On spectra of unitary groups arising from cocycles

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Let T be an invertible aperiodic measure preserving transformation on a probability space $(X, \mathscr{B}, m)$ where we assume that $(X, \mathscr{B})$ is a standard Borel space and m is continuous, i.e., assigns mass zero to singletons. Let φ be a Borel function on X of absolute value one and consider the unitary operator V^{φ} defined on $L_2(X, \mathscr{B}, m)$ by

$$(V^{\varphi}f)(x) = \varphi(x)f(Tx), \quad f \in L^2(X, \mathscr{B}, m).$$

In a paper entitled "cocycles and spectra" Helson and Parry prove that for every T there exists a φ such that V^{φ} has Lebesgue spectrum, moreover φ can be chosen to be real, i.e., taking values $+1$ and -1. The purpose of this paper is to extend this result to certain actions of countable groups which includes ergodic non-singular actions of countable abelian groups. We blend the method of Helson and Parry with the notions of weak equivalence and weak von Neumann transformations. In section 3 and 4 we discuss these results in connection with systems of imprimitivity.

The problem of extending the result quoted above to countable groups was raised by H. Helson to one of us. It is a pleasure to acknowledge his interest and encouragement in this work.

Section 1

Definition. A non-singular transformation τ on $(X, \mathscr{B}, m)$ is said to be weak von Neumann transformation if there exists a sequence $\mathscr{D}_k(\tau) = (D_1^k, \ldots, D_{2^k}^k)$ $k = 1, 2, \ldots$ of ordered partitions of X into measurable sets such that

(a) $$D_i^k = D_i^{k+1} \cup D_{i+2^k}^{k+1}, \quad i = 1, \ldots, 2^k; \; k = 1, 2, 3, \ldots$$

(b) $$D_i^k = \tau^{i-1} D_1^k, \quad i = 1, \ldots, 2^k, \;\; k = 1, 2, 3, \ldots$$

Remark 1. If the sets D_i^k, $i = 1, \ldots, 2^k$, $k = 1, 2, \ldots$ generate the σ-algebra $\mathscr{B}$ (modulo null sets) then τ is said to be von Neumann transformation. We will not need von Neumann transformation.

Remark 2. Let us write $\mathrm{Orb}(x, n)$ to mean the finite set $(x, \tau x, \ldots, \tau^{n-1}x)$ if $n \geqq 0$ and the set $(\tau^{-1}x, \ldots, \tau^{n}x)$ if $n<0$. If we put

$$F_k = D^k_{2^k-1}, \quad F_{k1} = D^{k+1}_{2^k-1}, \quad F_{k2} = D^{k+1}_{2^k-1+2^k}$$

then $F_k = F_{k1} \cup F_{k2}$. Further if $2^k \leqq |n| < 2^{k+1}$ then $\mathrm{Orb}(x, n)$ intersects F_k in atleast one point and it intersects F_{k1} and F_{k2} in atmost one point. This fact will be useful later.

It is known that every ergodic non-singular transformation is weakly equivalent to a weak von Neumann transformation (see Hajian, Ito, Kakutani [3] for the measure preserving case and K. Schmidt [5] for the non-singular case). More generally ergodic non-singular action of a countable abelian group is known to be hyperfinite, hence weakly equivalent to a von Neumann transformation (see Feldman and Lind [2]). Let G be a countable group, not necessarily abelian but written additively. Let T_g, $g \in G$, be a group of non-singular transformations on X weakly equivalent to a weak von Neumann transformation. This means that there exists a weak von Neumann transformation τ on X such that for almost every x, orbit of x under T_g, $g \in G$ is same as the orbit of x under τ. Define (almost everywhere) the function C: $G \times X \to Z$, Z denoting group of integers, by

$$C(g, x) = n \quad \text{if} \quad T_g x = \tau^n x.$$

For fixed g, $\{x\colon C(g, x) = n\} = \{x\colon T_g x = \tau^n x\}$ is a measurable set. Hence for every fixed g, the function $C(g, .)$ is measurable. Moreover it can be verified that C satisfies the cocycle identity:

$$C(g+h, x) = C(g, x) + C(h, T_g x) \quad \text{a.e. } x$$

for all $g, h \in G$. For $g \in G$ and $k \in Z$ let m^g and m^k denote measures defined by $m^g(B) = m(T_g B)$, $m^k(B) = m(\tau^k B)$, $B \in \mathscr{B}$. Let $m_g = \left(\frac{dm^g}{dm}\right)^{\frac{1}{2}} dm$, $m_k = \left(\frac{dm^k}{dm}\right)^{\frac{1}{2}} dm$, these being measures whose Radon—Nikodym derivatives with respect to m are $\left(\frac{dm^g}{dm}\right)^{\frac{1}{2}}$ and $\left(\frac{dm^k}{dm}\right)^{\frac{1}{2}}$ respectively. Schwarz inequality immediately gives

$$m_g(B) \leqq \sqrt{m(B)}, \quad m_g(X) \leqq 1$$

and similar inequalities for $m_k(B)$ and $m_k(X)$. If g and k are fixed and $B \subseteqq \{x\colon T_g x = \tau^k x\}$ then

$$\int_B \frac{dm^g}{dm}\, dm = m(T_g B) = m(\tau^k B) = \int_B \frac{dm^k}{dm}\, dm$$

from which we conclude that on the set $\{x\colon T_g x=\tau^k x\}$ we have $\dfrac{dm^g}{dm}=\dfrac{dm^k}{dm}$. Hence also

(1) $$m_g = m_k \quad \text{on the set} \quad \{x\colon T_g x = \tau^k x\}.$$

Lemma 1. *Given* $\varepsilon>0$ *and a positive integer* k *there exists a finite set* $S\subseteqq G$ *such that if*

$$Q(k, g) = \{x\colon |C(g, x)| < 2^k\}$$

then

$$\sum_{g\notin S} m_g(Q(k, g)) < \varepsilon.$$

Proof. Choose $\delta>0$ so small that $m(B)<\delta$ implies $m_j(B)<\frac{\varepsilon}{2^{k+1}}$, $-2^k<j<2^k$. Let $g_1, g_2, g_3, \dots$ be a denumeration of G. Then $|C(g_n, x)|\to\infty$ as $n\to\infty$ for a.e. x. By Egorov's theorem there exists a set B of measure less than δ such that $|C(g_n, x)|\to\infty$ uniformly on $X-B$. Choose N so large that if $g\notin\{g_1, g_2, \dots, g_N\}=S$ then $|C(g, x)|\geqq 2^k$ for $x\in X-B$. Now

$$\begin{aligned}\sum_{g\notin S} m_g(Q(k, g)) &= \sum_{g\notin S}\sum_{|j|<2^k} m_g(x\colon C(g, x) = j)\\ &= \sum_{|j|<2^k}\sum_{g\notin S} m_g(x\colon C(g, x) = j)\\ &= \sum_{|j|<2^k}\sum_{g\notin S} m_j(x\colon C(g, x) = j)\end{aligned}$$

where the last step follows from (1). For fixed j, the set $\{x\colon C(g, x)=j\}$ are all disjoint as g runs over G and for $g\notin S$ they are all contained in B. Hence

$$\sum_{g\notin S} m_g(Q(k, g)) \leqq \sum_{|j|<2^k} m_j(B) < \varepsilon$$

by choice of the set B. Q.e.d.

Remark 3. If S satisfies the conclusion of the above lemma then any finite subset of G containing S also satisfies the conclusion of the lemma. In view of this we have the following.

Corollary 1. *There exist finite sets* $S_k\subset G$, $k=0, 1, 2, \dots$ *such that*

(i) $S_0=\emptyset$, $S_k \subseteqq S_{k+1}$ *for all* k
(ii) $\bigcup_{k=0}^{\infty} S_k = G$
(iii) *if* $g\in S_{k+1}-S_k$ *and if we set*

$$Q(g) = \{x\colon |C(g, x)| < 2^k\} = Q(k, g)$$

then $\sum_{g\in G} m_g(Q(g))<\infty$.

Proof. Choose positive ε_k so that $\sum_{k=1}^{\infty} \varepsilon_k < \infty$. For each $k \geqq 1$, $S_k \supseteqq S_{k-1}$ be so chosen that $\sum_{g \notin S_k} m_g(Q(k, g)) < \varepsilon_k$. This is possible by lemma above and remark 3.

We may assume that $\bigcup_{k=1}^{\infty} S_k = G$. Then

$$\sum_{g \in G} m_g(Q(g)) = \sum_{k=0}^{\infty} \sum_{g \in S_{k+1} - S_k} m_g(Q(k, g))$$
$$\leqq \sum_{k=0}^{\infty} \sum_{g \notin S_k} m_g(Q(k, g)) < \infty$$

where we have used also the fact that $Q(0, g)$ is empty except for $g=0$. Q.e.d.

Section 2

Let p be a real measurable function on X. We shall write $\sum^n p(\tau^k x)$ to mean the sum

$$\sum_{k=0}^{n-1} p(\tau^k x) \quad \text{if} \quad n > 0, \quad -\sum_{k=1}^{-n} p(\tau^{-k} x)$$

if $n<0$ and zero if $n=0$. The function $\varphi(n, x) = \exp\left(i \sum^n p(\tau^k x)\right)$ is then a $Z \times X$ cocycle (relative to τ) taking values in the circle group. The function $A(g, x) = \varphi(C(g, x), x)$ can be verified to be a $G \times X$ cocycle (relative to T_g, $g \in G$), i.e., $A(g, .)$ is a Borel function for each g and satisfies

$$A(g+h, x) = A(g, x) A(h, T_g x), \quad g, h \in G, \quad x \in X.$$

Define the unitary group V_g, $g \in G$, on $L^2(X, \mathscr{B}, m)$ by

$$(V_g f)(x) = A(g, x) f(T_g x) \sqrt{\frac{dm^g}{dm}}(x). \tag{2}$$

We are now ready to state the generalization of the result of Helson and Parry mentioned in the introduction. The proof follows the pattern of the first construction of their paper [4].

Theorem 1. *Let T_g, $g \in G$, be a non-singular action of G on $(X, \mathscr{B}, m)$ which is weakly equivalent to a weak von Neumann transformation. Then there exists a $G \times X$ cocycle A taking values $+1$ and -1 such that for all*

$$f \in L^2(X, \mathscr{B}, m), \quad \sum_{g \in G} |(V_g f, f)|^2 < \infty.$$

Proof. Let τ be a weak von Neumann transformation on X such that for a.e. $x \in X \{\tau^n x \colon n \in Z\} = \{T_g x \colon g \in G\}$. Let E_j, $(j = 1, 2, 3, \ldots)$, be disjoint measurable sets in X with characteristic function h_j. Form the random set E whose characteristic function h is $\sum \eta_j h_j$, where η_j's are independent random variables taking values 0 and 1 each with probability $\frac{1}{2}$. Define $p=\pi h$ which is now a function of x and w, where w is in the space on which η_j's are defined. Set

$$\varphi(n, x) = \exp\left(i \sum^n p(\tau^k x)\right)$$

$$A(g, x) = \varphi(C(g, x), x)$$

which are now random $Z\times X$ and $G\times X$ cocycles respectively. For $f\in L^\infty(X, \mathscr{B}, m)$ let $\varrho(g)=(V_g f, f)$, where V_g, $g\in G$, is defined by (2) using the random cocycle A defined above. $\varrho(g)$ depends on w. We shall show that E_j's can be so chosen that for all $f\in L^\infty(X, \mathscr{B}, m)$ $\sum_{g\in G} |\varrho(g)|^2<\infty$ for a.e. w. A routine calculation shows that

$$|\varrho(g)|^2 = \iint (*) \exp \pi i\Big(\sum_{j=1}^\infty \sum^{C(g,x)} \eta_j h_j(\tau^k x)$$

$$-\sum_{j=1}^\infty \sum^{C(g,y)} \eta_j h_j(\tau^k y)\Big)(**)\, dm(x)\, dm(y)$$

where $*$ in the integrand stands for the expression $f(T_g x)\bar{f}(x)\bar{f}(T_g y) f(y)$ and $**$ for the expression $\left(\frac{dm^g}{dm}(x)\cdot\frac{dm^g}{dm}(y)\right)^{\frac{1}{2}}$. Integrating over the probability space on which η_j's are defined gives

$$\int |\varrho(g)|^2\, dw = \iint (*) \prod_{j=1}^\infty \tfrac{1}{2}\Big[\Big(1+\exp \pi i \sum^{C(g,x)} h_j(\tau^k x) \tag{3}$$

$$-\sum^{C(g,y)} h_j(\tau^k y)\Big)\Big](**)\, dm(x)\, dm(y).$$

The product on the right hand side takes values 0, 1, and equals 1 on the set in $X\times X$ consisting of all (x, y) such that

$$\text{parity of } \sum^{C(g,x)} h_j(\tau^k x) = \text{parity of } \sum^{C(g,y)} h_j(\tau^k y) \text{ for all } j = 1, 2, 3, \ldots.$$

Set

$$a_j^g(x) = \begin{cases} 0 \text{ if } \operatorname{Orb}(x,\ C(g, x)) \text{ intersects } E_j \text{ in even number of points} \\ 1 \text{ if } \operatorname{Orb}(x,\ C(g, x)) \text{ intersects } E_j \text{ in odd number of points}\end{cases}$$

Let $a^g(x)=(a_1^g(x), a_2^g(x), \ldots)$, a sequence of zeros and ones, terminating in zeros since for any x, $\operatorname{Orb}(x, C(g, x))$ is a finite set and $E_1, E_2, E_3, \ldots$ are pairwise disjoint non-empty sets. For each sequence a of zeros and ones terminating in zeros let G_a^g be the set of $x\in X$ such that $a^g(x)=a$. For a fixed g, G_a^g form a disjoint covering of X as a runs over all sequences of zeros and ones terminating in zeros. Evidently $a^g(x)=a=a^g(y)$ if and only if x and y belong to G_a^g. The $m_g \times m_g$ measure of this set of (x, y) is $(m_g(G_a^g))^2$. Thus by (3) we have

$$\int |\varrho(g)|^2\, dw \leqq \|f\|_\infty^4 \sum_a (m_g(G_a^g))^2.$$

Since τ is a weak von Neumann transformation we have sets F_{k1}, F_{k2}, as per remark 2, for $k=1, 2, 3, \ldots$. Let S_k's be as in corollary of lemma 1. Since S_k is finite and m_g, $g \in G$, are all absolutely continuous with respect to m we can decompose F_{k1}, F_{k2} into finitely many sets $F_k^1, \ldots, F_k^{l_k}$ such that for all $g \in S_k$ the sets $\bigcup_{|s|<2^{k+1}} \tau^s F_k^l$ have m_g measure less than δ_k, where δ_k will be chosen later. Let E_j, $j=1, 2, 3, \ldots$ be a denumeration of sets F_k^l, $l=1, \ldots, l_k$, $k=1, 2, 3, \ldots$. These are the sets needed to prove the theorem. The E_j's are disjoint by construction. Each E_j is contained in a unique $F_k = F_{k1} \cup F_{k2}$. If $2^k \leqq |n| < 2^{k+1}$, then by remark 2, for each x, Orb (x, n) intersects F_k and then such an orbit intersects each F_k^l in at most one point since each F_k^l is contained in only one of F_{k1} or F_{k2}.

Fix g and choose r such that $g \in S_r - S_{r-1}$. The measure of $m_g(G_a^g)$ is to be estimated. Now

$$m_g(G_a^g) \leqq \sum_{k=r}^{\infty} m_g\big(G_a^g \cap \{x\colon 2^k \leqq |C(g, x)| < 2^{k+1}\}\big) + m_g(Q(g))$$

where $Q(g)$ is as in corollary 1. We now estimate the k^{th} term under the summation. Let $a=(a_1, a_2, \ldots)$. Suppose $a_j=0$ for each j for which $E_j \subseteqq F_k$. If $2^k \leqq |C(g, x)| < 2^{k+1}$, then Orb $(x, C(g, x))$ intersects some $E_j \subseteqq F_k$ in exactly one point. Hence $G_a^g \cap \{x\colon 2^k \leqq |C(g, x)| < 2^{k+1}\}$ is empty if $a_j=0$ for each j with $E_j \subseteqq F_k$. For such a,

$$m_g\big(G_a^g \cap \{x\colon 2^k \leqq |C(g, x)| < 2^{k+1}\}\big) = 0.$$

Otherwise $a_j=1$ for at least one j such that $E_j \subseteqq F_k$. The set

$$G_a^g \cap \{x\colon 2^k \leqq |C(g, x)| < 2^{k+1}\}$$

is then contained in the set of all x such that Orb $(x, C(g, x))$ intersects E_j and $2^k \leqq |C(g, x)| < 2^{k+1}$. Thus

$$G_a^g \cap \{x\colon 2^k \leqq |C(g, x)| < 2^{k+1}\} \subseteqq \bigcup_{|s|<2^{k+1}} \tau^s E_j$$

$$m_g(G_a^g \cap \{x\colon \cap 2^k \leqq |C(g, x)| < 2^{k+1}\}) \leqq \delta_k$$

$$m_g(G_a^g) \leqq \sum_{k \geqq r} \delta_k + m_g(Q(g)) = \gamma_r + m_g(Q(g))$$

where we have put $\gamma_r = \sum_{k \geqq r} \delta_k$.

Now for a fixed $g \in S_r - S_{r-1}$, G_a^g form a disjoint covering of X as a runs over sequences of zeros and ones terminating in zeros. Hence

$$\sum_a (m_g(G_a^g))^2 \leqq (\gamma_r + m_g(Q(g)) \cdot \sum_a m_g(G_a^g)) \leqq \gamma_r + m_g(Q(g)).$$

Finally summing over $g\in G$ we get

(4) $\sum_{g\in G}(\sum_a(G_a^g))^2 \leqq \sum_{r=1}^{\infty}$ number of elements in $(S_r-S_{r-1})\cdot\gamma_\gamma+\sum_{g\in G}m_g(Q(g))$.

We now choose δ_k's in such a way that the first sum on the right hand side of (4) is convergent. The second sum is convergent by corollary 1. Thus $\sum_{g\in G}\int|\varrho(g)|^2dw<\infty$. Hence for almost every w, $\sum_{g\in G}|(V_gf,f)|^2<\infty$, the null set of w where $\sum_{g\in G}|(V_gf,f)|^2$ may not converge depends on f. But $L^2(X,\mathscr{B},m)$ is separable, hence there is a grand null set N of w points such that $\sum_{g\in G}|(V_gf,f)|^2$ converges for every $w\notin N$ and every $f\in L^\infty(X,\mathscr{B},m)$ hence also by approximation for every $f\in L^2(X,\mathscr{B},m)$. Q.e.d.

Remark 4. In case G is a countable abelian group the above theorem shows that A can be so chosen that the unitary group V_g, $g\in G$, has spectral measure absolutely continuous with respect to Haar measure on the compact dual of discrete G.

Remark 5. If T_g^1, $g\in G_1,\dots,T_g^n$, $g\in G_n$ are finitely many non-singular actions of countable groups $G_1,\dots,G_n$ all having orbits same as a single weak von Neumann transformation τ on X, then there exists a single $Z\times X$ cocycle φ relative to τ such that the associated V_g^j, $g\in G_j, j=1,\dots,n$ defined by (2) all satisfy $\sum_{g\in G_j}|V_gf,f)|^2<\infty$, for $f\in L^2(X,\mathscr{B},m)$. This can be accomplished by choosing δ_k's in the proof of the theorem suitably small.

Section 3

In this section we give an application of theorem 1 to systems of imprimitivity (see [1], [6]).

Let X denote the circle group. Let $G\subseteqq X$ be a countably infinite subgroup with discrete topology. Let K be the compact dual of G. The identity map e: $e(g)=g$, $g\in G$, is a character of G, hence an element of K. Translation by e is an ergodic action on K equipped with Haar measure h on its Borel sets. Let H be a complex separable Hilbert space and consider $L^2(K,H,h)$, the space of H valued square integrable functions on K. Let φ be a Borel function on K whose values are unitary operators on H. Finally let unitary operator V^φ be defined by

$$(V^\varphi f)(x)=\varphi(x)f(x+e),\quad f\in L^2(K,H,h).$$

It is known that the spectral measure E^φ of V^φ is of uniform multiplicity. Further any non-negative finite measure m on X having the same null sets as E^φ is quasi-invariant and ergodic under translation action of the group G on X.

Theorem 2. *Given a non-negative finite measure m on X quasi-invariant and ergodic under action of G by translation, there exists a φ such that the spectral measure E^φ has multiplicity one and has same null sets as m.*

Proof. As permitted by theorem 1 let A be a $G\times X$ cocycle of absolute value one such that V_g, $g\in G$, as defined by (2) has absolutely continuous spectrum. It is known that such a V_g, $g\in G$, has uniform multiplicity, say $n\leqq\aleph_0$. Let H be an n-dimensional complex Hilbert space and let S be an isometry from $L^2(X, \mathcal{B}, m)$ onto $L^2(K, H, h)$ such that for all g, $SV_gS^{-1}=$multiplication by χ_g, where χ_g is the character on K corresponding to $g\in G$. If U is the operator on $L^2(X, \mathcal{B}, m)$ given by $(Uf)(x)=xf(x)$, $f\in L^2(X, \mathcal{B}, m)$ then it is known that $SUS^{-1}=V^\varphi$ for some φ. But U has multiplicity one and its spectral measure has the same null sets as m. Q.e.d.

Section 4

In this section we prove that the result of Helson and Parry quoted in the introduction is best possible in a sense made precise by the following theorem.

Theorem 3. *If μ is a continuous probability measure on the circle group singular with respect to Haar measure, then there is an ergodic T such that μ does not appear in the spectrum of V^φ for any φ, i.e., there is a support S of μ such that $E^\varphi(S)=0$, where E^φ is the spectral measure of V^φ.*

Proof. Let S be a Borel set of Haar measure zero on which μ is supported. Then the set $F=\{\lambda: \mu(S\cap\lambda S)>0\}$ has Haar measure zero, for otherwise the Haar measure of S is seen to be positive (contrary to assumption) by convoluting μ with Haar measure. The sets $F_n=\{\lambda: \lambda^n\in F\}$, $n\neq 0$, then have Haar measure zero. If $\lambda\notin\bigcup_{n\neq 0} F_n$, then $\mu(S\cap\lambda^n S)=0$ for all $n\neq 0$. Choose $\lambda\notin\bigcup_{n\neq 0} F_n$ such that $\{\lambda^n: n\in Z\}$ is dense in X, the circle group. Define T on X by $Tx=\lambda x$. T acts ergodically on X equipped with its Haar measure m. Let φ be any Borel function on X of absolute value one. Define V^φ on $L^2(X, \mathcal{B}, m)$ by $(V^\varphi f)(x)=\varphi(x)f(Tx)$, for a.e. $x\in X$ with respect to Haar measure. Let E^φ denote the spectral measure of V^φ. Then it is known that E^φ is ergodic under T, which means that if a Borel set B is invariant under T then $E^\varphi(B)$ is either zero or the identity operator. Since the support S of μ is a wandering set under T, i.e., $\mu(S\cap T^n S)=0$ for all $n\neq 0$, it follows that $E^\varphi(S)=0$ if E is continuous. If E is discrete, then the result is obvious, since μ is continuous. Q.e.d

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