $U_k \to J$ lifts to a morphism $M_k \to P$. The latter extends to a morphism $d\colon \bigcup_{r>0} M_r \to P$.

(2) The morphism d descends to a morphism det: $U_k \cup U_{k-1} \to \bar{J}$. But d does not induce a morphism on $\cup U_r$ for $r<k-1$ extending the det morphism.

Having found a negative answer to our first question, further questions arise: What is the closure of the graph of the det morphism in $U \times \bar{J}$? What is the closure of a fibre of the det morphism in U? Let U_L be the closed subset of U_k corresponding to vector bundles with a fixed determinant L and let $\bar{U}_L$ be its closure in U. We show that (3.20, 4.9) $\bar{U}_L \subset U_L \cup U_0$, and in case of rank two $\bar{U}_L = U_L \cup \{\pi_* E | \det E = \pi^* L(-x_1-x_2),\ E \text{ stable}\}$.

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1. Generalised parabolic bundles

Notation 1.1.

Let X be an irreducible curve with only nodes as singularities over an algebraically closed field k. Let π: $\tilde{X} \to X$ be the normalisation map. For simplicity of exposition we shall assume that X has a single node x_0, the results can be seen to generalise easily to the general case. Let x_1, x_2 be the two points of $\tilde{X}$ lying over x_0, $D = x_1 + x_2$. Let θ_{x_0}, m_0 denote the local ring and its maximum ideal at x_0.

We want to study the moduli space U of semistable torsion free sheaves of rank two and degree d on X. This space has been studied by Seshadri [S] and Gieseker [G]. Our approach is different from either of them, it is closer to the former. One has a stratification of U given by $U = \bigcup_{a=0}^{2} U_a$, where U_a denotes the subset of U consisting of points corresponding to sheaves F such that $F_{x_0} \approx a\theta_{x_0} \oplus (2-a)m_0$; U_2 is an open dense subset of the (irreducible) complete variety U corresponding to locally free sheaves F. Let U_2^L denote the subset of U_2 corresponding to F such that determinant of F is a fixed line bundle L. We are particularly interested in studying U_2^L and its closure in U. It can be shown that the determinant morphism from U_2 to the generalised Jacobian of X can be extended to $U_1 \cup U_2$, it seems that it is not extendable to U_0. In [S], a bijective correspondence between sheaves F corresponding to elements in U_a and bundles on $\tilde{X}$ with additional structures at x_1 and x_2 is given (theorem 17, p. 178, [S]). But this correspondence is different on each stratum and does not preserve degrees. Hence it is not of much use in studying the moduli space U as a whole. In our approach, we get sheaves F in U from "generalised parabolic bundles" E on $\tilde{X}$ of same degree as F.

Definition 1.2. A ***Generalised parabolic vector bundle*** of rank 2 on $\tilde{X}$ is a vector bundle E of rank two on X together with a two-dimensional k-subspace $F_1(E)$ of $E_{x_1} \oplus E_{x_2}$.

Definition 1.3. A generalised parabolic vector bundle E is ***stable*** (***semistable***) if for every line subbundle L of E,

$$\text{degree } L + \dim\left(F_1(E) \cap (L_{x_1} \oplus L_{x_2})\right) <_{(\leq)} \tfrac{1}{2}\left(\text{degree } E + \dim F_1(E)\right)$$

i.e.

$$\deg \cdot L + \dim\left(F_1(E) \cap L_D\right) <_{(\leq)} \mu(E) + 1.$$

Remark 1.4. If degree E is odd, then stability is equivalent to semistability for the generalised parabolic bundle of rank two.

Definition 1.5. A homomorphism of generalised parabolic bundles E_1, E_2 of rank two is a vector-bundle homomorphism of E_1 into E_2 which maps $F_1(E_1)$ into $F_1(E_2)$.

1.6. We now want to associate to a generalised parabolic bundle E of rank 2 and degree d on $\tilde{X}$ a torsionfree sheaf F on X of rank two and degree d. We have $\pi_*(E)\otimes k(x_0)=E_{x_1}\oplus E_{x_2}$ (p. 175, [S]) and hence a surjective morphism $\pi_*(E)\to E_{x_1}\oplus E_{x_2}/F_1(E)$. Define F to be the kernel of this surjection i.e. F is given by

$$0\to F\to \pi_*E\to \pi_*(E)\otimes k(x_0)/F_1(E)\to 0. \tag{1.7}$$

Proposition 1.8. *Let p_1 and p_2 denote the canonical projections from $F_1(E)$ to E_{x_1} and E_{x_2} respectively.*

(1) *If p_1 and p_2 are both isomorphisms, then F corresponds to an element in U^2 i.e. F is locally free.*
(2) *If only one of p_1 or p_2 is an isomorphism and the other is of rank one, then F corresponds to an element in U^1.*
(3) *If p_1 and p_2 are both of rank one or one of them is zero, then F corresponds to an element in U^0.*

Proof. (3) Note that if neither of p_1 or p_2 is an isomorphism, then p_1, p_2 satisfy the conditions of (3).

In case both p_1, p_2 are of rank 1, $F_1(E)=k_1\oplus k_2$, $k_i\subset E_{x_i}$, $i=1, 2$. Then clearly $F=\pi_*(E_0)$, where E_0 is defined by

$$0\to E_0\to E\to E_{x_1}/k_1\oplus E_{x_2}/k_2\to 0.$$

If $p_2=0$, $F_1(E)=E_{x_1}$ and $F=\pi_*(E_0)$, with E_0 defined by $0\to E_0\to E\to E_{x_2}\to 0$ i.e. $E_0=E(-x_2)$. Similarly, if $p_1=0$, $F=\pi_*(E(-x_1))$.

(1) and (2). In cases (1) and (2), one of p_1 and p_2 say p_1 is an isomorphism. Then using p_1, $F_1(E)$ can be regarded as the graph of a homomorphism $\sigma\colon E_{x_1}\to E_{x_2}$, σ being an isomorphism in case (1) and of rank one in case (2). Since $F|X-x_0\approx \pi_*(E)|X-x_0$ is locally free, our problem is local at x_0. So we are reduced to the following situation. Let A be the local ring at x_0, it is a Gorenstein local ring with maximum ideal m, $\bar{A}$ is a semi local ring with two maximum ideals m_1, m_2; $\sigma\colon \bar{A}/m_1\oplus\bar{A}/m_1\to\bar{A}/m_2\oplus\bar{A}/m_2$ a nonzero linear map with graph Γ_σ. We write $k_i=\bar{A}/m_i$, $\bar{A}_i=\bar{A}$, $i=1, 2$ and $n_i\colon \bar{A}_i\to k_1\oplus k_2$ canonical maps, for $i=1, 2$. F is an A-module given by

$$0\to F\to \bar{A}_1\oplus\bar{A}_2\to_p ((k_1\oplus k_2)\oplus(k_1\oplus k_2))/\Gamma_\sigma\to 0$$

where p is the composite of the map $(n_1\oplus n_2)$ with the quotient map $k_1\oplus k_2\oplus k_1\oplus k_2\to(k_1\oplus k_2\oplus k_1\oplus k_2)/\Gamma_\sigma$. Thus $F=(n_1\oplus n_2)^{-1}\Gamma_\sigma$. We want to show that $F\approx A\oplus A$ or $A\oplus\bar{A}$ according as σ is of rank two or one. Note that $\bar{A}$, m, m_1 and m_2 are all isomorphic. Fix a basis e_1, e_2 of k^2. With respect to the basis e_1, e_2, let the matrix of σ be $\begin{pmatrix} g & b\\ c & d\end{pmatrix}$ and let the matrix of σ^{-1} be $\begin{pmatrix} G & B\\ C & D\end{pmatrix}$ if σ is of rank two. Since

$n=n_i$: $\bar{A}\to k_1\oplus k_2$ is a surjection, there exist α, β, γ, δ in $\bar{A}$ such that $n(\alpha)=(1, G)$, $n(\beta)=(0, B)$, $n(\gamma)=(0, C)$ and $n(\delta)=(1, D)$. Then the matrix $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in GL(\bar{A})$ as $n(\alpha\delta-\beta\gamma)=(1, GD-BC)$ is a unit in $\bar{A}$ modulo the conductor m, $\bar{A}/m\approx\bar{A}/m_1\oplus\bar{A}/m_2$. This matrix defines an automorphism φ of $\bar{A}\oplus\bar{A}$ which induces the homomorphism ψ: $k_1\oplus k_2\oplus k_1\oplus k_2\to k_1\oplus k_2\oplus k_1\oplus k_2$ given by

$$\psi(x_1, y_1, x_2, y_2) = (x_1, Gy_1+By_2, x_2, Cy_1+Dy_2).$$

We have $\Gamma_\sigma=\{(x_1, gx_1+bx_2, x_2, cx_1+dx_2)|(x_1, x_2)\in k_1\oplus k_2\}$. Since $\sigma^{-1}\circ\sigma=\mathrm{Id}$ it follows that $\psi(\Gamma_\sigma)=\Gamma_{\mathrm{Id}}$. Since ψ lifts to the automorphism φ i.e. $\psi(n_1\oplus n_2)=(n_1\oplus n_2)\circ\phi$, it follows that $(n_1\oplus n_2)^{-1}\Gamma_\sigma\approx(n_1\oplus n_2)^{-1}\Gamma_{\mathrm{Id}}\approx A\oplus A$.

Now let σ be of rank one. In the above proof, we lifted the homomorphism ψ defined by $\sigma^{-1}\in GL(k^2)$ to an automorphism φ of $\bar{A}\oplus\bar{A}$. We can do it for any $f\in GL(k^2)$; then ψ will map Γ_σ into $\Gamma_{f\circ\sigma}$. Hence we can replace Γ_σ by $\Gamma_{f\circ\sigma}$. Since $\sigma\to f\circ\sigma$ is equivalent to change by row transformations of the matrix of σ, we may replace the matrix of σ by any matrix obtained by doing row transformations. (Note that column transformations are not allowed e.g. ψ: $(x_1, y_1, x_2, y_2)\to(x_1, y_1-x_2, x_2, y_2)$ cannot be lifted to an automorphism of $\bar{A}\oplus\bar{A}$.) By row transformations, any matrix σ of rank 1 can be reduced to one of the following forms

$$\text{(i)}\ \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\quad \text{(ii)}\ \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\quad \text{(iii)}\ \begin{pmatrix}1 & b\\ 0 & 0\end{pmatrix},\quad b\neq 0.$$

The following can be seen easily. In case (i), $(n_1\oplus n_2)^{-1}\Gamma_\sigma=A\oplus m_2$. In case (ii) $(n_1\oplus n_2)^{-1}\Gamma_\sigma=m_1\oplus A$. In case (iii), we need a little more work. We have $\Gamma_\sigma=\{(x_1, x_1+bx_2, x_2, 0)|(x_1, x_2)\in k_1\oplus k_1\}$. Consider

$$\begin{array}{ccc}\bar{A}_1\oplus\bar{A}_2 & \xrightarrow{n_1\oplus n_2} & k_1\oplus k_2\oplus k_1\oplus k_2\\ \Big\downarrow{\scriptstyle p_2} & & \Big\downarrow{\scriptstyle p_2'}\\ \bar{A}_2 & \xrightarrow[n_2]{} & k_1\oplus k_2\end{array}$$

Then $p_2'\circ(n_1\oplus n_2)F=p_2'(\Gamma_\sigma)=\{(x_2, 0)|x_2\in k_1\}=k_1\oplus\{0\}$. Now L.H.S.$=n_2\circ p_2(F)$, so $n_2\circ p_2(F)=k_1\oplus 0$ i.e. $p_2(F)=m_2$. Let $K=\mathrm{Ker}\, p_2|F=(\bar{A}_1\oplus 0)\cap F$. As $(n_1\oplus n_2)(K)=(n_1\oplus n_2)(\bar{A}\oplus 0)\cap\Gamma_\sigma=\{(x_1, x_1, 0, 0)|x_1\in k_1\}$ we have $K=A\oplus 0$. Thus we have an exact sequence $0\to A\to F\xrightarrow{p_2}\bar{A}\approx m_2\to 0$. Since $\mathrm{Ext}^1_A(\bar{A}, A)=0$, this sequence splits giving $F\approx A\oplus\bar{A}$.

This finishes the proof of the proposition.

Proposition 1.9. *If F is a semistable (respectively stable) torsionfree sheaf on X, then E is a semistable (respectively stable) generalised parabolic bundle on $\tilde{X}$. The converse is also true.*

Proof. Suppose that F_1 is stable. Let $L\subset E$ be a line subbundle. We want to show that $\deg L+\dim(F_1(E)\cap(L_{x_1}\oplus L_{x_2}))<\mu(E)+1$. Let $\dim(F_1(E)\cap(L_{x_1}\oplus L_{x_2}))=a$, $a=0, 1$ or 2.

(i) $a=0$: One has an exact sequence on X

$$0\to L_1\to \pi_* L\to (L_{x_1}\oplus L_{x_2})\to 0,\quad (L_{x_1}\oplus L_{x_2})\approx \pi_* L\oplus k(x_0),$$

with $L_1\subset F$. The stability of F implies that $\deg\cdot L_1<\mu(F)$ i.e. $\deg L-1<\mu(E)$ i.e. $\deg L+a<\mu(E)+1$.

(ii) $a=1$: One has $0\to L_1\to\pi_* L\to\pi_* L\otimes k(x_0)/k^a\to 0$, with $L_1\subset F$, $\deg\cdot L_1=\deg L$. Hence $\deg L_1<\mu(F)$ implies that $\deg\cdot L+a<\mu(E)+1$.

(iii) $a=2$: In this case, $L_1=\pi_* L$ so that $\deg\cdot L_1=\deg\cdot L+1$. The stability of F implies that $\deg\cdot L+2<\mu(E)+1$.

Thus F is stable implies that E is a stable generalised parabolic bundle. The proof in the semistable case is obtained by replacing '$<$' by '$\leqq$' in the above proof.

We now prove the converse. Let L_1 be a torsionfree subsheaf of F of rank 1. One has $\pi^* L_1/\text{torsion}\subset\pi^* F/\text{torsion}$ and (a sheaf inclusion) $\pi^* F/\text{torsion}\to E$. Let L be the line subbundle of E generated by $\pi^* L_1/\text{torsion}$; $a=\dim(F_1(E)\cap(L_{x_1}\oplus L_{x_2}))$. As seen above, if $a=0$, $L_1=\pi_*(L(-x_1-x_2))$ so that $\deg\cdot L<\mu(E)+1$ implies that $\deg\cdot L_1<\mu(F)$. If $a=1$, as seen above, L_1 is locally free and $\deg L_1=\deg L$. Hence $\deg L+a<\mu(E)+1$ implies that $\deg L_1<\mu(F)$. If $a=2$, $L_1=\pi_* L$, $\deg\cdot L_1=\deg L+1$ and we again get $\deg L_1<\mu(F)$. Thus F is stable (semistable) if E is stable (semistable) generalised parabolic bundle.

Remark 1.10. In 1.6, we defined a mapping f from the set S of isomorphism classes of generalised parabolic vector bundles of rank 2 and degree d on $\tilde{X}$ to the set $\mathbf{R}$ of isomorphism classes of torsionfree sheaves of rank 2 and degree d on X. Proposition 1.9 shows that $f(E, F_1(E))=F$ is semistable (stable) *iff* $(E, F_1(E))$ is so. Let $\tilde{U}^2$, $\tilde{U}^1$ *and* $\tilde{U}^0$ be the subsets of S corresponding to generalised parabolic bundles which satisfy the conditions (1), (2) and (3) respectively in proposition 1.8. Then f maps $\tilde{U}^i$ into U^i, $i=0, 1, 2$. Here U^i denotes the subset of R consisting of torsionfree sheaves F such that the stalk F_{x_0} of F at x_0 is isomorphic to $i\theta_{x_0}\oplus(2-i)m_0$.

Proposition 1.11. (1) *f maps $\tilde{U}^2$ bijectively onto U^2,* (2) *f maps $\tilde{U}^0$ onto U^0,* (3) *f maps $\tilde{U}^1$ onto U^1.*

Proof. (1) We give the inverse of f on U^2. Let $F\in U^2$. Define $E=\pi^* F$, $F_1(E)=F\otimes k(x_0)\subset F\otimes\pi_*\theta_{\tilde{X}}\otimes k(x_0)=\pi_*(E)\otimes k(x_0)$. It is easy to see that $(E, F_1(E))$ is a generalised parabolic bundle which maps to F under f.

(2) Let $F\in U^0$. Then $F=\pi_* E_0$ for a unique vector bundle E_0 on X (proposition 10, p. 174 [S]). The fibre of f over F consists of generalised parabolic bundles of the following type.

a) $E=E_0(x_2)$, $F_1(E)=E_{x_1}$.
b) $E=E_0(x_1)$, $F_1(E)=E_{x_2}$.
c) E given by an extension of the type $0\to E_0\to E\to k(x_1)\oplus k(x_2)\to 0$, $F_1(E)=\operatorname{Ker}(E\otimes\theta_{x_1+x_2}\to k(x_1)\oplus k(x_2))$.

Now, $\operatorname{Ext}^1(k(x_1)\oplus k(x_2), E_0)\approx(E_0\otimes(\Omega^1)^{-1})\otimes\theta_{x_1+x_2}\approx(E_0)_{x_1}\oplus(E_0)_{x_2}$ and given $k_1\subset(E_0)_{x_1}$, $k_2\subset(E_0)_{x_2}$, $k_1\approx k_2\approx k$, there exists a unique extension of the above type with kernel $((E_0)_{x_i}\to E_{x_i})=k_i$, $i=1,2$. Thus the set of generalised parabolic bundles of type c) is isomorphic to $\mathbf{P}^1\times\mathbf{P}^1(=\mathbf{P}((E_0)_{x_1})\times\mathbf{P}((E_0))_{x_2})$.

(3) Before proving that $\varphi|\tilde{U}^1$ is a surjection onto U^1, let us analyse $\varphi|\tilde{U}^1$. In this case, we can write $F_1(E)$ as the graph Γ_σ of a homomorphism $\sigma\colon E_{x_1}\to E_{x_2}$ of rank one if p_1 is an isomorphism, p_1 being the projection of $F_1(E)$ to E_{x_1}. (The case when p_2 is an isomorphism can be dealt with similarly.) Let $F=f(E, F_1(E))$, $E_0=\pi^*(F)/\text{torsion}$. Then one has exact sequences $0\to E_0\to E\to E_{x_2}/\text{Image }\sigma\to 0$ and

$$0\to F\to \pi_* E\to E_{x_1}\oplus E_{x_2}/\Gamma_\sigma\to 0.$$

Hence $(E_0)_{x_1}\xrightarrow{\sim}E_{x_1}$ canonically, let N_1 denote the isomorphic image of kernel σ in $(E_0)_{x_1}$. Since $0\to k\to(E_0)_{x_2}\to E_{x_2}\to E_{x_2}/\text{Image }\sigma\to 0$, $(E_0)_{x_2}$ contains a one dimensional N_2 such that $(E_0)_{x_2}/N_2\approx\text{Image }\sigma$. Let $\bar\sigma$ denote the isomorphism $(E_0)_{x_1}/N_1\xrightarrow{\sim}(E_0)_{x_2}/N_2$ induced by the composite $(E_0)_{x_1}\xrightarrow{\sim}E_{x_1}\xrightarrow{\sigma}\text{Image }\sigma\xrightarrow{\sim}(E_0)_{x_2}/N_2$. F is defined by

$$\begin{aligned}\Gamma(U,F) &= \{s\in\Gamma(\pi^{-1}U,E)\,|\,s(x_2)=\sigma s(x_1)\}\\ &= \{s\in\Gamma(\pi^{-1}U,E_0)\,|\,s(x_2)\bmod N_2=\bar\sigma(s(x_1)\bmod N_1)\}.\end{aligned}$$

Now start with an $F\in U^1$. Define $E_0=\pi^*F/\text{torsion}$. Since the stalk $F_{x_0}\approx m_0\oplus\theta_{x_0}$, $(E_0)_{x_i}\approx N_i\oplus\Delta_i$, $N_i\approx m_0\bar\theta_x\otimes k(x_i)$, $\Delta_i\approx\bar\theta_{x_0}\otimes k(x_i)$, $i=1,2$, $\bar\theta_x$ being the normalisation of θ_x. Define the vector bundle E on $\tilde X$ by $0\to E_0\to E\to k(x_2)\to 0$ with the condition $\operatorname{Ker}((E_0)_{x_2}\to E_{x_2})=N_2$, it is easy to see that such E exists. By theorem 17, p. 178 [S], there is a natural bijection between the set of isomorphism classes of torsionfree sheaves F of rank 2, degree d on X with $\mathscr{F}_{x_0}\approx\theta_{x_0}\oplus m_0$ and the set of isomorphism classes of triples $(E_0,\Delta_1,\Delta_2,\bar\sigma)$, E_0 being a vector bundle of rank 2 on $\tilde X$ of degree $d-1$, Δ_i are one-dimensional subspaces of $(E_0)_{x_i}$, $i=1,2$ and $\bar\sigma$ is an isomorphism $\Delta_1\to\Delta_2$. Since Δ_1, Δ_2 both come from $\theta_{x_0}\subset F_{x_0}$, we have an isomorphism $\bar\sigma\colon\Delta_1\to\Delta_2$. Define σ as the composite $E_{x_1}\xrightarrow{\sim}(E_0)_{x_1}\to\Delta_1\xrightarrow{\bar\sigma}\Delta_2\to E_{x_2}$ and $F_1(E)=\Gamma_\sigma$. From our analysis of $f|\tilde U^1$, it is easy to see that $f(E,F_1(E))=F$ i.e. f maps $\tilde U^1$ onto U^1.

Lemma 1.12. *If E is a stable generalised parabolic vector bundle of rank 2, then either E is stable as a vector bundle or E has a unique (maximum) line subbundle L of degree d_1, where $d_1=\mu(E)$ if degree of E is even and $d_1=\mu(E)+\frac{1}{2}$ if degree of E is odd. Moreover one has $F_1(E)\cap(L_{x_1}\oplus L_{x_2})$ is zero.*

Proof. Let L be a line subbundle of E and $a=\dim(F_1(E)\cap(L_{x_1}\oplus L_{x_2}))$. If $a=1$ or 2, stability of E as a generalised parabolic bundle implies that degree $L<\mu(E)$. If $a=0$, it implies that degree $L<\mu(E)+1$. Hence if E is not a stable vector bundle, there must exist a line subbundle L of degree d_1 such that $\mu(E)\leqq d_1<\mu(E)+1$ and $a=0$. It is easy to see that such a line bundle is unique, even the former condition suffices for uniqueness.

Lemma 1.13. (i) *If E is a generalised parabolic bundle (of rank two). Then the following condition* (C) *is satisfied.*

(C) *For any line subbundle L of E with*

$$\deg L = \begin{cases} \mu(E)-\frac{1}{2} & \text{if } \deg E \text{ is odd} \\ \mu(E)-1 & \text{if } \deg E \text{ is even,} \end{cases}$$

one has $a(L)<2$. Here $a(L)=\dim F_1(E)\cap(L_{x_1}\oplus L_{x_2})$.

(ii) *If E is a stable vector bundle satisfying condition* (C) *for $F_1(E)\subset E_{x_1}\oplus E_{x_2}$ then E together with $F_1(E)$ is a stable generalised parabolic vector bundle.*

Proof. Proofs are straightforward (using definitions).

Remark 1.14. ($g\geqq 2$). Given a stable vector bundle E of odd degree there exists $F_1(E)=k_1\oplus k_2$, $k_i\subset E_{x_i}$ such that $a(L)\neq 2$ i.e. $k_1\oplus k_2\neq L_{x_1}\oplus L_{x_2}$ for any line subbundle L of degree$=\mu(E)-\frac{1}{2}$. In fact if degree E is odd, rank $E=2$, E can have at most 4 line subbundles of degree $\mu(E)-\frac{1}{2}$ (proposition 4.2 [L]). By corollary 4.6 [L], the variety of maximal line subbundles of E has dimension $\leqq 1$ for any vector bundle E of rank 2.

2. Generalised parabolic line bundles and extension of the determinant map

Definition 2.1. A generalised parabolic line bundle on $\tilde{X}$ is a line bundle L on $\tilde{X}$ together with a one dimensional subspace $F_1(L)$ of $L_{x_1}\oplus L_{x_2}$.

Proposition 2.2. *The moduli space P of generalised parabolic line bundles on $\tilde{X}$ of fixed degree d* (degree $L=d$) *is a $\mathbf{P}^1$-bundle over the Jacobian $J(\tilde{X})$ of $\tilde{X}$ of line bundles of degree d. The variety P is a desingularisation of the compactified Jacobian $\bar{J}(X)$ of X.*

Proof. Let V be the Poincaré bundle on $J(\tilde{X})\times\tilde{X}$. Let $\mathscr{F}(V)$ denote the flag variety over $J(\tilde{X})\times\tilde{X}$ of type determined by the generalised parabolic structure (i.e. $k^2\supset k\supset 0$) and let P denote its restriction to $J(\tilde{X})\times\{x_1, x_2\}$. Let $p\colon P\to J(\tilde{X})$ be the composite $P\to J(\tilde{X})\times(x_1, x_2)\to J(\tilde{X})$. Clearly $p\colon P\to J(\tilde{X})$ is a $\mathbf{P}^1$-bundle over $J(\tilde{X})$, and P is the moduli space of generalised parabolic line bundles of degree d.

Consider the universal bundle $(p\times\mathrm{id})^*V$ on $P\times\tilde{X}$. We have a surjection $(p\times\mathrm{id})^*V\to(p\times\mathrm{id})^*(V|J(\tilde{X})\times\{x_1, x_2\})$. Let p_1 be the projection $P\times\tilde{X}\to P$. On P, there is a surjection $V|J(\tilde{X})\times\{x_1, x_2\}\to\theta_p(1)\to 0$. Since $p_1^*(V|J(\tilde{X})\times\{x_1, x_2\})=(p\times\mathrm{id})^*V|J(\tilde{X})\times\{x_1, x_2\}$, we get a surjection

$$\varphi\colon (\mathrm{id}\times\pi)_*(p\times\mathrm{id})^*V\to (\mathrm{id}\times\pi)_*(p\times\mathrm{id})^*V|P\times x_0\to p_1^*\theta_P(1)|P\times x_0, p'\colon P\times X\to P.$$

Since $\theta_P(1)$ is free over P, it follows that $K=\mathrm{Kernel}\,\varphi$ is flat over P. For every $g=(L, F_1(L))\in P$, we have an exact sequence

$$(S)\qquad 0\to K|g\times X\to\pi_*L\xrightarrow{\varphi_g}(\pi_*L)\otimes k(x_0)/F_1(L)\to 0.$$

Thus K is a family of torsionfree sheaves on X of degree d flat over P, so it gives a morphism h of P to the compactified Jacobian $\bar{J}(X)$ of X. $\bar{J}(X)$ contains $J(X)$, the generalised Jacobian of X as a dense open subset. We shall now show that h is a surjective morphism which is an isomorphism from $h^{-1}(J(X))$ onto $J(X)$ and fibre over each point in $\bar{J}(X)-J(X)$ consists of two points. In the sequence (S), write $L_1=K|j\times X$. It is easy to see that if $F_1(L)\neq L_{x_1}$ or L_{x_2}, then L_1 is obtained by identifying fibres L_{x_1} and L_{x_2} by an isomorphism σ whose graph is $F_1(L)$ and L_1 is locally free with $\pi^*L_1=L$. In case $F_1=L_{x_1}$, $L_1=\pi_*(L(-x_2))$ and $\pi^*L_1/\mathrm{torsion}=L(-x_2)$. If $F_1=L_{x_2}$, $L_1=\pi_*(L(-x_1))$, $\pi^*L_1/\mathrm{torsion}\approx L(-x_1)$. Thus if L_1 is locally free it comes from a unique generalised parabolic line bundle $(L=\pi^*L_1, F_1(L))$, $F_1(L)=\Gamma_\sigma$, $\sigma\colon(\pi^*L_1)_{x_1}\xrightarrow{\sim}(\pi^*L_1)_{x_2}$ canonical isomorphism. If L_1 is not locally free, the fibre over L_1 consists of two points viz.

$$\big((\pi^*L_1/\mathrm{torsion})(x_2)=L, F_1(L)=L_{x_1}\big), \big(L=(\pi^*L_1/\mathrm{torsion})(x_1), F_1(L)=L_{x_2}\big).$$

Thus P is the disjoint union of $J(X)$ and two copies of $\{J(\tilde{X})\approx\bar{J}(X)-J(X)\}$. This finishes the proof of the proposition.

2.3. *Extension of the determinant map to* $\tilde{U}^1$ *and* U^1.

Consider a generalised parabolic vector bundles $(E, F_1(E))$ on $\tilde{X}$. If we fix E, then $F_1(E)$ varies over $G(2,4)=$the grassmannian of 2-dimensional subspaces of $E_{x_1}\oplus E_{x_2}$. $G(2,4)$ is embedded as a quadric in $\mathbf{P}^5=P(\Lambda^2(E_{x_1}\oplus E_{x_2}))$. Fixing a basis (e_1, e_2) of E_{x_1} and (e_3, e_4) of E_{x_2}, a basis of $\mathbf{P}^5$ is given by $(e_i\Lambda e_j)_{i<j}$. An element of $\mathbf{P}^5$ is of the form $\sum_{i<j}P_{ij}e_i\Lambda e_j$, P_{ij} being the Plücker coordinates. Then $G_{2,4}\cap(P_{12}\neq 0, P_{34}\neq 0)\subset G_{2,4}$ is the open subset corresponding to elements $(E, F_1(E))$

in $\tilde{U}^2$. $A=G(2,4)\cap\{(P_{12}\neq 0)\cup(P_{34}\neq 0)\}$ corresponds to elements $(E, F_1(E))$ in $\tilde{U}^1\cup\tilde{U}^2$. $A^c=(P_{12}=0=P_{34})\cap G(2,4)$ corresponds to elements $(E, F_1(E))$ in $\tilde{U}^0$. The set $(P_{12}\neq 0)$ (respectively $(P_{34}\neq 0)$) can be identified with $\mathrm{Hom}(E_{x_1}, E_{x_2})$ (respectively $\mathrm{Hom}(E_{x_2}, E_{x_1})$) by identifying $\sigma\in\mathrm{Hom}(E_{x_1}, E_{x_2})$ with its graph Γ_σ. If $\sigma(e_1)=\alpha e_3+\gamma e_4$, $\sigma(e_2)=\beta e_3+\delta e_4$, then Γ_σ as an element of $\mathbf{P}^5$ has coordinates $P_{12}=1$, $P_{34}=\det\sigma$, $P_{13}=\beta$, $P_{14}=\delta$, $P_{23}=-\alpha$, $P_{24}=-\gamma$ i.e. it is a point with homogeneous coordinates $(1, \det\sigma, \beta, \delta, -\alpha, -\gamma)$ in $\mathbf{P}^5$. If σ is an isomorphism, the graph of σ^{-1} is the point $(d^{-1}, 1, \beta d^{-1}, \delta d^{-1}, -\alpha d^{-1}, -\gamma d^{-1})$, $d\equiv\det\sigma$, which is the same point as Γ_σ. Thus, for $(E, F_1(E)=\Gamma_\sigma)$ in $\tilde{U}^1\cup\tilde{U}^2$ we can define its determinant as the pair $(\det E, p(\Gamma_\sigma))$ where $p\colon \mathbf{P}^5\to\mathbf{P}^1$ is defined by the projection $(P_{ij})_{i<j}\to(P_{12}, P_{34})$ in homogeneous coordinates i.e. $\det(E, \Gamma_\sigma)$ is the generalised parabolic line bundle $(\det E, \Gamma_{\det\sigma})$. Thus we get a map from $\tilde{U}^1\cup\tilde{U}^2$ onto the variety P of generalised parabolic line bundles.

Now consider the subset of $G(2,4)$ defined by $(P_{12}(x)\neq 0)\cap(P_{34}(x)\neq 0)\cap(p(x)=\text{fixed})$. Let (x_0, y_0) be the homogeneous coordinates of $p(x)$ for an x in this set. Thus a point in this set looks like $(tx_0, ty_0, *, *, *, *)$ showing that the closure of this set in $G(2,4)$ is given by (this set)$\cup(G(2,4)\cap(P_{12}=0=P_{34})$. The subset $(P_{12}=0=P_{34})\cap G(2,4)$ corresponds to elements $(E, F_1(E))$ in $\tilde{U}^0$. Notice also that fixing the determinant of $(E, F_1(E))$ is equivalent to fixing the determinant of $F=f(E, F_1(E))$ for $(E, F_1(E))$ in $\tilde{U}^2$. We clearly have a commutative diagram

$$\begin{array}{ccc}\tilde{U}^2 & \xrightarrow{\det} & h^{-1}(J(X)) \\ {\scriptstyle f}\downarrow & & \wr\downarrow{\scriptstyle h} \\ U^2 & \xrightarrow{\det} & J(X).\end{array}$$

We now want to show that the determinant map $\tilde{U}^1\to P-h^{-1}(J(X))$ goes down to a map $U^1\to\bar{J}(X)-J(X)$.

If $F\in U^1$, any $(E, F_1(E))\in\tilde{U}^1$ mapping to F is obtained either from an extension of type

(i) $$0\to\pi^*F/\text{torsion}\to E\to k(x_1)\to 0$$

or of type

(ii) $$0\to\pi^*F/\text{torsion}\to E\to k(x_2)\to 0$$

and one has

(i)′ $$\det(E, F_1(E)) = (L = (\det\pi^*F/\text{torsion})(x_1), F_1(L) = L_{x_2})$$

or

(ii)′ $$\det(E, F_1(E)) = (L = (\det\pi^*F/\text{torsion})(x_2), F_1(L) = L_{x_1}).$$

(See the proof of proposition 1.11(3).) As seen in the proof of proposition 2.2, R.H.S. of both (i)′ and (ii)′ map into the same point in $\bar{J}(X)-J(X)$ under h, we define this point as the determinant of F. Thus we have the required commutative diagram

$$\begin{array}{ccc} \tilde{U}^1 & \xrightarrow{\det} & P-h^{-1}(J(X)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ U^1 & \xrightarrow{\det} & \bar{J}(X)-J(X). \end{array}$$

For simplicity, let $(x_1, x_2, x_3, x_4, x_5, x_6)$ denote the homogeneous coordinates in $\mathbf{P}^5$, with $x_1=P_{12}$, $x_2=P_{34}$ and let $G(2,4)$ be defined by the quadratic equation $x_1x_2+x_3x_4+x_5x_6=0$. Fix a point (y_1, y_2) in $\mathbf{P}^1$, we normalise y_1, y_2 by $y_1y_2=-1$. Let $\alpha, \beta, \gamma, \delta, \lambda, t \in k$. Define $C_{1,\lambda}(t)\in\mathbf{P}^5$ by $C_{1,\lambda}(t)=(ty_1, ty_2, \lambda t+\alpha, t+\beta, (\lambda-1)t+\gamma, -t+\delta)$. Then $C_{1,\lambda}(t)\in G(2,4)$ iff $\lambda(\beta+\delta)+\alpha-\gamma-\delta=0$, $\beta\alpha+\gamma\delta=0$, $C_{1,\lambda}(t)\in G(2,4)\cap\{(x_1\neq 0)\cap(x_2\neq 0)\}$ for $t\in k^*$ and $C_{1,\lambda}(0)\in A^c$. Hence $\{C_{1\lambda}(t)\}_{t\in k^*}$ parametrizes a family of parabolic vector bundles on $\tilde{X}$ with a fixed determinant or equivalently a family of vector bundles on X with a fixed determinant (2.4) and the limit point $C_{1,\lambda}(0)=(0,0,\alpha,\beta,\gamma,\delta)$ corresponds to an element of $\tilde{U}^0$. Define $D_{1,\lambda}(t)=(t,0,\alpha,\beta,\gamma,\delta)$, $\{D_{1,\lambda}(t)\}_{t\in k^*}$ parametrices a family of elements in $\tilde{U}^1$ with the same limit and with a fixed determinant. It is easy to see that any point $(0,0,\alpha,\beta,\gamma,\delta)$ in A^C is of the form $C_{i,\lambda}(0)$ for some i, λ where $C_{2,\lambda}(t)=(ty_1, ty_2, \lambda t+\alpha, t+\beta, t+\gamma, \delta-(\lambda y_t))$, $C_{3,\lambda}(t)=(ty_1, ty_2, \alpha, \beta, t+\gamma, t-\gamma)$, $C_{4,\lambda}(t)=(ty_1, ty_2, t+\alpha, \lambda t+\beta, t+\gamma, \delta-(\lambda-1)t)$.

A point in U_0 corresponds to a torsionfree sheaf π_*E_0, E_0 being a stable vector bundle on $\tilde{X}$. Let E be a bundle ocurring in an extension of the form $0\to E_0\to E\to k(x_1)\oplus k(x_2)\to 0$. Then $(E, F_1(E))$ with $F_1(E)=(0,0,\alpha,\beta,\gamma,\delta)\in G(2,4)$ is a point in $\tilde{U}^0$ lying over the point $[\pi_*E_0]$ in U_0. Let $L=(\det\cdot E_0)(x_1+x_2)=\det\cdot E$. Let $p\colon P\to\operatorname{Pic}\tilde{X}$ and $h\colon \mathbf{P}\to\bar{J}$ be as in proposition 2.2. Varying (y_1, y_2) over $\mathbf{P}^1$ in the above discussion, we see that $C_{i,\lambda}$'s parametrise families of bundles on X with determinant a fixed line bundle M, where M varies over $h(p^{-1}(L))\cap\operatorname{Pic}X$. Define $D_{2,\lambda}(t)=(0,t,\alpha,\beta,\gamma,\delta)$. Then $D_{1,\lambda}$ (respectively $D_{2,\lambda}$) parametrices a family of torsionfree sheaves (which are not locally free) on X with a fixed determinant $\pi_*(L(-x_2))$ (respectively $\pi_*(L(-x_1))$) belonging to $h(p^{-1}(L))$. This shows that the fibre over (π_*E_0) of the closure (in $U\times\bar{J}$) of the graph of the determinant map (which is a rational morphism) contains $h(p^{-1}(L))\approx\mathbf{P}^1$.

2.4. We now want to "globalise" the construction of 1.6. Let $\mathscr{E}\to T\times\tilde{X}$ be a family of vector bundles on $\tilde{X}$ of rank 2, degree d flat over T. Let $G(\mathscr{E})$ be the Grassmannian bundle over $T\times D$, $D=x_1+x_2$, such that $G(\mathscr{E})_t\cong G(2,(\mathscr{E}_t)_D)$, the Grassmannian of two dimensional subspaces of $\mathscr{E}|t\times D$. On $G(\mathscr{E})$, we have an exact sequence $0\to U\to\mathscr{E}|T\times D\to Q\to 0$, Q being the universal quotient bundle. Let $p\colon G(\mathscr{E})\to T\times D\to T$, $p_1\colon G(\mathscr{E})\times\tilde{X}\to G(\mathscr{E})$, $p_1'\colon G(\mathscr{E})\times X\to G(\mathscr{E})$ be the natural maps. The above sequence gives a surjection $p_1^*(\mathscr{E}|T\times D)\to p_1^*Q$ and hence $(1\times\pi)_*(p_1^*\mathscr{E}|T\times D)\to$

$(1\times\pi)_* p_1^* Q$. One has $(1\times\pi)_* p_1^* Q = p_1'^* Q$; also $(1\times\pi)_* p_1^* \mathscr{E}|T\times D \approx ((1\times\pi)_*\mathscr{E})|T\times x_0$. The restriction map $\mathscr{E}\to\mathscr{E}|T\times D$ gives a homomorphism

$$(1\times\pi)_*(p\times 1)^*\mathscr{E} \to (1\times\pi)_*(p\times 1)^*(\mathscr{E}|T\times D) = (1\times\pi)_* p_1^*(\mathscr{E}|T\times D).$$

Composition gives a homomorphism $(1\times\pi)_*(p\times 1)^*\mathscr{E}\to p_1'^* Q$. Let $\mathscr{F}$ be defined by the exact sequence

$$0 \to \mathscr{F} \to (1\times\pi)_*(p\times 1)^*\mathscr{E} \to p_1'^* Q \to 0. \tag{2.5}$$

Since π is a finite morphism and $\mathscr{E}$ is flat over T, it follows that $(1\times\pi)_*(p\times 1)^*\mathscr{E}$ is flat over $G(\mathscr{E})$. Since Q is locally free over $G(\mathscr{E})$, $p_1'^* Q$ is flat over $G(\mathscr{E})$. It follows that $\mathscr{F}$ is flat over $G(\mathscr{E})$. Thus $\mathscr{F}$ is a flat family of torsionfree sheaves of rank two, degree d on X parametrised $G(\mathscr{E})$. Let $G(\mathscr{E})_{ss}(G(\mathscr{E})_s)$ be the open subset of $G(\mathscr{E})$ corresponding to $g\in G(\mathscr{E})$ such that $\mathscr{F}_g$ is semistable (stable). Then we have a morphism $\varphi\colon G(\mathscr{E})_{ss}\to U$ mapping $G(\mathscr{E})_s$ to stable points in U.

We have

$$G(\mathscr{E}) \subset \mathbf{P}(\Lambda^2(\mathscr{E}|T\times D)) = P(\Lambda^2(\mathscr{E}|T\times x_1\oplus\mathscr{E}|T\times x_2))\cdot \Lambda^2(\mathscr{E}|T\times x_1\oplus\mathscr{E}|T\times x_2)$$

has $\Lambda^2\mathscr{E}|T\times x_1\oplus\Lambda^2\mathscr{E}|T\times x_2$ as a direct summand and hence a projection onto it. Hence we get a rational morphism $G(\mathscr{E})\to\mathbf{P}(\det\mathscr{E}|T\times x_1\oplus\det\mathscr{E}|T\times x_2)$, this is nothing but the extended determinant map of (2.3), as $\det\mathscr{E}|T\times\tilde{X}$ and $V|J(\tilde{X})\times\tilde{X}$ are locally isomorphic, V being the universal bundle on $J(\tilde{X})\times\tilde{X}$.

2.5. In the notations of 2.4, let now $T=M$, where M is the moduli space of stable vector bundles of rank two and odd degree on X and let $\mathscr{E}$ be the universal bundle on $M\times\tilde{X}$. Let L be a fixed line bundle on X and let M^0 denote the subvariety of M corresponding to bundles E with determinant π^*L. Let $G=G(\mathscr{E})_s=G(\mathscr{E})_{ss}$ (remark 1.4). Let $G_i=\varphi^{-1}(U_i)$, $i=0,1,2$, $G_2^L=\varphi^{-1}(U_2^L)$ (notations 1.1) G_2 is a fibration over M with fibre $GL(2)$ and hence is of dimension $4g-3$ and G_2^L is a closed subvariety of G_2 of $\dim\cdot 3g-3$. The restriction of φ to G_2 is an isomorphism onto an open dense subset U_2' of U_2, mapping G_2^L isomorphically onto $U_2'^L$ contained in U_2^L; $U_2'^L$ being open and dense in U_2^L. Using 2.3, it follows that the closure of G_2^L in $G=\overline{G_2^L}=\{(E, F_1(E))|E\in M^0, F_1(E)=k_1\oplus k_2,\ k_i\subset E_{x_i},\ (E, F_1(E))$ parabolic stable i.e. E has no line subbundles L' of degree $(\mu(E)-\frac{1}{2})$ such that $L'_{x_1}\oplus L'_{x_2}=k_1\oplus k_2\}$ and $\varphi(\overline{G_2^L})=\{\pi_*(E_0)|E_0$ (stable) bundle given by an extension of the form

$$0\to E_0\to E\to E_{x_1}\oplus E_{x_2}/F_1(E)\to 0,\ (E, F_1(E))\in\overline{G_2^L}\}.$$

Note that $\det E_0=(\pi^*L)(-x_1-x_2)$. We claim that any stable bundle E_0 can be obtained by an extension of the above form. Now, the extensions of the above form (i.e. $0\to E_0\to E\to k(x_1)\oplus k(x_2)\to 0$) are parametrised by $(E_0\otimes K_{\tilde{X}}^{-1})_{x_1}\oplus(E_0\otimes K_{\tilde{X}}^{-1})_{x_2}\approx$

$(E_0)_{x_1}\oplus(E_0)_{x_2}$ and given $k_i\subset(E_0)_{x_i}$ one dimensional subspaces, there is a (unique) extension such that Ker $((E_0)_{x_i}\to E_{x_i})=k_i, i=1,2$. Choose k_1, k_2, such that $k_1\oplus k_2\neq L_{x_1}\oplus L_{x_2}$ for any line subbundle L' of E_0 of degree $\mu(E)-\frac{3}{2}$ (remark 1.14). Then E obtained for such a choice is stable and parabolic stable. Thus $\varphi(\overline{G_2^L})=\{\pi_* E_0|E_0$ stable bundle on $\tilde{X}$ with determinant $(\pi^*L)(-x_1-x_2\}$.

2.6. *The case* $g(\tilde{X})=1$. In this case M^0=a point corresponding to a stable bundle E. Then $(E, F_1(E))$, $F_1(E)=k_1\oplus k_2$, all give the same bundle E_0 as there is a unique stable vector bundle E_0 of rank 2 and fixed determinant $(\pi^*L)(-x_1-x_2)$ on $\tilde{X}$. Moreover, $(E, F_1(E))$ with $E=N\oplus(\pi^*L\otimes N^{-1})$, $F_1(E)=k_1\oplus k_2$, degree $N=\frac{1}{2}$ (degree $L+1$) also give the same E_0 for the same reason.

Lemma 2.7. *Let X be an irreducible complete curve with the only singularity a single node at x_0. Let R be a discrete valuation ring, $T=\operatorname{spec} R$, T_0 the closed point of T. Let $F\to X\times T$ be a family of torsionfree sheaves on X, flat over T, with the generic member locally free and $F|x_0\times T_0\approx a\theta_{x_0}\oplus bm_0$, $a>0$, m_0 being the maximum ideal of θ_{x_0}. Assume that $H^0(F)$ generates F. Then one can find an exact sequence $0\to\theta\to F\to G\to 0$, where G is a family of torsionfree sheaves on X flat over T and G is a torsionfree sheaf.*

Proof. Write $F_{(x_0,T_0)}=\theta_{x_0}\oplus M$, M is the direct sum $(a-1)\theta_{x_0}\oplus bm_0$. Since $H^0(F)$ generates $F_{(x_0,T_0)}$, there exists e_1 in $H^0(F)$ such that $e_1(x_0, T_0)=(1,0)$, $1\in\theta_{x_0}$. Define $V=\{s\in H^0(F)|s=\sum c_ie_i,\ c_1\neq 0\}$. Then for any s in the open set V, s maps into θ_{x_0} at x_0. Since $F|(X-x_0)\times T$ is locally free, there exists an open set $W\subset H^0(F)$ such that for s in W, the map $\theta|(X-x)\times T\xrightarrow{s}F|(X-x_0)\times T$ is injective. Then for any s in $V\cap W$, we have an exact sequence

$$(I)\qquad 0\to\theta\to F\to G\to 0.$$

We shall now check that G is torsionfree and is flat over T. Since R is a discrete valuation ring, to check that G is flat over T, it suffices to check that G is flat over T_0. Tensorising the sequence (I) by θ_{T_0}, we have $0\to\operatorname{Tor}_1(G,\theta_{T_0})\to\theta_{T_0}\to F|T_0\to G|T_0\to 0$. Since, by our construction, $\theta\to F|T_0$, is an injection, it follows that $\operatorname{Tor}_1(G,\theta_{T_0})=0$ i.e. G is flat over T_0.

Since G is flat over T, it has no T-torsion. So G can have only X-torsion, say G'; so that G/G' is torsionfree. Since G and G/G' are flat over T, it follows that G' if flat over T. This implies that $0\to G'|T_0\to G|T_0\to(G/G')|T_0\to 0$ is exact. By our choice of s, G/T_0 is torsionfree, so that $G'|T_0=0$ and hence $G'=0$. Thus G is torsionfree.

Remark 2.8. If F is of rank two, we can define the determinant of $F|X\times T_0$ as $G|X\times T_0$.

In the general case i.e. rank $F=n$, $F_{x_0, T_0} \approx (n-1)\theta_{x_0} \oplus m_0$ write $G=G_1$. Applying the above lemma to G_1, we get a torsionfree quotient G_2 flat over T_0. Repeating the process, we get a torsionfree rank one sheaf G_{n-1} flat over T. We can define the determinant of $F|X\times T_0$ as $G_{n-1}|X\times T_0$.

3. Generalisations and construction of the moduli space

The generalised parabolic bundles defined before (definitions 1.2, 1.3 and 2.1) are special cases of the more general definition below (3.1). A good generalisation of the concept of a parabolic structure at a point seems to be a parabolic structure on a divisor. On singular curves one seems to get naturally vector bundles E with flags on $E|D$, D being a Cartier divisor concentrated at the singular point. Definition 1.2 is obtained from 3.1 by taking $D=x_1+x_2$ and weights $(\alpha_1, \alpha_2)=(0, 1)$.

Definition 3.1. Let E be a vector bundle on an irreducible nonsingular curve X over an algebraically closed base field k.

A generalised parabolic structure σ on E over a Cartier divisor D consists of

(1) a flag $\mathscr{F}$ of vector subspaces of $E|D$, $\mathscr{F}$: $F_0=E|D\supset F_1\supset F_2\supset\ldots\supset F_r=0$, where $E_{|D}:=H^0(E\otimes \mathcal{O}_D)$

(2) real numbers $\alpha_1, \ldots, \alpha_r$, $(0\leqq\alpha_1<\alpha_2<\ldots<\alpha_r<1)$ called weights.

Let $m_i=\dim F_{i-1}/F_i$, $i=1, \ldots, r$. Define

$$wt\cdot E|D = \sum_{i=1}^r m_i\alpha_i.$$

If E has generalised parabolic structure over finitely many divisors D_j, we call E with this structure a generalised parabolic vector bundle. Define $wt\cdot E=\sum_j wt\cdot E|D_j$, parabolic degree of $E=$degree of $E+wt\cdot E$.

Definition 3.2. Every subbundle K of E gets a natural structure of a generalised parabolic bundle. The induced flag is given by $\mathscr{F}(K)=K|D\supseteqq(F_1\cap K|D)\supseteqq\ldots\supseteqq F_r=0$, if β_j is the weight associated to $F_j\cap K|D$, then $\beta_j:=\alpha_j$ where F_j is the smallest subspace in $\mathscr{F}$ containing $F_j\cap K|D$. By a subbundle of a generalised parabolic bundle E we will always mean a subbundle with this induced parabolic structure.

Definition 3.3. A generalised parabolic vector bundle E is semistable (respectively stable) if for every (respectively proper) subbundle K of E, parabolic degree of K/rank of $K\leqq$(resp. $<$) parabolic degree of E/rank of E.

Definition 3.4. ***Induced parabolic stucture on a quotient bundle.*** Let p: $E\to Q$ be a quotient of E. The parabolic structure on E over D induces one on Q as follows.

Let $\mathscr{F}=\{F_i(E)\}$ be the flag on $E|D$ with weights $\{\alpha_i(E)\}$, $i\in I$. Let $\bar{p}=p|D$. Then $\bar{p}(\mathscr{F})$ induces a flag $\bar{\mathscr{F}}$ on $Q|D$, $\bar{\mathscr{F}}=\{F_j(Q)\}$, $j\in J\subseteqq I$. The weights $\{\alpha_j(Q)\}$ for this flag are determined as follows. Given $F_j(Q)$, there exists $F_i(E)$ such that $\bar{p}(F_i(E))=(F_j(Q))$, take i_0 largest such i and define $\alpha_j(Q)=\alpha_{i_0}(E)$.

Definition 3.5. A generalised parabolic vector bundle E is semistable (respectively stable) if for every nonzero quotient bundle Q of E, parabolic degree of $E/\text{rank}\, E\leqq$ (respectively $<$) parabolic degree of $Q/\text{rank}\, Q$.

Remark. It is easy to see that Definitions 3.3 and 3.5 are equivalent.

Definition 3.6. Let $i=1,2$ and let E_i be a generalised parabolic bundle with parabolic structure over D with flag $\{F_j(E_i)\}$ and weights $\{\alpha_j(E_i)\}$. A morphism of generalised parabolic bundles is a homomorphism $f\colon E_1\to E_2$ of vector bundles such that for all j, $\bar{f}(F_j(E_1))\subset F_{l+1}(E_2)$ whenever $\alpha_j(E_1)>\alpha_l(E_2)$, where $\bar{f}=f|D$.

Lemma 3.7. *Let E be a semistable (resp. stable) generalised parabolic bundle. If* $\text{par}\,\mu(E)=$*parabolic degree of* $E/\text{rank}\, E>$(resp. $\geqq$)$2g-1$, *then* $(H^1(E))=0$.

Proof. Suppose that $H^1(E)\neq 0$. By Serre' duality, this implies that there exists a nonzero homomorphism $f\colon E\to K$, K being the canonical line bundle. Then one has

$$\text{par}\,\mu(E)\leqq\text{par}\,\mu(K)=2g-2+\omega t\,K\leqq 2g-1.$$

if E is semistable (resp. $<$ for E stable). Hence if E is semistable (or stable) with par $\mu>$(or $\geqq$)$2g-1$ then $f=0$, i.e. $H^1(E)=0$.

Lemma 3.8. *Let $f\colon E_1\to E_2$ be a morphism of semistable generalised parabolic bundles (D fixed) of same rank and same parabolic degree. Then f is of constant rank. Further, if one of E_1 or E_2 is stable, then either $\alpha=0$ or α is an isomorphism.*

Proof. The morphism f factors through a generic isomorphism h as follows.

$$\begin{tikzcd} 0 \arrow[r] & K_1 \arrow[r] & E_1 \arrow[r] \arrow[d, "f"] & I_1 \arrow[r] \arrow[d, "h"] & 0 \\ 0 & I_2 \arrow[l] & E_2 \arrow[l] & K_2 \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

Let $\mu=\text{par}\,\mu(E_1)=\text{par}\,\mu(E_2)$. By semistability of E_1, E_2 one has $\mu=\text{par}\,\mu(E_1)\leqq$ par $\mu(I_1)$, par $\mu(K_2)\leqq\mu$. Since h is a generic isomorphism $\deg\cdot I_1\leqq\deg K_2$, also $wt\,I_1\leqq wt\,K_2$, hence par $\mu(K_2)$. Thus $\mu\leqq\text{par}\,\mu(I_1)\leqq\text{par}\,\mu(I_1)\leqq\text{par}\,\mu(K_2)\leqq\mu$, i.e. par $\mu(I_1)=\text{par}\,\mu(K_2)=\mu$. Thus parabolic degrees of I_1 and K_2 are same, it follows that degree $I_1=$degree K_2, $wt\,I_1=wt\,K_2$ and so h is an isomorphism i.e. f is of constant rank. The last assertions of the lemma are now clear.

Corollary 3.9. *If E is a stable generalised parabolic vector bundle, then any morphism of E into itself is a scalar.*

Proof. Lemma 3.8 shows that any nonzero morphism f of E into itself is an isomorphism. Let $x \in X$ and c be an eigenvalue of f_x. Then the morphism $f - cId$ is not an isomorphism and hence must be zero.

Proposition 3.10. *The category S of all semistable generalised parabolic bundles E on X with parabolic structure on a divisor D and with fixed* par $\mu = \mu_0$ *is an abelian category. The simple objects in this category are the stable generalised parabolic bundles. By Jordan—Hölder theorem, for $E \in S$, there exists a filtration in S*

$$E = E_n \supset E_{n-1} \supset \dots \supset E_0 = 0$$

such that E_i/E_{i-1} is a stable generalised parabolic bundle with par $\mu = \mu_0$ *for all i and* $\mathrm{gr}\, E = \bigoplus_i E_i/E_{i-1}$ *is unique upto isomorphism.*

Proof. This follows from 3.8 and 3.9.

Definition 3.11. We define E_1, E_2 in S to be equivalent if $\mathrm{gr}\, E_1 \approx \mathrm{gr}\, E_2$.

Theorem 1. *Let X be an irreducible nonsingular projective curve over an algebraically closed field. Then there exists a course moduli space M for equivalence classes of semistable generalised parabolic bundles E of rank k on X with fixed degree and parabolic structure given by* $\deg D = 2$, *weights* $(\alpha_1, \alpha_2) \equiv (0, \alpha)$ *and* $\mathscr{F}: F_0(E) = E|_D \supset F_1(E) \supset 0$. *The space M is a normal projective variety. If rank and degree of E are coprime, α is close to 1 and* $\dim F_1(E) = k$ *then M is nonsingular. One has* $\dim M = k^2(g-1) + 1 + \dim F$, *$F$ being flag variety of type $\mathscr{F}$.*

The proof of this theorem is on similar lines as that of the main theorem in [V]. The construction uses geometric invariant theory, the choice of weights and degree of D corresponds to the choice of a polarisation. This choice is a bit tricky. A choice similar to the one in [SM] [V] fails for degree $D > 1$, so we have to look for a new candidate. This was the main difficulty in the construction below. Note that unlike in [SM], [V] we do not assume here that parabolic degree of $E = 0$.

Let S denote the set of all semistable generalised parabolic bundles E of the type specified in the statement of the theorem. Let b denote the fixed parabolic degree of $E \in S$, without loss of generality, may assume $b \leqq k$. Then S is bounded, there exists m_0 such that for $m \geqq m_0$, one has $H^1(E(m)) = 0$ and the canonical map $H^0(E(m)) \to H^0(E(m)/D)$ is a surjection. By arguments similar to those on p. 226, [SM] we can choose an integer $m \gg g$, $g =$ genus of X, such that $H^1(F(m)) = 0$ and $H^0(F(m)) \to H^0(F(m) \otimes \mathcal{O}_D)$ is surjective for $F \in S$ or $F \subset E$, E in S and parabolic degree of $F > (b - (g + 2\alpha)k)$. Let P be the Hilbert polynomial of E in S and let $n = \dim \cdot H^0(E(m))$. Denote by Q the Quot scheme i.e. the Hilbert scheme of coherent sheaves on X which are quotients of $\mathcal{O}_X^n$ and have Hilbert polynomial P. Let U denote the universal family on $Q \times X$ and R denote the subscheme $\{q \in Q | H^1(U_q)=0, H^0(U_q) \approx \mathcal{O}_X^n, U_q$ is locally free and generically generated by global sections$\}$. R is a nonsingular variety and contains the subset determined by $E(m)$, $E \in S$ by our choice of m. Let $V=(p_1)_*(U|R \times D)$, p_1: $R \times D \to R$. Let $G(V)$ be the flag bundle over R of the type determined by the parabolic structure and let $\tilde{R}$ be the total space of $G(V)$. It is easy to see that $\tilde{R}$ has the local universal property for generalised parabolic bundles. Let the subsets of $\tilde{R}$ corresponding to semistable (respectively stable) generalised parabolic bundles be denoted by $\tilde{R}^{SS}(\tilde{R}^S)$. The group $SL(n)$ acts on R, $\tilde{R}^{SS}$ and $\tilde{R}^S$ via its action on $\mathcal{O}_X^n$. We want to give an affine injective $SL(n)$-equivariant morphism from $\tilde{R}$ to a projective variety Y with $SL(n)$-action such that the geometric invariant theoretic quotient $Y/SL(n)$ is known to exist.

For a while, let us forget about the parabolic structure. Following Gieseker, we define a 'good pair' (F, φ) to be a flat family $F \to T \times X$ of vector bundles on X such that F_t is generated by its global sections at the generic point of $t \times X$ and φ: $\mathcal{O}_X^n \to p_*(F)$ is an isomorphism. Let $c = \text{degree}\, E(m)$, $E \in S$, $A = \text{Pic}^c(X)$, g: $X \times A \to A$ projection and M the Poincare' bundle on $X \times A$. Let $Z = \mathbf{P}(\text{Hom}(\Lambda^k \mathcal{O}_A^n, g_* M)^*)$. Given a good pair (F, φ) one gets a morphism $T(F, \varphi)$: $T \to Z$. For $t \in T$, $T(F, \varphi)(t)$ is the composite $\Lambda^k K^n \to \Lambda^k H^0(F_t) \xrightarrow{\psi} H^0(\Lambda^k F_t)$, where the first map is $\Lambda^k \varphi$ and the second map ψ is the natural map $\psi(s_1 \Lambda \ldots \Lambda s_k) = s$, where $s(x) = s_1(x) \Lambda \ldots \Lambda s_k(x)$. $SL(n)$ acts on Z preserving the fibres over A.

If, in addition, F is a family of generalised parabolic vector bundles the flag on $F_t|D$ induces, via φ, a flag on $K^n = H^0(F_t)$

$$K^n = F_0(H^0(F_t)) \supset F_1(H^0(F_t)) \supset F_2(H^0(F_t)),$$

$F_2(H^0(F_t))$=kernel of e: $H^0(F_t) \to F_t|D$ and $F_1(H^0(F_t)) = e^{-1}(F_1(F_t))$. Hence we have a morphism f from T into the flag variety G of flags in K^n. Thus the good pair (F, φ) determines a morphism $\tilde{T}(f, \varphi)$: $T \to Z \times G$, $\tilde{T}(F, \varphi) = T(f, \varphi) \times f$. Let T: $\tilde{R} \to Z \times G$ be the induced morphism. T maps $\tilde{R}^{SS}$ into $Gr = \prod G_{n, f_i}$ where $G_{n,i}$ denotes the Grassmannian of f_i-dimensional subspaces of K^n, $f_i = \dim F_i(H^0(E))$, $i=0, 1, 2$. On $Z \times Gr$ we take the polarisation $L^{\otimes \alpha k} \otimes \mathcal{O}_Z(k(m+1-2\alpha-g)+b))$, where L is the generator of $\text{Pic}(G_{n,1})$, b=parabolic degree of E in S. For this polarisation, a point $(\tau, (F_i))$ in $Z \times Gr$ is semistable (or stable) if and only if for any subspace $W \subset V$, $K^n = V$, one has

$$\begin{aligned}\sigma_W &= [k(m-1-g)+b](d \dim V - k \dim W) \\ &\quad + k\alpha\left[\dim W \dim F_1(V) - \dim V \dim(W \cap F_1(V))\right] \geqq 0 \quad (\text{or} > 0)\end{aligned}$$

where d is the maximum of the cordinalities of τ-independent subsets of W. Let $(Z \times G_r)^{SS}$ (or $(Z \times G_r)^S$) denote the set of semistable (or stable) points in $Z \times Gr$.

Proposition 3.12.

(a) $q \in \tilde{R}^{ss} \Rightarrow T(q) \in (Z \times Gr)^{ss}$

(b) $q \in \tilde{R}^{s} \Rightarrow T(q) \in (Z \times Gr)^{s}$

(c) $q \in \tilde{R}, T(q) \in Z \times Gr, q \notin \tilde{R}^{ss} \Rightarrow T(q) \notin (Z \times Gr)^{ss}$

(d) $q \in \tilde{R}^{ss} - \tilde{R}^{s} \Rightarrow T(q) \notin (Z \times Gr)^{s}$

Proof. For $F \subset E$, define

$$\chi_F = [k(m+1-g)+b]\left[rk\, F \cdot h^0(E(m)) - rk\, E \cdot h^0(F(m))\right] + k\left[h^0(F(m)) \cdot wt\, E - h^0(E(m))\, wt\, F\right].$$

We first make a few observations.

(1) For E with $h^1(E)=0$,

$$\chi_F = n\,(db - k \text{ parabolic degree } F) - nkh^1(F(m))$$

where $d = \text{rank } F$, $n = h^0(E(m))$.

Proof. Rearranging the terms one has

$$\begin{aligned}\chi_F &= h^0(E(m))\left[(k(m+1-g)+b)d - k\, wt\, F\right] \\ &\quad kh^0(F(m))(wt\, E - k(m+1-g) - b) \\ &= h^0(E(m))\left[k(m+1-g)d + bd - k\, wt\, F - kh^0(F(m))\right],\end{aligned}$$

since $\text{wt}\, E - k(m+1-g) - b = h^0(E(m))$ by Riemann—Roch theorem. Similarly $h^0(F(m)) - h^1(F(m)) = \text{parabolic degree } F - \text{wt } F + d(m+1-g)$, hence one gets

$$\chi_F = n\left[bd - k \text{ parabolic deg} \cdot F - kh^1 F((m))\right].$$

(2) If $h^1(E(m)) = 0 = h^1(F)(m))$, par $\mu = \frac{\text{parabolic degree}}{\text{rank}}$, *then*

$$\chi_F = n\, dk\,(\text{par}\,\mu(E) - \text{par}\,\mu(F)).$$

Proof. Obvious.

(3) If $W = H^0(F(m))$, $V = H^0(E(m))$, $H^0(F(m)) \to F(m) \otimes \mathcal{O}_D$, $H^0(E(m)) \to E(m) \otimes \mathcal{O}_D$ *are surjections and* $h^1(F(m)) = 0 = h^1(E(m))$, *then*

$$\sigma_W = \chi_F.$$

This follows by straightforward computation. We now come to the proof of the proposition. Assertions (c) and (d) follow exactly as in the proofs of proposition 2(c), (d) in [V] using (2) and (3) above.

Proof of (a) *and* (b). Let E be a generalised parabolic semistable (or stable) bundle ($E \in S$). Let W be a subspace of V and let $F(m)$ be the subbundle of $E(m)$ generically generated by W.

Case (i). If W satisfies the conditions of (3) above, we have $\sigma_W = \chi_F \geqq 0 (>0)$ if E is semistable (stable) as a generalised parabolic bundle.

Case (ii). ***Parabolic degree*** $F > b-(g+2\alpha)k$. By our choice of m, $H^1(F(m))=0$ and $H^0(F(m)) \to F(m) \otimes \mathcal{O}_D$ is surjective. Let $W' = H^0(F(m))$. If $W'=W$, we are through by case (i); so may assume $W' \supsetneqq W$. By (2) above, $\chi_F \geqq 0 (>0)$ if E is semistable (stable) as a generalised parabolic bundle. Therefore, it suffices to show that $\sigma_W - \chi_F \geqq 0$. It is easy to see that

$$\begin{aligned}\tfrac{1}{k}(\sigma_W - \chi_F) &= k(\dim W' - \dim W)\big[(k(m+1-g-2\alpha)+b) - \alpha \dim F_1(V)\big]\\ &\quad + \alpha \dim V\big(\dim W' \cap F_1(V) - \dim W \cap F_1(V)\big) \geqq 0\end{aligned}$$

as, by Riemann—Roch theorem, the term in the square bracket is $(1-\alpha)\dim V$ while the terms in round brackets are nonnegative.

Case (iii). ***Parabolic degree*** $F \leqq b-(g+2\alpha)k$. Let $W' = H^0(F(m))$, then $W' \supseteqq W$. Regrouping terms and after simplifications one gets $\sigma_W - \chi_F \geqq -2\alpha nkd$ as follows.

$$\begin{aligned}\sigma_W - \chi_F &= k \dim V\big(-2d\alpha + wt\, F - \alpha \dim F_1(V) \cap W\big)\\ &\quad + k(\dim W' - \dim W)\big(k(m+1-2\alpha-g)+b-\alpha F_1(V)\big)\\ &\quad + k \dim W'(2k\alpha - wt\, E + \alpha \dim F_1(V)).\end{aligned}$$

Using the fact that $2k\alpha - \mathrm{wt}\, E + \alpha \dim F_1(V) = \alpha \dim V$, we have

$$\sigma_W - \chi_F = k \dim V\big(-2d\alpha + wt\, F + \alpha \dim W' - \alpha \dim F_1(V) \cap W\big) \geqq -2k\alpha\, dn,$$

$$\text{since} \quad wt\, F \geqq 0 \quad \text{and} \quad \dim W' - \dim F_1(V) \cap W \geqq 0.$$

Now, since $F(m)$ is generically generated by sections, one has $h^0(F(m)) \leqq \deg F(m) + d$ or equivalently, $-h^1(F(m)) \geqq -gd$. By (1) above

$$\chi_F = nbd - nk\, h^1(F(m)) - nk \operatorname{par\,deg}(F) \geqq n\,db - ng\,dk - nk \operatorname{par\,deg}(F).$$

If $\operatorname{par\,deg}(F) \leqq b-(g+2\alpha)k$, we have

$$\sigma_W = (\sigma_W - \chi_F) + \chi_F \geqq n\big[((g+2\alpha)k-b)(k-d)\big] \geqq 0.$$

Thus the proof of the proposition is completed.

Proposition 3.13. *The morphism* $T\colon \tilde{R}^{ss} \to (Z \times Gr)^{ss}$ *is proper and injective.*

Proof. The properness of T can be proved exactly as in proposition 3 [V]. The injectivity of T follows from the fact that $\tilde{R}$ is a bundle over R with fibre flag variety corresponding to the parabolic structure and the morphism $T: \tilde{R}^{ss} \to Z$ is injective (lemma 4.3 [G 2]).

We are now in a position to complete the proof of the theorem. Since a proper injective morphism is affine, T is an affine morphism. Since the existence of a good quotient of $(Z \times Gr)^{ss}$ modulo $SL(n)$ is well-known, the existence of a good quotient M of $\tilde{R}^{ss}$ modulo $SL(n)$ follows as T is an affine morphism. Since $\tilde{R}$ is a nonsingular projective variety of dimension $k^2(g-1)+1+n^2-1+\dim F$, M is a normal projective variety of dimension $k^2(g-1)+1+\dim F$.

If rank E and degree of E are coprime and $F_1(E)$ has dimension equal to rank of E, then E is parabolic semistable if and only if E is parabolic stable i.e. $\tilde{R}^{ss} = \tilde{R}^{s}$. Also, by corollary 3.19 if E is stable then the only automorphisms (keeping the generalised parabolic structure invariant) of the generalised parabolic bundle E are scalars. Hence it follows that in this case there exists a nonsingular geometric quotient M of $\tilde{R}^{ss} = \tilde{R}^{s}$.

Lemma 3.14. *Let C be a nonsingular curve, $\mathscr{E} \to C \times X$ a flat family of generalised parabolic vector bundles in S. Let P be a point in C and $\mathscr{E}_q \approx E = \mathrm{gr}\, E$ for all $q \neq P$ in C. Then $\mathscr{E}_P \cong E$.*

Proof. This follows as in lemma 4.7 [G 2] using lemma 3.8.

Proposition 3.15. *Let h be the canonical morphism from $\tilde{R}^{ss}$ onto M. Let $\mathscr{E}$ denote the pull back to $\tilde{R}^{ss}$ of the universal family U on $R \times X$. Then for p, q in $\tilde{R}^{ss}$, $h(p) = h(q)$ if and only if $\mathrm{gr}\, \mathscr{E}_p = \mathrm{gr}\, \mathscr{E}_q$.*

Proof. By construction $h(p) = h(q)$ if and only if closures of $SL(n)-$orbits of p and q intersect. Lemma 3.14 implies that $SL(n)-$orbit of $E = \mathrm{gr}\, E$ is closed. If $E \cong \mathrm{gr}\, E$, then $\mathrm{gr}\, E$ is in the closure of the orbit of E. Since E is a successive extension of stable generalised parabolic bundles (proposition 3.10) there exists a family $\{\mathscr{E}_t\}$ with $\mathscr{E}_t \approx E$ for $t \neq 0$, $t \in \mathrm{A}^1$ and $\mathscr{E}_0 \approx \mathrm{gr}\, E$. Thus, if [E] denotes a point in $\tilde{R}^{ss}$ corresponding to a generalised parabolic bundle E, then $h([E]) = h([\mathrm{gr}\, E])$. If $p, q \in \tilde{R}^{ss}$ are such that $\mathrm{gr}\, \mathscr{E}_p \approx \mathrm{gr}\, \mathscr{E}_q$, then $h(p) \equiv h([\mathscr{E}_p]) = h(\mathrm{gr}\, [\mathscr{E}_p]) = h(\mathrm{gr}\, (\mathscr{E}_q)) = h([\mathscr{E}_q]) \equiv h(q)$ *i.e.* $h(p) = h(q)$. Conversely, $h(p) = h(q) \Rightarrow h([\mathrm{gr}\, \mathscr{E}_p]) = h([\mathrm{gr}\, \mathscr{E}_q])$. Since $SL(n)$-orbit of any $\mathrm{gr}\, E$ is closed, this implies that $\mathrm{gr}\, \mathscr{E}_p \approx \mathrm{gr}\, \mathscr{E}_q$.

Proposition 3.16. *If rank and degree are coprime,* $\dim F_1(E) = \mathrm{rank}\, E$ (degree $D=2$, *and weights are* $(0, \alpha)$) *then the moduli space M of stable generalised parabolic bundles* (*theorem* 1) *is a fine moduli space.*

Proof. The proof is exactly as in § 5, Chapter 5 of [N], so we only indicate the necessary modifications. In lemma 5.10 [N], $\mathrm{Hom}\,(E_1, E_2)$ has to be replaced by $\mathrm{Mor}\,(E_1, E_2)$ Mor denoting homomorphisms of parabolic bundles, and one uses lemma 3.8 and corollary 3.9 to prove that if E_1, E_2 are two families of stable generalised parabolic bundles as above, with $(E_1)_s \cong (E_2)_s$ $\forall s \in S$ then there exists a line bundle L on S such that $E_2 \cong E_1 \otimes p_S^* L$, in fact one takes $L = (p_s)_* \operatorname{Mor}(E_1, E_2)$. It remains to prove the existence of a universal family on $M \times X$. The universal family $\mathscr{E} \to \tilde{R}^{ss} \times X$ has a $GL(n)$ action, but no $PGL(n)$-action as the matrix $\lambda\,\mathrm{Id}$ acts on it by scalar λ. As in lemma 5.11 [N], if rank and degree are coprime, one can find a line bundle L on $\tilde{R}^{ss}$ such that $\lambda\,\mathrm{Id}$ acts on it by scalar λ^{-1}. Then $PGL(n)$ action on $\tilde{R}^{ss}$ lifts to $\mathscr{E} \otimes L$ and the quotient gives required universal bundle on $M \times X$. We need lemma 3.7 here to construct L. This completes the sketch of the proof of the proposition.

3.17. Henceforth we restrict ourselves to semistable generalised parabolic bundles E of rank 2, degree d, with parabolic structure over $D = x_1 + x_2$ given by $E|D \supset F_1(E) \supset 0$, $\dim \cdot F_1(E) = 2$, and weights $(\alpha_1, \alpha_2) = (0, \alpha)$ α near 1. The moduli space M of equivalence classes (definition 3.11) of such bundles is a normal projective variety which is nonsingular if d is odd. Let p_1 and p_2 denote the projections from $F_1(E)$ to E_{x_1} and E_{x_2} respectively. Let M_2 be the open subset of M corresponding to generalised parabolic bundles such that p_1 and p_2 are both isomorphisms. Let M_1 be the subset of M defined by the condition that only one of p_1 and p_2 is an isomorphism and the other is of rank one. Let M_0 be the subset of M defined by the condition that either p_1 and p_2 are both of rank one or $p_1 = 0$ or $p_2 = 0$. Clearly, M is the disjoint union of M_a, $a = 0, 1, 2$. We can now sum up the main results of sections 1 and 2 (particularly 1.11, 2.3, 2.4) as follows.

Theorem 2. *Let X be an irreducible projective curve defined over an algebraically closed field with only one node x_0 as a singularity. Let $\pi\colon \tilde{X} \to X$ be its normalisation. Let M be the moduli space of bundles on $\tilde{X}$ as in theorem* 1 *with $D = \pi^{-1}(x_0)$. Let U be the moduli space of semistable torsionfree sheaves of* rank 2, degree d *on X, $U = \bigcup_{a=0}^{2} U_a$ where $U_a = \{F \mid F_{x_0} \approx a\mathcal{O}_{x_0} \oplus (2-a) m_{x_0}\}$. Then one has the following.*

(I) *There exists a surjective morphism $f\colon M \to U$ such that $f^{-1}(U_a) = M_a$, $a = 0, 1, 2$ and the restriction of f gives an isomorphism of M_2 onto U_2.*

(II) *Let $\bar{J}$ be the compactified Jacobian of X and let P be its desingularisation (proposition 2.2). Then there exist morphisms φ, ψ extending the determinant morphisms such that the diagram*

$$\begin{array}{ccc} M_1 \cup M_2 & \xrightarrow{\varphi} & P \\ \downarrow & & \downarrow \\ U_1 \cup U_2 & \xrightarrow{\psi} & \bar{J} \end{array}$$

commutes.

Remark 3.18. (a) For α near 1, stability for weights $(0, 1) \Rightarrow$ stability for weights $(0, \alpha) \Rightarrow$ semistability for $(0, \alpha) \Rightarrow$ semistability for $(0, 1)$.

(b) In general, when X has more than one node, say y_i, $i=1, \ldots, m$. M in the above theorem will be replaced by the moduli space M of equivalence classes of semistable generalised parabolic bundles of rank 2, degree d with parabolic structure over $D_i=\pi^{-1}(y_i)=x_{i,1}+x_{i,2}$ given by $E|D_i=E_{x_{i},1}\oplus E_{x_{i},2}\supset F_1^i(E)\supset 0$, where $\dim F_1^i(E)=2$ and weights $(\alpha_{i,1}, \alpha_{i,2})=(0, \alpha)$. One still has semistable=stable in this case if d is odd. M is the disjoint union of $M_{i,a}$, $i=1, \ldots, m$; $a=0, 1, 2$. M_{ia} is defined by the conditions on p_1, p_2 at D_i as in 3.17. One also has $U=\bigcup_{i,a} U_{ia}$ where $U_{i,a}=\{F|F_{y_i}\approx a\mathcal{O}_{y_i}\oplus(2-a)m_{y_i}\}$, $f^{-1}(U_{ia})=M_{ia}$ and f: $\bigcup_i M_{i2}\to\bigcup_i U_{i2}$ is an isomorphism.

Remark 3.19. If d is odd, then M (in 3.17) is a desingularisation of U.

Remark 3.20. Let U_2^L be the subset of U_2 corresponding to vector bundles on X with fixed determinant L and $\overline{U_2^L}$ its closure in U. Let $M_2^L=f^{-1}(U_2^L)$. Clearly, $f(\overline{M_2^L})\subset(f(M_2^L))^-$. Now, $f(M_2^L)\subset f(\overline{M}_2^L)$ and f being proper $f(\overline{M}_2^L)$ is closed, it follows that $(f(M_2^L))^-\subset f(\overline{M}_2^L)$. Thus $f(\overline{M}_2^L)=(f(M_2^L))^-=\overline{U}_2^L$. Hence to find $\overline{U}_2^L$, suffices to determine the closure of its isomorphic copy in M. The considerations in 2.3 and Part (II) of theorem 2 show that $\overline{M_2^L}\cap M_1=\phi$ i.e. $\overline{U}_2^L\cap U_1=\phi$ and $\overline{U}_2^L$ contains all points corresponding to torsionfree sheaves of the form $\pi_*(E_0)$, where E_0 is a stable rank two vector bundle on $\tilde{X}$ with $\det E_0\approx(\pi^*L)(-x_1-x_2)$. $\overline{U}_2^L-U_2^L$ consists of only such sheaves, and in general when X has many nodes, $\overline{U}_2^L-U_2^L$ consists of points corresponding to direct images on X of stable vector bundles with suitable determinants on partial normalisations of X.

4. Generalisation to rank n

4.1. It is possible to generalise our results to rank n sheaves. We consider generalised parabolic vector bundles (E, σ) on $\tilde{X}$, E of rank n, degree d and σ is given by $D=x_1+x_2, \mathscr{F}$: $F_0=E|D\supset F_1(E)\supset 0$, $\dim F_1(E)=n$, $(\alpha_1, \alpha_2)=(0, \alpha)$ $\alpha\leqq 1$. (Definition 3.1.) To (E, σ), we associate a torsionfree sheaf F of rank n and degree d on X defined by

$$0 \to F \to \pi_*E \to \pi_*(E)\otimes k(x_0)/F_1(E) \to 0.$$

Proposition 4.2. (a) *If F is a stable torsionfree sheaf then (E, σ) is a stable generalised parabolic bundle (with cots* (0, 1)*).*

(b) *Converse of* (a) *holds.*

(c) *Statements* (a) *and* (b) *are true for 'stable' replaced by 'semistable'.*

Proof. (a) Let K be a subbundle of E of rank r. Let $F_1(K)=F_1(E)\cap(K_{x_1}\oplus K_{x_2})$ have dimension s. Define K_1 on X by $0\to K_1\to\pi_*K\to(\pi_*K)\otimes k(x_0)/F_1(K)\to 0$, K_1

is a subsheaf of F. One has $\deg K_1=\deg K+r-(2r-s)=\deg K+s-r$. Stability of F implies that $(\deg K_1)/r<(\deg F)/n$. This last condition holds if and only if $(\deg K+s-r)/r<(\deg F)/n$ i.e. $(\deg K+s)/r<(\deg E+n)/n$ i.e. (E, σ) is a stable parabolic bundle (definition 3.3). (b) and (c) follow similarly noting that K is the subbundle of E generated by the image of $\pi^* K_1$/torsion in E.

Proposition 4.3. *Let p_1 and p_2 be the canonical projections from $F_1(E)$ to E_{x_1} and E_{x_2} respectively.*

(1) *If p_1 and p_2 are both isomorphisms, then $F_{x_0}\approx n\mathcal{O}_{x_0}$ i.e. F is locally free.*
(2) *If only p_1* or *p_2 is an isomorphism and the other is of* rank r, *then*

$$F_{x_0} \approx r\mathcal{O}_{x_0}\oplus(n-r)\, m_0 .$$

(3) *If $F_1(E)=M_1\oplus M_2$, $M_i\subset k(x_i)^n$, then $F_{x_0}\approx nm_0$.*

Proof. In cases (1) and (2), at least one of p_1 or p_2 is an isomorphism. Suppose that p_1 is an isomorphism. Then $F_1(E)$ is the graph Γ_σ of the homomorphism $\sigma=p_2\circ p_1^{-1}$ from E_{x_1} to E_{x_2}. In case (1), σ is an isomorphism while in case (2), σ is of rank r. For simplicity of notations, let $(\mathcal{O}_{x_0}, m_0)=(A, m)$. Let $\bar{A}$ denote the normalisation, it is a semilocal ring with two maximum ideals m_1 and m_2. A is a Gorenstein local ring of dimension one with $m^*\approx\bar{A}$, $m_1\approx m_2\approx m$, also $m\approx\bar{A}$ (p. 164, [S]). We have a nonzero k-linear map $\sigma\colon k_1^n\to k_2^n$ where $k_i=\bar{A}/m_i$, $i=1,2$. Let $g\colon \bar{A}\to\bar{A}\otimes_A k=k_1\oplus k_2$ be the natural map. F is defined by the exact sequence

$$0 \to F \to \bar{A}^n \xrightarrow{p} (k_1\oplus k_2)^n/\Gamma_\sigma \to 0,$$

where p is the composite of ng with the natural map $(k_1\oplus k_2)^n\to(k_1\oplus k_2)^n/\Gamma_\sigma$ i.e. $F=(ng)^{-1}\Gamma_\sigma$, $ng=g\oplus\ldots\oplus g$ n-times. We want to show that $F\approx A^r\oplus\bar{A}^{n-r}$, $r=$ rank of σ.

Proof of (1). Suppose first that σ is an isomorphism. Let $\{x_i\}$, $\{y_i\}$ denote the coordinates in k_1^n and k_2^n respectively. Let (B_{ij}) be the matrix of σ and let (B_{ij}^{-1}) be the inverse matrix. Define $\psi\colon (k_1\oplus k_2)^n\to(k_1\oplus k_2)^n$ by $\psi(x_1, y_1, x_2, y_2, \ldots)=(x_1, \sum B_{1j}^{-1}y_j, x_2, \sum B_{2j}^{-1}y_j, \ldots)$. Then one has $\Gamma_\sigma=(x_1, \sum B_{1l}x_l, x_2, \sum B_{2l}x_l, \ldots)$ and $\psi(\Gamma_\sigma)=(x_1, x_1, x_2, x_2, \ldots)=$graph Γ_{Id} of the identity automorphism of k^n. Choose $C_{ij}\in\bar{A}$ such that $g(C_{ij})=(\delta_j^i, B_{ij}^{-1})$. Then $(C_{ij})\in GL(\bar{A})$ as $g(\det(C_{ij}))=(1, \det\cdot B_{ij}^{-1})$ is a unit. The automorphism φ of $\bar{A}^n$ defined by (C_{ij}) lifts ψ i.e. $\psi\circ ng=ng\circ\varphi$. It follows that $p^{-1}(\Gamma_{Id})\approx p^{-1}(\Gamma_\sigma)=F$. Since g^{-1} (diagonal in $k_1\oplus k_2)=A$, it follows that $F\approx A^n$.

Proof of (2). The above proof shows that given any $f\in GL(k^n)$ (replacing σ^{-1} by f in the above proof), one can define a homomorphism $\psi\colon (k_1\oplus k_2)^n\to(k_1\oplus k_2)^n$ which lifts to an automorphism φ of $\bar{A}^n$. Since ψ maps Γ_σ onto $\Gamma_{f\circ\sigma}$, we can replace

Γ_σ by $\Gamma_{f\circ\sigma}$. Now, changing σ to $f\circ\sigma$ is equivalent to changing the matrix of σ by row transformations. We now need the following lemma.

Lemma 4.4. *Let B be a nonzero $n\times n$ matrix of* rank *r. Then by row transformations B can be transformed to a matrix of the form*

$$\begin{pmatrix} I_{r_1} & * & 0 & * & 0 \dots \\ & 0 & 0 & 0 & 0 \dots \\ & & I_{r_2} & * & 0 \dots \\ 0 & & & 0 & 0 \dots \\ - & - & - & - & - \end{pmatrix}$$

where I_t denotes the identity matrix of rank *t, $0\leqq r_i\leqq\sum r_i=r$.*

Proof. We shall prove the result by induction on n. We write $B\sim C$ if C can be obtained from B by row transformations.

Case (*i*). Suppose that the first column of A is not identically zero. By row transformations we may assume that $B_{11}=1$, $B_{j1}=0$ $\forall j>1$, i.e. $B\sim\begin{pmatrix}1 & * \\ 0 & C\end{pmatrix}$. If M is an $s\times s$ submatrix of C, then B has an $(s+1)\times(s+1)$ submatrix of the form $N=\begin{pmatrix}1 & * \\ 0 & M\end{pmatrix}$ and $\det M=\det N$. So if all the minors of B of size $(s+1)$ vanish, then all the minors of C of size s also vanish. Hence $\operatorname{rank} C\leqq\operatorname{rank} B-1=r-1$. Also, by above, no $r\times r$ submatrix of B is contained in C. Hence any $r\times r$ submatrix of B is of the form N. It follows that C has a nonzero minor of size $r-1$ and $\operatorname{rank} C=r-1$. By induction, the result is true for C. Thus

$$B\sim\begin{pmatrix} 1 & & * & & \\ & I_{s_1} & * & 0 & * \\ 0 & & 0 & 0 & 0 \\ & & & I_{s_2} & * \\ & 0 & & & \ddots \end{pmatrix}$$

$0\leqq s_i\leqq r-1$, $\sum s_i=r-1$. Then

$$B\to\begin{pmatrix} 1 & 0 & * & 0 & * \dots & \\ & I_{s_1} & * & 0 & * \dots & \\ & & 0 & 0 & 0 \dots & \\ & 0 & & I_{s_2} & * \dots & \\ & & & & & \dots \end{pmatrix}$$

Letting $s_1+1=r_1$ and $s_i=r_i$ for $i>1$ we get the result.

Case (ii). Suppose that the first column of A is identically zero. By switching rows if necessary we have $B\sim\begin{pmatrix}0 & * \\ 0 & C\end{pmatrix}$, rank $C=r$.

Applying induction to C, we get

$$B \sim \begin{pmatrix} 0 & 0 & b_1 & 0 & b_2 & \cdots \\ & I_{r_1} & * & 0 & * & \cdots \\ 0 & & 0 & 0 & 0 & \cdots \\ & & & I_{r_2} & * & \cdots \\ & 0 & & & & \cdots \end{pmatrix}, \quad \sum r_i = r, \quad 0 \leqq r_i \leqq r.$$

Consider the minor

$$\det \begin{pmatrix} 0 & b_1 & 0 & 0 & - & - \\ I_{r_1} & * & 0 & 0 & - & - \\ & & I_{r_2} & 0 & & 0 \\ & & & I_{r_3} & & \\ & 0 & & & \ddots & \\ & & & 0 & & I_{r_m} \end{pmatrix} = \pm b_1$$

of size $r+1$. Since A has rank r, it follows that $b_1=0$. Similarly, $b_i=0$ for all i. Thus we have

$$B \sim \begin{pmatrix} 0 & 0 & 0 & 0 & - & - \\ & I_{r_1} & * & 0 & * & - & - \\ & & 0 & 0 & 0 & - & - \\ & & & I_{r_2} & * & - & - \\ 0 & & & & & - & - \end{pmatrix}, \quad \sum r_i = r, \quad 0 \leqq r_i \leqq r.$$

Proof of 4.3 (2) (continued). In view of lemma 4.4 we may assume that the matrix B of σ is of the form given by lemma 4.4. Then there exist coordinates $\{u_i\}$, $\{w_j\}$ of k^n $(i=1, \ldots, r;\ j=1, \ldots, n-r)$ such that $\sigma u_i=u_i+\sum b_{ij}w_j$, $\sigma(w_j)=0$ so $\Gamma_\sigma=(x_1, y_1, x_2, y_2, \ldots)$ where if $x_i=u_i$, $y_i=u_i+\sum b_{ij}w_j$, if $x_i=w_i$, $y_i=0$. Let pr be the projection $(k_1\oplus k_2)^n\to(k_1\oplus k_2)^{n-r}$ corresponding to $x_i=w_i$ coordinates. Then $pr(\Gamma_\sigma)=(k_1\oplus 0)^{n-r}$, $\operatorname{Ker} pr\cap\Gamma_\sigma=\{(x_i, y_i)_i, |$ if $x_i=w_i$, then $x_i=0=y_i$, if $x_i=u_i$, $y_i=u_i\}=\Delta^r$, where Δ denotes the diagonal of $k_1\oplus k_2$, Δ^r is embedded in $(k_1\oplus k_2)^r$ corresponding to $\{u_i\}$ coordinates. Let $\bar{p}$ denote the projection $\bar{A}^n\to\bar{A}^{n-r}$ lifting pr; one has $pr\circ ng=(n-r)g\circ\bar{p}$. Now, $(n-r)g\cdot\bar{p}(F)=pr\circ ng(F)=pr(\Gamma_\sigma)=(k_1\oplus 0)^{n-r}$, so that $\bar{p}(F)=m_2^{n-r}$. If K is the kernel of the restriction of $\bar{p}$ to F, $K=\operatorname{Ker}\bar{p}\cap F=(\bar{A}^r\oplus 0)\cap F=(\bar{A}^r\oplus 0)\cap(ng)^{-1}\Gamma_\sigma$. Hence $(ng)K=\Delta^r$ or $K\approx A^r$. Thus we have an exact sequence $0\to A^r\to F\to m_2^{n-r}\to 0$. Since $\operatorname{Ext}^1_A(m_2, A)\approx\operatorname{Ext}^1_A(\bar{A}, A)=0$ this sequence splits giving the required result.

Proof of (3). In the above notations, in this case, we have an exact sequence

$$0 \to F \to \bar{A}^n \xrightarrow{p} (k_1 \oplus k_2)^n / M_1 \oplus M_2 \to 0$$

and we want to determine F up to isomorphism. Let $\dim M_1 = r$. Let $h_i \in \operatorname{Aut} k_i^n$, $i=1,2$ be such that $h_1(M_1)=(k_1 \oplus 0)^r \oplus 0$, $h_2(M_2)=0 \oplus (0 \oplus k_2)^{n-r}$, $h_1(M_1)$ (resp. $h_2(M_2)$) mapping in the first r factors (resp. last $n-r$ factors) in $(k_1 \oplus k_2)^n$. Let (a_{ij}) and (b_{ij}) be the matrices of h_1 and h_2 (with respect to the canonical basis). Let $a_{ij} \in \bar{A}$ be such that $g(c_{ij})=(a_{ij}, b_{ij})$. Then $g(\det(c_{ij}))=(\det(a_{ij}), \det(b_{ij}))$ and hence $(c_{ij}) \in GL(\bar{A}^n)$. Thus $h=h_1 \oplus h_2$ lifts to an automorphism of $\bar{A}^n$. Hence one can replace $M_1 \oplus M_2$ by $h(M_1 \oplus M_2)$ i.e. $F \approx (ng)^{-1}(h(M_1 \oplus M_2)) = m_1^{n-r} \oplus m_2^r$.

4.5. Let M be the moduli space of semistable generalised parabolic bundles on $\tilde{X}$ of type described in 4.1. For $r=1, \ldots, n$ let $M_r \subset M$ be the subset of M corresponding to (E, σ) such that at least one of p_1, p_2 is an isomorphism and the other is of rank r. Let M_0 be the subset of M corresponding to (E, σ) such that none of p_1, p_2 is an isomorphism or $p_1=0$ or $p_2=0$. Clearly $M=\bigcup_{r=0}^n M_r$. As in 2.4, one can obviously globalise the construction in 4.1 to get a morphism $f\colon M \to U$, U being the moduli space of semistable torsionfree sheaves of rank n, degree d on X. One has $U=\bigcup_{r=0}^n U_r$, where U_r corresponds to torsionfree sheaves F such that $F_{x_0} \approx r\mathcal{O}_{x_0} \oplus (n-r) m_0$. In particular, U_n is the open subset of U corresponding to locally free sheaves. Proposition 4.3 shows that $f(M_r) \subset U_r$ for $r=1, \ldots, n$. In fact one has the following theorem.

Theorem 3. *Let X be an irreducible projective curve defined over an algebraically closed field, with only one node x_0 as a singularity. Let $\pi\colon \tilde{X} \to X$ be the normalisation. Let M be the moduli space of semistable generalised parabolic bundles E of* rank n, degree d *and parabolic structure given by* $D=\pi^{-1}(x_0)$, $E|_D \supset F_1(E) \supset 0$, $\dim F_1(E)=n$, $(\alpha_1, \alpha_2)=(0, \alpha)$ α near 1. *Let U be the moduli space of semistable torsionfree sheaves of* rank n, degree d *on X, $U=\bigcup_{r=0}^n U_r$ where $U_r=\{F \mid F_{x_0} \approx r\mathcal{O}_{x_0} \oplus (n-r) m_0\}$. Then there exists a surjective morphism $f\colon M \to U$ such that $f(M_r) \subseteqq U_r$ for all $r=1, \ldots, n$ and the restriction of f gives an isomorphism of M_n onto U_n. In particular, if n and d are coprime, then M is a desingularization of U.*

Proof. We have only to check that (i) $f|M_r$ is a surjection for all r and (ii) $f|M_n$ is an isomorphism onto U_n. This can be done on similar lines as in proposition 1.11, so we only sketch the proof with necessary modifications. For (ii), the inverse f^{-1} is given as follows. For $F \in U_n$ (i.e. corresponding to an element of U_n) define $E=\pi^* F$, $F_1(E)=F \otimes k(x_0) \subset F \otimes \pi_* \mathcal{O}_{\tilde{X}} \otimes k(x_0) = \pi_* E \otimes k(x_0)$. Since the above inclusion is essentially given by the inclusion $\mathcal{O}_{x_0} \subset \bar{\mathcal{O}}_{x_0}$ and $\mathcal{O}_{x_0}$ maps onto the diagonal in $k^2=\bar{\mathcal{O}}_{x_0} \otimes k(x_0)$, it follows that p_1 and p_2 are isomorphisms. Define $f^{-1}(F)=$

$(E, F_1(E))$. (i) If $F \in U_0$, $F = \pi_*(E_0)$ for a unique vector bundle E_0 on $\tilde{X}$. Take any E given by an extension of the form

$$0 \to E_0 \to E \xrightarrow{h} k(x_1)^r \oplus k(x_2)^{n-r} \to 0, \quad 0 \leqq r \leqq n$$

and $F_1(E)$=kernel of $h|x_1+x_2$. Then $f(E, F_1(E)) = F$. If $F \in U_r$, $0<r<n$, the result can be proved as in proposition 1.11(3). In this case, $E_0 = \pi^* F/\text{torsion}$, E is given by

$$0 \to E_0 \to E \to k(x_2)^{n-r} \to 0$$

or

$$0 \to E_0 \to E \to k(x_1)^{n-r} \to 0.$$

*4.6. **The determinant map.***

Let (E, σ) be as in 4.1. We shall generalise the results of 2.3 to define the "determinant" of (E, σ) when at least one of p_1 or p_2 (see 4.3) is an isomorphism. The space $F_1(E)$ is an element of the Grassmanian of n dimensional subspaces of $E_{x_1} \oplus E_{x_2}$. By Plücker embedding, G is embedded in $P(\Lambda^n(E_{x_1} \oplus E_{x_2}))$. Now, $\Lambda^n(E_{x_1} \oplus E_{x_2})$ contains $\Lambda^n E_{x_1} \oplus \Lambda^n E_{x_2}$ as a direct summand, let d be the projection $\mathbf{P}(\Lambda^n(E_{x_1} \oplus E_{x_2})) \to \mathbf{P}(\Lambda^n E_{x_1} \oplus \Lambda^n E_{x_2}) = \mathbf{P}^1$. Let $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ be the bases of E_{x_1} and E_{x_2}. Then a basis of $F_1(E)$ is of the form $(u_i = \sum a_{ij} e_j + \sum b_{ij} f_j)_{i=1,\ldots,n}$. The point P in G corresponding to $F_1(E)$ is given by $u_1 \Lambda \ldots \Lambda u_n = \det(a_{ij}) e_1 \Lambda \ldots \Lambda e_n + \det(b_{ij}) f_1 \Lambda \ldots \Lambda f_n +$ other mixed terms. Hence $d(P) = (\det(a_{ij}), \det(b_{ij})) = (\det p_1, \det p_2)$. We define $\det(E, \sigma) = (\det E; (\det p_1, \det p_2))$, p_1 and p_2 being the projections from $F_1(E)$ to E_{x_1} and E_{x_2} respectively. Note that $(\det p_1, \det p_2)$ defines a one dimensional subspace of $(\det E)_{n_1} \oplus (\det E)_{x_2}$, so $\det(E, \sigma)$ is a generalised parabolic line bundle.

It is easy to see that (see 2.4) this construction gives a morphism det: $\bigcup_{i=1}^n M_i \to P$, P being the moduli space of generalised parabolic line bundles (2.1, 2.2). We shall show that det goes down to a morphism det: $U_n \cup U_{n-1} \to \bar{J}(X)$. Let $F \in U_n$, $f(E, F_1) = F$. Then $F_1(E)$ is the graph of a morphism say g: $E_{x_1} \to E_{x_2}$ and F is obtained by identifying E_{x_1} with E_{x_2} via g. Hence $\det F$ is obtained by identifying $\det E_{x_1}$ with $\det E_{x_2}$ via $\det g$ i.e. it is the generalised parabolic line bundle $(\det E, \Gamma_{\det g})$. Note that $g = p_2 \circ p_1^{-1}$ so $\det g$ is the point $(1, \det g) \sim (\det p_1, \det p_2)$ in $\mathbf{P}^1$. Thus $\det|M_n$ is the same as the determinant morphism $U_n \to J(X)$ under the identification by $f|M_n$. By the proof of theorem 3, $F \in U_r$, and element $(E, F_1(E))$ in M_r on the fibre of f over F is obtained from an extension of the type

(a) $0 \to E_0 \to E \to k(x_1)^{n-r} \to 0$ or

(b) $0 \to E_0 \to E \to k(x_2)^{n-r} \to 0.$

Let $L=\det E_0=\det(\pi^* F/\text{torsion})$. Then one has either

(c) $\det(E, F_1(E)) = (L((n-r)x_1),\ F_1(L) = L_{x_2})$ or

(d) $\det(E, F_1(E)) = (L((n-r)x_2).\ F_1(L) = Lx_1)$.

If $n-r=1$ i.e. $F\in U_{n-1}$, then (c) and (d) map into the same element of $\bar{J}(X)-J(X)$ under the normalisation morphism $P\to\bar{J}(X)$. Thus det induces a morphism det: $U_{n-1}\bar{J}(X)-J(X)$. Note that det does not induce a morphism on U_r, $r\leqq n-1$ as (c) and (d) give different elements in $\bar{J}(X)-J(X)$. Thus we have proved the following.

Proposition 4.7. (1) *The morphism* det: $U_n\to J(X)$ *lifts to a morphism* $M_n\to P$. *The latter extends to a morphism* d: $\bigcup_{r>0} M_r\to P$.

(2) *The morphism d descends to a morphism* det: $U_n\cup U_{n-1}\to\bar{J}(X)$. *But d does not induce a morphism on* $\bigcup_{r<n-1} U_r$ *extending* det.

Examples 4.8. Consider the rank two torsionfree sheaf $\mathcal{O}\oplus\mathcal{M}$. We claim that $\Lambda^2(\mathcal{O}\oplus\mathcal{M})/\text{torsion}\approx\mathcal{M}$. Since both $\mathcal{O}$ and $\mathcal{M}$ are trivial outside x_0, the problem is local at x_0. Let $(\mathcal{O}_{x_0}, m_0)=(A, m)$. One has the inclusion i: $A\oplus m\to A\oplus A$. Let (e_1, e_2) be the canonical basis of $A\oplus A$ and let x, y be the generators of m, $i(e_1)=e_1$, $i(x)=xe_2$, $i(y)=ye_2$. Then $\Lambda^2 i$: $\Lambda^2(A\oplus m)\to A$ maps the torsion to zero and $\Lambda^2(A\oplus m)/\text{torsion}\approx I=\text{Image of }\Lambda^2 i$. The three generators $e_1\Lambda x$, $e_1\Lambda y$, $x\Lambda y$ of $\Lambda^2(A\oplus m)$ map respectively to $xe_1\Lambda e_2$, $ye_1\Lambda e_2$ and 0. Thus $I=m$ and hence $\Lambda^2(\mathcal{O}\oplus\mathcal{M})/\text{torsion}\approx\mathcal{M}$. Similarly, $\Lambda^n(\mathcal{O}^{n-1}\oplus\mathcal{M})/tor\approx\mathcal{M}$. Notice that $\text{degree}(\mathcal{O}^{n-1}\oplus\mathcal{M})=\text{degree}\,\mathcal{M}=-1$.

(2) Consider now the rank two torsionfree sheaf $\mathcal{M}\oplus\mathcal{M}$. As above, we need only to compute $\Lambda^2(m\oplus m)/\text{torsion}$. Writing $m\oplus m=m_1\oplus m_2$, let (x_j, y_j) be the generators of m_j, $j=1, 2$, i: $m_1\oplus m_2\to A\oplus A$ the inclusion, $i(x_j)=x_j e_j$, $i(y_j)=y_j e_j$, (e_1, e_2) being the canonical basis of $A\oplus A$. One sees that $\text{Ker}(\Lambda^2 i)$ is generated by $x_1\Lambda y_1$, $x_2\Lambda y_2$ while $I=\text{Im}(\Lambda^2 i)$ is generated by $x^2 e_1\Lambda e_2$, $xye_1\Lambda e_2$, $y^2 e_1\Lambda e_2$ i.e. $I=m^2$. Thus $\Lambda^2(\mathcal{M}\oplus\mathcal{M})/\text{torsion}\approx m^2$. Note that $\text{degree}(\Lambda^2(\mathcal{M}\oplus\mathcal{M}))/\text{torsion}=-3$ while $\text{degree}(\mathcal{M}\oplus\mathcal{M})=-2$. Similarly, $\Lambda^{r+s}(\mathcal{O}^r\oplus\mathcal{M}^{\oplus s})/\text{torsion}\approx\mathcal{M}^s$ and $\text{degree}(\mathcal{O}^r\oplus\mathcal{M}^{\oplus s})=-s$ while degree $\mathcal{M}^s\neq -s$ if $s>1$. This also explains why the determinant morphism does not extend to U_r, $r<n-1$.

Remark 4.9. Let U_L be the subset of U_n corresponding to vector bundles on X with a fixed determinant L and let $\bar{U}_L$ be its closure in U. Let M_L be the isomorphic image of U_L under $(f|U_n)^{-1}$. Since f is proper and $f|U_n$ is an isomorphism, as in remark 3.20, we see that $f(\bar{M}_L)=\bar{U}_L$. From proposition 4.7(1) it follows that $\bar{M}_L\cap(\bigcup_{r>0} M_r)=\phi$ i.e. $\bar{M}_L\subset M_L\cup M_0$ and hence $\bar{U}_L\subset U_L\cup U_0$.

References

G. GIESEKER, D.: A degeneration of the moduli space of stable bundles, *J. Differential Geom.* **19** (1984), 173—206.

G 2. GIESEKER, D.: On the moduli of vector bundles on algebraic surfaces, *Ann. Math.*, **106** (1977), 45—60.

L. LANGE, H., NARASIMHAN, M. S.: Maximal subbundles of rank two vector bundles on curves. *Math. Ann.* **266** (1983), 55—72.

N. NEWSTEAD, P. E.: *Introduction to moduli problems and orbit spaces*. Tata Institute, Bombay, Springer-Verlag (1978).

S. SESHADRI, C. S.: Fibres vectoriels sur les courbes algébriques, *Astérisque* **96** (1982).

SM. MEHTA, V. B. and SESHADRI, C. S.: Moduli of vector bundles on curves with parabolic structures. *Math. Ann.* **248** (1980), 205—239.

SN. SUNDARAM, N.: Special divisors and vector bundles, *Tohoku Math. Journ.* **39** (1987), 175—213.

V. BHOSLE, U. N.: Parabolic vector bundles on curves, *Arkiv för Matematik* vol. **27** (1989) No. 1, 15—22.

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U. Bhosle
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay 400 005
INDIA