# A note on Euler's $\varphi$-function

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Let $\varphi(n)$ denote the Euler $\varphi$-function; we define the error term $E(x)$ by the relation

$$\sum_{n \leqq x} \varphi(n) = \frac{3}{\pi^2} x^2 + E(x)$$

By a simple elementary argument (cf. [2] p. 268) it may be shown that $E(x) \ll x \lg x$, while A. Walfisz (cf. [6] p. 114) used Vinogradov's method to show that

$$E(x) \ll x (\lg x)^{2/3} (\lg \lg x)^{4/3}$$

In the opposite direction, S. S. Pillai and S. D. Chowla (cf. [3]) proved that

$$E(x) = \Omega(x \lg \lg \lg x)$$

and P. Erdős and H. N. Shapiro (cf. [1]) proved that

$$E(x) = \Omega_{\pm}(x \lg \lg \lg \lg x).$$

Concerning the average of $E(n)$, Pillai and Chowla showed that

(1) $$\sum_{n \leqq x} E(n) = \frac{3}{2\pi^2} x^2 + o(x^2)$$

and conjectured that (1) may be "as deep as the prime number theorem" (cf. [3] p. 95). In this Arkiv, D. Suryanarayana and S. Sitaramachandra Rao (cf. [4]) showed that this latter error term can be replaced by

(2) $$O\big(x^2 \exp(-c(\lg x)^{3/5} (\lg \lg x)^{-1/5})\big)$$

and that, if the Riemann hypothesis is assumed, then it may be replaced by

(3) $$O(x^{9/5+\varepsilon})$$

The object of this note is to give a proof of (1) without using the fact that $\zeta(1+it)\neq 0$ and to show by standard techniques of analytic number theory that, if $E_1(x)$ denotes the error term in (1), then the infimum of those $\alpha$'s for which $E_1(x)\ll x^{1+\alpha}$ is equal to the supremum of the real parts of the zeros of $\zeta(s)$. Thus the Riemann hypothesis is equivalent to $E_1(x)\ll x^{3/2+\varepsilon}$. Moreover, it is shown that estimates of type (2) can easily be obtained by classical methods.

Let $E_1(x)$ denote the error term in (1). It is easily seen that

$$E_1(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta(s-1)}{\zeta(s)}\cdot\frac{x^{s+1}}{s(s+1)}\,ds+O(x\lg x) \tag{4}$$

for $1<c<2$. Now the functional equation for $\zeta(s)$ relates $\zeta(s-1)$ to $\zeta(2-s)$; whence we deduce that $\zeta(s-1)/\zeta(s)$ is regular in the half-plane $\operatorname{Re} s\geqq 1$. The contour integral above can therefore be moved onto the line $\operatorname{Re} s=1$. (The integrals over the horizontal sides of the rectangle $c\pm iT$, $1\pm iT$ tend to zero for $T\to\infty$, cf. [5], p. 185.) The resulting integral is absolutely convergent (cf. [5], p. 81) and thus by the Riemann—Lebesgue lemma we obtain $E_1(x)=o(x^2)$. This gives a proof of (1) without using the fact that $\zeta(1+it)\neq 0$, i.e. without using the prime number theorem.

If we use the classical zero-free region for $\zeta(s)$ and move the contour further to the left in the usual way, we find that

$$E_1(x)\ll x^2\exp\left(-c(\lg x)^{1/2}\right) \tag{5}$$

(Similarly, using the best known result on the zero-free region for $\zeta(s)$, cf. [6], p. 226, one can establish (2).)

To sharpen (3), we note that if $\theta$ denotes the supremum of the real parts of the zeros of $\zeta(s)$, then we can take the contour to be the line $\operatorname{Re} s=\theta+\varepsilon$ and deduce that $E_1(x)\ll x^{1+\theta+\varepsilon}$. On the other hand it is easy to see that

$$\int_1^{+\infty}\left(\int_1^t E(u)\,du\right)\frac{1}{t^{s+2}}\,dt=\frac{\zeta(s-1)}{\zeta(s)}\frac{1}{s(s+1)}-\frac{3}{\pi^2}\frac{1}{(s-2)(s+1)}. \tag{6}$$

In view of the relation $E_1(x)=\int_1^x E(u)\,du+O(x\lg x)$ the estimate $E_1(x)\ll x^{1+\alpha}$ then implies that $\zeta(s)\neq 0$ for $\operatorname{Re} s>\alpha$. This establishes the desired equivalence. It should also be noted that (6) implies the estimate $E_1(x)=\Omega\pm(x^{3/2})$ also by standard means.

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## References


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