Removable singularities for analytic or subharmonic functions

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Abstract

In this paper, results on removable singularities for analytic functions, harmonic functions and subharmonic functions by Besicovitch, Carleson, and Shapiro are extended. In each theorem, we need not assume that f has the global property at any point, so we are able to allow dense sets of singularities. We do not state our results in terms of exceptional sets, but each one leads to a series of results implying that certain sets are removable for appropriate classes of functions.

1. Analytic functions

In 1931, Besicovitch obtained a generalization of Painlevé's theorem: let f be a bounded function defined on an open set W, and L a subset of linear (i.e. one-dimensional) measure 0. Assume that at each point z of $W-L$, f admits a Taylor expansion $f(z+w)=f(z)+wf'(z)+o(|w|)$. Then f can be continued over L to be analytic. Besicovitch's theorem is noteworthy in two respects: the set of singularities is allowed to be everywhere dense, and the concept of holomorphy is replaced by that of a complex derivative. At present, these hypotheses are further reduced: differentiability is not required at any point in the domain, and approximate differentiability (wherever it is required) is understood in the space L^1. The difficulties attendant upon covering a dense set of singularities are finessed (or evaded) with Whitney's partition of unity, and line integrals are avoided. In consequence of the last point, we are able to obtain integrability theorems that are close to best possible (as in [4]).

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To formulate our theorem precisely, we define the reduced norm $N_0(f, B)$, for a measurable function f defined on an open ball B: this is the infimum of integrals

$$\iint_B |f(z)-g(z)|\,dx\,dy,$$

extended over functions $g(z)$, analytic on B. The infimum is attained. (If necessary we set $f=0$ outside W.)

Theorem 1. *Let f be measurable on a bounded open set W, and for each $\varepsilon>0$, suppose there is a covering*

$$W = \bigcup_1^\infty B(z_i, r_i) \quad \textit{such that} \quad \sum_1^\infty N_0(f, B(z_i, 2r_i))r_i^{-1} < \varepsilon.$$

Then f can be corrected on a set of measure 0, to become analytic on W.

Example 1, Besicovitch's theorem. Let f be bounded a.e. on W, and suppose that L is a set of linear measure 0. At each point z in $W-L$, we suppose

$$f(z+w) = f(z)+wc(z)+o(|w|), \quad \text{for small } w.$$

(Here $o(|w|)$ may depend on z).

Inasmuch as f is bounded a.e., and

$$N_0(f, B(z_i, 2r_i))r_i^{-1} \leqq 4\pi r_i \|f\|_\infty,$$

the set L can be accounted for. To each $\varepsilon>0$, we can cover $W-L$ by disks $B(z_j, r_j)$ so that

$$N_0(f, B(z_j, 8r_j)) \leqq \varepsilon r_j^3,$$

and we can choose a subsequence $B(z_k, r_k)$, still covering $W-L$, while $\Sigma_1^\infty r_k^2$ is bounded by a constant depending only on the area of W.

Example 1'. We can add an exceptional set L_1 to L, provided f is continuous at each point of L_1, or that L_1 is contained in the set of Lebesgue points of f. Indeed, we have

$$N_0(f, B(z, 2r)) = o(r^2)$$

at each Lebesgue point z of f. (See Besicovitch [1].)

Example 2. Suppose for simplicity that W is convex, and set

$$h_1(\delta) = \sup |f(z_1)-f(z_2)|,$$

over the pairs $z_i \in W$ with $|z_1-z_2| \leqq \delta$. (Observe that $h_1(2\delta) \leqq 2h_1(\delta)$.) If the set L of Example 1 is allowed to have Hausdorff measure 0 for the measure-function $h(\delta)=\delta h_1(\delta)$, then f is analytic. Indeed

$$N_0(f, B(z, r)) = O(r^2 h(r))$$

for any z in W. Compare [2].

Example 3. Let f belong to $L^p(|z|<1)$, and suppose that the exceptional set L has finite Hausdorff α-measure, with $p>2$ and $\alpha=1-(p-1)^{-1}$. Then f is equal a.e. to an analytic function on W.

To verify that our theorem can be applied, we cover L by disks $B(z_i, r_i)\equiv B_i$, such that $\Sigma r_i^\alpha \leqq C$, while $\max r_i$ is small. Writing B_i^* for the double of B_i, we have the inequalities

$$\sum r_i^{-1}\int_{B_i^*}|f| \leqq \sum r_i^{-1}\Big(\int_{B_i^*}|f|^p\Big)^{\frac{1}{p}} m(B_i^*)^{\frac{1}{q}} < \pi \sum r_i^{-1+\frac{2}{q}}\Big(\int_{B_i^*}\Big)^{\frac{1}{p}}.$$

This can be handled by Hölder's inequality, because $q(-1+2q^{-1})=\alpha$. The sum actually tends to zero with $\max r_i$, because $\cup B_i^*$ has small area when $\max r_i$ is small. With a little more effort we can treat a set L of σ-finite Hausdorff α-measure.

In the case of sets L geometrically like the Cantor set, each covering is composed of $O(r_j^{-\alpha})$ cubes Q_i of side r_j, for some sequence $r_j\to 0$.

Now the set $B_j=\cup Q_j^*$ has measure $O(r_j^{2-\alpha})$, so what must be proved is that

$$\int_E |f| = o\big(m(E)\big)^{\frac{1}{q}} \quad \text{for sets} \quad E, \quad m(E)\to 0.$$

But this is nothing but the condition

$$m\{|f|>\lambda\} = o(\lambda^{-p}), \quad \lambda \to +\infty.$$

We shall now prove that when a closed set L has positive α-measure, $0<\alpha<1$, there is an analytic function f on $W-L$, such that

$$m\{|f|>\lambda\} = O(\lambda^{-p}), \quad \lambda \to +\infty.$$

Having positive α-measure, L carries a positive measure μ, such that $\mu\big(B(r)\big)\leqq Cr^\alpha$ for every ball B of radius r. For f we take $\int(\zeta-z)^{-1}\mu(d\zeta)$, so that f is analytic off L, and admits no extension to an entire function, as its primitive is multiply-valued. If $z\in\mathbf{R}^2$ and $\delta>0$ is a small number

$$\int_{|\zeta-z|>\delta}|\zeta-z|^{-1}\mu(d\zeta) = O(\delta^{\alpha-1})$$

as can be seen by a Stieltjes integral. Fubini's theorem shows that

$$\int_{|z|\leqq 100}\int_{|\zeta-z|\leqq\delta}|\zeta-z|^{-1}\mu(d\zeta)\,dx\,dy = O(\delta).$$

To confirm that

$$m\{|z|<100,\ |f(z)|>\lambda\} = O(\lambda^{-p}),$$

we choose δ a large multiple of $\lambda^{1/(\alpha-1)}$, so that $\lambda^{-1}\delta=O(\lambda^{-p})$. (For similar results, see [4, VI].)

Proof of Theorem 1. Before writing the formulas necessary in the proof of the theorem, we construct a refinement of the covering $B(z_i, r_i)$ of W. We will be interested in covering a compact subset K of W, so we work with a finite set of balls $B(z_i, r_i)$, $1 \leqq i \leqq N$, and assume that $r_1 \geqq r_2 \geqq \ldots \geqq r_N$. We expand $B(z_1, r_1)$ to $B(z_1, 3r_1/2)$ and discard any ball $B(z_i, r_i)$, $(i \geqq 2)$, contained in this one; that is, we discard any ball for which $|z_1 - z_i| \leqq 3r_1/2 - r_i$. If any ball $B(z_i, r_i)$ remains, we expand the radius of the largest one as before and discard superfluous sets. We continue this process until no ball remains and obtain a covering $B(z_i, 3r_i/2)$, $i \in T$, in which $|z_i - z_j| > r_i/2$, whenever $i < j$. We shall work with this covering only.

Next, we find a function Φ in $C^1(\mathbf{R}^2)$, such that $0 \leqq \Phi \leqq 1$, $\Phi = 0$ on each set $B(z_i, r_i/8)$, $\Phi = 1$ outside the union of all the sets $B(z_i, r_i/4)$, and $|\nabla \Phi| \leqq c r_i^{-1}$ on the annular regions defined by $r_i/8 \leqq |z - z_i| \leqq r_i/4$.

Let now H be a function of class $C^1(W)$, with compact support in W. Then f is integrable on its support, call it K, and we shall prove that

$$\iint_W f(z) \bar{\partial} H(z)\, dx\, dy = 0.$$

By a classical method (Weyl's lemma), f is equal a.e. to a function analytic on W. Here $\bar{\partial}$ is defined by the equations

$$\partial + \bar{\partial} = \frac{\partial}{\partial x}, \quad \partial - \bar{\partial} = -i \frac{\partial}{\partial y}.$$

In the proof that

$$\iint_W f \bar{\partial} H = 0,$$

we can assume that $|H| < 1$, $|\partial H/\partial x| < 1$, $|\partial H/\partial y| < 1$.

First we estimate

$$\iint f(\bar{\partial} H - \bar{\partial}(\Phi H)).$$

We recall that the support of $1 - \Phi$ is contained in disjoint balls $B(z_i, r_i/4)$ and $|\nabla \Phi| \leqq c r_i^{-1}$ on $B(z_i, r_i/4)$. Hence

$$\bar{\partial} H - \bar{\partial}(\Phi H) = O(1) + c r_i^{-1}$$

and

$$\iint_{B\left(z_i, \frac{r_i}{4}\right)} f(\bar{\partial} H - \bar{\partial}(\Phi H))$$

is at most

$$\left(O(1) + c r_i^{-1}\right) \times N_0(f, B(z_i, 2r_i)).$$

Summing over i, we obtain a quantity tending to 0 with ε.

Now let $1 = \Sigma_1^\infty \varphi_k$ be Whitney's partition of unity for the complement of the set $F = \{z_i, i \in T\}$. The standard procedure [6] shows that for the support Q_k of φ_k, we have $\operatorname{diam} Q_k \leqq 3d(Q_k, F)$. We can improve this to $d(Q_k, F)/4$. The sums $\Phi H = \Sigma\, \Phi H_k$ and $\bar{\partial}(\Phi H) = \Sigma\, \bar{\partial}(\Phi H \varphi_k)$ are actually finite, and we shall now consider some k so that $\Phi H \varphi_k \neq 0$. If Q_k meets $B(z_i, 3r_i/2)$ for a certain i, then Q_k is entirely contained in $B(z_i, 1.875r_i)$ and Q_k is not covered by $B(z_i, r_i/8)$ and therefore fails to meet $B(z_i, r_i/16)$. Let us choose one $i=i(k)$ so that $d(Q_k, F)=d(Q_k, z_i)$. Then each index i occurs at most c times (an absolute constant) in view of our assertion that $Q_k \subseteq B(z_i, 2r_i)-B(z_i, r_i/16)$. In case Q_k meets some ball $B(z_j, r_j/4)$, then $r_i/16 \leqq r_j/4$, whence $|\bar{\partial}\Phi| \leqq 4cr_i^{-1}$. Now

$$\iint_W f\bar{\partial}(\Phi H\varphi_k)$$

can be evaluated over $B(z_i, 2r_i)$ and is bounded, as before, by

$$O(r_i^{-1})N_0(f, B(z_i, 2r_i)).$$

In view of the remark on the function $i(k)$, this leads to the conclusion that

$$\iint_W f\bar{\partial}H = 0.$$

This completes the proof.

In the proof, we used a quantity $N(f)$ smaller than $N_0(f)$, defined as follows:

$$\left|\int_B f\bar{\partial}h\right| \leqq N(f)\sup|\partial h| + N(f)\sup|\bar{\partial}h|$$

for functions $h \in C^1$, vanishing near the boundary of a ball B. The functional $N(f)$ is genuinely smaller than $N_0(f)$. Indeed, let

$$f_\alpha(z) = |z|^\alpha z^{-2}, \quad 0 < \alpha < 1.$$

Then

$$f_\alpha(e^{i\theta} z) = e^{-2i\theta} f_\alpha(z),$$

so the best analytic approximation to f_α is 0. However, $N(f_\alpha)$ is bounded, because

$$\int f_\alpha\bar{\partial}h = -\tfrac{1}{2}\int \alpha r^{\alpha-2}z^{-1}(h(z)-h(o)).$$

If, then, $|h(z)-h(o)| \leqq C|z|$, we obtain

$$\left|\int f_\alpha\bar{\partial}h\right| \leqq \pi C.$$

The limit as $\alpha \to 0+$, is $-\pi(\partial h)(o)$.

2. Subharmonic functions

Let W be a bounded open set with smooth boundary in $\mathbf{R}^n$ $(n \geqq 2)$, L be a closed set in W. In [3], Carleson proved the following

Theorem *(Carleson). Let H_α be the class of harmonic functions in $W \setminus L$ which satisfy a Hölder condition of order α $(0<\alpha<1)$,*

$$|u(x)-u(x')| \leqq C|x-x'|^\alpha \quad \textit{whenever} \quad x, x' \in W \setminus L.$$

Then L is removable for the class H_α if and only if L has $(n-2+\alpha)$-dimensional measure zero.

The proof is ingenious, however, it does not seem applicable for studying removable singularities of subharmonic functions. In [5], V. L. Shapiro studied removable sets for subharmonic functions. After a reformulation to suit our purpose, one of his results states that

Theorem *(Shapiro). Let $S_\alpha(-(n-2)\leqq\alpha<1)$ be the class of $L^1(W\setminus L)$ functions with property*

$$\sup_{\substack{B(x,\varrho)\subseteq W\\ 0<\varrho<r}} \varrho^{-\alpha-n}\int_{B(x,\varrho)}|u(y)-u_{B,\varrho}(x)|\,dy = O(1) \tag{1}$$

as $r\to 0$, where $u_{B,\varrho}(x)$ is the average on u of ball $B(x,\varrho)$. Then L is removable for subharmonic functions in class S_α if and only if L has $(n-2+\alpha)$-dimensional measure zero.

Shapiro's theorem implies the harmonic result by Carleson. However, the condition (1) seems too restrictive for studying subharmonic functions, because (1) is on smoothness of a function, which does not hold even for the most fundamental subharmonic function $u(x)=-|x|^{-(n-2)}$ (when $n\geqq 3$). Because the sub mean value property is a one-sided inequality, a one-sided control of u from its mean or a two-sided control of u from other subharmonic functions would be more natural. Theorem 2 is an analogue of Theorem 1 for subharmonic functions. Theorem 3 shows how a one-sided control of a function and its mean gives subharmonicity. We observe that every subharmonic function on W must satisfy (2) and (3). Moreover, the sufficiencies for Carleson's theorem and Shapiro's theorem follow immedicately from Theorems 3 and 2 respectively.

For a measurable function f defined on an open ball $B=B(x,r)$ we define $N_1(f,B)$ to be the infimum of integrals

$$\int_B r^{-n}|f(x)-u(x)|\,dx$$

extended over functions $u(x)$, subharmonic on B.

Theorem 2. *Let f be measurable on a bounded open set W and for each $\varepsilon>0$, there is a covering $W=\bigcup_1^\infty B(x_i,r_i)$ such that $W=\bigcup_1^\infty \bar{B}(x_i,2r_i)$ and*

$$\sum_1^\infty r_i^{n-2}N_1(f,B(x_i,2r_i))<\varepsilon. \tag{2}$$

Then f can be corrected on a set of measure zero to be subharmonic on W.

Proof. Let H be a non-negative function in $C_0^2(W)$; the theorem will follow if we can show that

$$\int_W f(y)\Delta H(y)\,dy \geqq 0.$$

Let K be the support of f and let the covering $B(x_i, r_i)$ $i\in T$, Φ and $\{\varphi_k\}$ corresponding to K be as in the proof of Theorem 1. Observe that $0\leqq 1-\Phi\leqq 1$.

We assume that

$$|H|,\ \left|\frac{\partial}{\partial_{x_i}}H\right|,\ \left|\frac{\partial^2}{\partial_{x_i}\partial_{x_j}}H\right| < 1.$$

Hence:

$$\Delta H-\Delta(\Phi H) = O(1)(1+r_i^{-2}) \quad \text{on} \quad B(z_i, r_i/4), \quad i\in T.$$

Thus if u is subharmonic on W, then

$$\int_{B\left(x_i,\frac{r_i}{4}\right)} f(y)(\Delta H-\Delta(\Phi H))(y)\,dy \geqq \int_{B\left(x_i,\frac{r_i}{4}\right)} (f-u)(\Delta H-\Delta(\Phi H))\,dy.$$

Also, taking infimum over subharmonic functions on $B(x_i, r_i/4)$, we have

$$\inf\left|\int_{B\left(x_i,\frac{r_i}{4}\right)} (f-u)(\Delta H-\Delta(\Phi H))\,dy\right| \leqq Cr_i^{n-2}N_1(f, B(x_i, 2r_i)).$$

Summing over i and noticing that $1-\Phi$ is supported in the disjoint balls $B(z_i, r_i/4)$, $i\in T$, we obtain

$$\int_W f(\Delta H-\Delta(\Phi H))\,dy$$

bounded below by

$$-C\sum_1^\infty r_i^{n-2}N_1(f, B(z_i, 2r_i))$$

which tends to zero with ε.

For each k with $\Phi H\varphi_k\not\equiv 0$, we choose $i=i(k)$ as in Theorem 1. Reasoning similarly as before, we have

$$\int_{B(x_i, 2r_i)} f\Delta(\Phi H\varphi_k)\,dy \geqq -Cr_i^{n-2}N_1(f, B(z_i, 2r_i)).$$

Summing over k and recalling that each index i occurs at most c times, we obtain

$$\int_W f\Delta(\Phi H)\,dy \geqq 0.$$

Thus

$$\int_W f\Delta H\,dy \geqq 0$$

and the theorem is proved.

For an upper semi-continuous function f defined on the closure of the ball $B=B(x, r)$, we define

$$N_2(f, B) = \sup_{y\in B\left(x,\frac{r}{2}\right)} f(y)-PI_{x,r,f}(y)$$

where $PI_{x,r,f}$ is the Poisson integral on $B(x, r)$ with boundary function f.

Theorem 3. *Let f be an upper semi-continuous function on a bounded open set W and for each $\varepsilon > 0$, there is a covering*

$$W = \bigcup_1^\infty B(x_i, r_i) \quad \text{so that} \quad W = \bigcup_1^\infty \bar{B}(x_i, 2r_i)), \quad r_i < \varepsilon,$$

and

$$\sum_1^\infty r_i^{n-2} N_2^+ (f, B(x_i, 2r_i)) < \varepsilon. \tag{3}$$

Then f can be corrected on a set of $(n-2)$-dimensional measure zero, to become subharmonic on W.

Proof. Corresponding to a fixed ball $B(x, 2r)$ for which $N_2 = N_2(f, B(x, 2r))$ is positive, we define a subharmonic function u as follows:

$$u(y) = 2N_2 \log |y-x|, \qquad \text{if} \quad n = 2$$

$$u(y) = -2N_2 r^{n-2} |y-x|^{2-n}, \quad \text{if} \quad n \geqq 3$$

The Poisson integral of u on $B(x, 2r)$ exceeds u by at least N_2 on $B(x, r)$. Thus

$$f(y) + u(y) \leqq PI_{x, 2r, u+f}(y), \quad y \in B(x, r). \tag{4}$$

Let $\{B(x_i, 2r_i)\}$ be a covering of W for which

$$\sum r_i^{n-2} N_2(f, B(x_i, 2r_i))$$

is finite, and construct the corresponding functions u_i. The sum $\Sigma_i u_1$ converges in L^1 on any bounded set, and for $n \geqq 3$ each term is negative, so the sum is again subharmonic. When $n=2$ we can use the fact that $\log |y-x| \leqq C$ when $x, y \in W$, and $\Sigma N_2 < +\infty$ to conclude the subharmonicity.

For each positive integer k, choose a covering

$$\{B_{k,i}\} = \{B(x_{k,i}, r_{k,i})\}_1^\infty$$

corresponding to $\varepsilon = k^{-2}$, with the properties stated in the theorem, in particular

$$\sum_{i=1}^\infty r_{k,i}^{n-2} N_2(B(x_{k,i}, 2r_{k,i})) < k^{-2}.$$

Construct the corresponding function $u_{k,i}$ and let

$$v_K = \sum_{k=K}^\infty \sum_{i=1}^\infty u_{k,i}.$$

For each k the series converges everywhere to a function, finite or $-\infty$, and the double sum $\Sigma\,\Sigma\, |u_{k,i}|$ converges in $L^1(W)$. Moreover, $v_k \to 0$ in L^1 as $k \to \infty$. We intend to show that $f + v_k$ is subharmonic for each k, and the case $k=1$ is typical.

For each $x \in W$ there exist $\{B_{k,i(k)}\}$ so that $x \in B_{k,i(k)}$ and $r_{k,i(k)} \to 0$ as $k \to \infty$. In view of (4), we have

$$f + u_{k,i(k)}(x) \leqq PI_{x_{k,i(k)}, 2r_{k,i(k)}, f+u_{k,i(k)}}(x).$$

From the subharmonicity of $u_{k,i}$'s, it follow that

(5) $$f+v_1(x) \leqq PI_{x_{k,i(k)},\, 2r_{k,i(k)},\, f+v_1}(x),$$

that is, for each $x \in W$ there exists a ball in W containing x with arbitrarily small radius so that $f+v_1(x)$ is no greater than the value of the harmonic function in the ball, with boundary values $f+v_1$, evaluated at x. Let B be any ball with $\bar{B} \subseteqq W$ and g be any harmonic function on B with boundary values no less than $f+v_1$ on ∂B. To show $f+v_1$ is subharmonic it suffices to show $f+v_1-g \leqq 0$ on B. Let A be the closed subset of $\bar{B}$, where

$$\max \{f+v_1-g(x)\ \ x \in \bar{B}\}$$

is attained and x_0 be the point in A closest to ∂B. If $x_0 \in B$, in view of (5) and the extremum at x_0, there exists a ball B_0 containing x_0, with $\bar{B}_0 \subseteqq B$ and $\partial B_0 \subseteqq A$. This contradicts our choice of x_0 as the closest one in A from B. Thus $x_0 \in \partial B$. Therefore

$$f+v_1-g(x) \leqq f+v_1-g(x_0) \leqq 0 \quad \text{for} \quad x \in \bar{B}.$$

This shows that $f+v_1$ is subharmonic.

The sum

$$\sum_i \sum_k |u_{k,i}|$$

belongs to $L^1(W)$ and also to $L^1(d\sigma)$, σ being the surface measure of any ball. Thus $v_k \to 0$ in $L^1(d\sigma)$ as $k \to +\infty$. We know also that there is a subharmonic function f_1 on W, equal to f almost everywhere on W. When $B(x, r)$ is a ball contained in W

$$f_1(x) \leqq m(B(x, r))^{-1} \int\int_B f(y)\, dV;$$

letting $r \to 0+$ we obtain $f_1(x) \leqq f(x)$, because f is upper semi-continuous. Using the mean-value inequality for the functions $f+v_k$ and making $k \to \infty$ we find that

$$f(x) \leqq m(B(x, r))^{-1} \int\int_B f_1(y)\, dV$$

if

$$\sum_i \sum_k |u_{k,i}(x)| < +\infty.$$

Thus $f \geqq f_1$ everywhere and $f = f_1$ except on the set where $\Sigma\Sigma |u_{k,i}(x)| = +\infty$. But if $y \in B(x, r)$, then

$$f(y) \leqq N_2 + PI_{x, 2r, f}(y) = N_2 + PI_{x, 2r, f_1}(y).$$

We find that if y belongs to a sequence of balls $B(x_k, r_k)$ for which

$$\liminf N_2(f, B(x_k, 2r_k)) = 0,$$

then $f(y) \leqq f_1(y)$. Consequently $f = f_1$ except on a set of Hausdorff $(n-2)$-dimensional measure 0. The proof is complete.

Observe that if F is a closed set in W of $(n-2)$-measure 0, χ_F fulfills the hypotheses of the theorem, as does $\chi_F + g$, whenever g is subharmonic.

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