Mappings that preserve Sidon sets in R

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0. Introduction and statement of results. A closed subset E of R is a *Helson set* if for every $f \in C_0(E)$ there exists a Fourier transform $g \in A(R)$ such that $g|_E = f$. A *Sidon set* is a countable Helson set. Every function h: $R \to R$ that induces (by composition) an automorphism of $A(R)$ clearly maps Helson sets to Helson sets; such h are exactly the affine functions [BH]. Since the union of two Helson sets is again a Helson set (that is due to Drury and Varopoulos; see [GM, chapter 2] for a proof), every continuous function f: $R \to R$ whose graph consists of a finite number of straight line segments (possibly of infinite length) maps Helson sets to Helson sets.

It is therefore reasonable to ask whether functions f that map Helson sets to Helson sets have graphs consiting of a finite number of straight line segments. Theorem 1 shows that if the homeomorphism f: $R \to R$ maps Sidon sets to Sidon sets, then the graph of f consists of a countable number of straight line segments having a finite number of distinct slopes. Theorem 1 and its proof appear in Section 1. In Section 2 we give an example that shows that the condition of Theorem 1 cannot be improved globally. Whether Sidon set-preserving homeomorphisms are locally piecewise affine is unknown. Section 3 contains our concluding remarks.

1. Proof of the main result. We will use the following definitions. A continuous function f: $R \to R$ is c.p.a. *(countably piecewise affine)* if the set of x such that f is affine in a neighborhood of x is dense in R. The c.p.a. function f *has a finite number of slopes* if the slopes of the segments of the graph of f belong to a finite set.

Theorem 1. *Let f: $R \to R$ be a homeomorphism such that $f(E)$ is a Sidon set whenever E is a Sidon set. Then f is c.p.a. with a finite number of slopes.*

We will prove Theorem 1 by proving a slightly stronger result (Theorem 2). For that we need the following.

It is well-known that a Sidon or Helson set cannot contain a sequence of subsets of the form $A_n + B_n$ where $\text{Card}\, A_n = \text{Card}\, B_n = n$ for $n \geqq 1$. A set E is *dissociate* if for all $n \geqq 1$, distinct elements $x_1, \ldots, x_n \in E$ and choices $\varepsilon_1, \ldots, \varepsilon_n \in \{0, \pm 1, \pm 2\}$, $\sum \varepsilon_j x_j = 0$ only if all the ε_j are zero. A countable closed dissociate set is necessarily Sidon. If $E_1, E_2, \ldots$ is a sequence of disjoint subsets of R, we say that $\cup E_j$ is an *independent union* if $\mathrm{Gp}\, E_j \cup \mathrm{Gp} \bigcup_{k \neq j} E_k = \{0\}$. If a countable union E of dissociate sets is an independent union and has at most one cluster point, then E is a Sidon set. For proofs see [LR] or [GM].

Theorem 2. (i) *Let $g\colon R \to R$ be a continuous function that is not c.p.a. Then there exist sequences $\{A_n\}$, $\{B_n\}$ of subsets of R such that for all n,* Card $A_n =$ Card $B_n = n$ *and $g(A_n + B_n)$ is dissociate and such that $\bigcup_1^\infty g(A_n + B_n)$ is an independent union with at most one cluster point.*

(ii) *Let $g\colon R \to R$ be c.p.a. with an infinite number of distinct slopes. Then there exist sequences $\{A_n\}$ and $\{B_n\}$ with the same properties of* (i).

How Theorem 2 *implies Theorem* 1. In both cases (i) and (ii) of Theorem 2 we set $g = f^{-1}$. Let $E = g(A_n + B_n)^-$, where the A_n and B_n are given by Theorem 2 (i) or (ii), assuming that f is not c.p.a. with a finite number of slopes. Then E is a Sidon set. But $f(E) = \bigcup_1^\infty (A_n + B_n)^-$ contains arbitrarily large squares, so f cannot preserve Sidon sets. Theorem 1 thus follows from Theorem 2.

We shall need the following Lemma for the proof of Theorem 2 (i).

Lemma 3. *Let $g\colon R \to R$ be a continuous. Let $m \geqq 1$. Suppose that there exist integers $\{n_{i,j}\}_{i,j=1}^m$ such that the function $h(u_1, v_1, \ldots, u_m, v_m) = \sum_{i,j=1}^m n_{i,j} g(u_i + v_j)$ from R^{2m} to R is constant on an open set. Then either all the $n_{i,j}$ are zero, or g agrees with an affine function on an open set.*

Proof. Let $\varepsilon > 0$ and $I_\varepsilon = \{(x_1, \ldots, x_{2m}) \in R^{2m}\colon |x_j| < \varepsilon$ for all $j\}$. Suppose that h is constant on $(u_1', v_1', \ldots, u_m', v_m') + I_\varepsilon$, and that not all $n_{i,j} = 0$. Let $\{f_k\}$ be any bounded approximate identity for $L^1(R)$ such that each f_k is twice continuously differentiable and has support in $I_{\varepsilon/2}$. Then the function defined by

$$h_k(u_1, \ldots, v_m) = \sum_{i,j}^1 n_{i,j} (f_k * g)(u_i + v_j)$$

is constant in $X = (u_1', \ldots, v_m') + I_{\varepsilon/2}$ and is twice continuously differentiable.

Suppose that $n_{i,j} \neq 0$. Then $\dfrac{\partial^2 h_k}{\partial v_j \partial u_i} = 0$ in X. But

$$\frac{\partial^2 h_k}{\partial v_j \partial u_i} = \frac{\partial}{\partial v_j} \sum_l n_{i,l} (f_k * g)'(u_i + v_l) = n_{ij} (f_k * g)''(u_i + v_j).$$

Therefore $f_k * g$ is affine in the interval $I = (u_i' + v_j' - \varepsilon/2,\ u_i' + v_j' + \varepsilon/2)$. Since $f_k * g \to g$ uniformly in $I_{\varepsilon/2}$, g is affine in $I_{\varepsilon/2}$. The Lemma follows.

Proof of Theorem 2 (i). Suppose that g is not c.p.a. Then there exists an open interval $(a, b)\neq 0$ such that g is not affine on any open non-empty subset of (a, b). Without loss of generality, we may assume that $a<0<b$ (translate) and that $b>1$ (change scale). Let $A_1=B_1=\{1\}$.

Suppose that $n\geqq 1$ and that sets $A_1, B_1, \ldots, A_n, B_n$ have been found such that, for all $1\leqq m\leqq n$,

(1) $A_m\cup B_m\subseteq[0, 2^{-m}]$, Card $A_m=$Card $B_m=m$;
(2) $g(A_m+B_m)$ is dissociate; and
(3) $\bigcup_{k=1}^{m} g(A_k+B_k)$ is an independent union.

Let $H=\bigcup_{j=1}^{\infty}\mathrm{Gp}\left(\frac{1}{j}\bigcup_{k=1}^{n} g(A_k+B_k)\right)$.
For all sets $\eta=\{n_{ij}\}_{i,j=1}^{n+1}$ of integers, let

$$X(\eta)=\{(u_1, v_1, \ldots, u_{n+1}, v_{n+1})\in[0,2^{-n-1}]^{2n+2}\colon \sum n_{ij}g(u_i+v_1)-H\}.$$

Since H is countable, Lemma 3 implies that $X(\eta)$ is a union of closed sets, each having no interior. Let X be the union of the $X(\eta)$ as η ranges over all subsets $\{n_{i,j}\colon i,j=1, \ldots, n+1\}$ of $(n+1)^2$ integers. Then X is also of first (Baire) category, so there exists $(u_1, v_1, \ldots, u_{n+1}, v_{n+1})\in[0, 2^{-n-1}]^{2n+2}\setminus X$. Then, for $A_{n+1}=\{u_1, \ldots, u_{n+1}\}$ and $B_{n+1}=\{v_1, \ldots, v_{n+1}\}$, we see that (1)—(3) hold for $m=n+1$. Now Theorem 2 (i) follows.

Proof of Theorem 2 (ii). Suppose that g is c.p.a. having the form $g(x)=a_kx+b_k$ in the interval $[l_k, r_k]$, $k=1, 2, \ldots$. Suppose also that $\{a_k\}$ contains an infinite set of distinct numbers.

Let $\{a_{k(j)}\}_{j=1}^{\infty}$ be such that the $a_{k(j)}$ are distinct. Without loss of generality we may assume that $l_{k(j+1)}<l_{k(j)}$ for $j\geqq 1$ (we may need to replace $g(x)$ by $g(-x)$). If $\lim l_{k(j)}=l$ exists, we may assume that $l=0$. By passing to a subsequence $\{a_{k(j)}\}$ we may assume that for all choices $m\geqq 1$ and $n_j=\{0, \pm 1, \pm 2\}$ for $1\leqq j\leqq m$.

$$\sum_1^m n_j a_{k(j)} = 0 \quad \text{only if} \quad n_1 = \ldots = n_m = 0. \tag{4}$$

(That is merely the routine matter of extracting a dissociate subsequence from $\{a_{k(j)}\}$.)

We now consider sums of the form $S=\sum_{i,j=1}^{m} n_{i,j}(a_{k(i)}(u_i+v_j)+b_i)$, where the $n_{i,j}$ are chosen from $\{0, \pm 1, \pm 2\}$, the u_i from $\left[l_{k(i)}, \frac{1}{2}(l_{k(i)}+r_{k(i)})\right]$, and the v_j from $\left[0, \frac{1}{2}(r_{k(i)}-l_{k(i)})\right]$. The set X of $(u_1, v_1, \ldots, u_m, v_m)$ such that $S=0$ has no interior, for if X had interior then on varying the u_i and v_j, we would conclude that $\sum_i n_{i,j}\, a_{k(i)}=0$ for each j, thus contradicting (4). The argument now proceedes exactly as in the proof of Part (i). The remaining details are left to the reader.

2. An example. The following example shows that we cannot conclude that a mapping preserving Helson sets is affine on neighborhoods of $+\infty$ and $-\infty$. We define $f\colon R\to R$ as follows.

$$(5)\qquad f(x)=\begin{cases}\frac{1}{2}x+n & 2n\leqq x\leqq 2n+1\\ \frac{3}{2}x-n-1 & 2n+1\leqq x\leqq 2n+2 \quad \text{for} \quad n=1,2,\ldots\\ x & \text{otherwise.}\end{cases}$$

We claim that f preserves Helson sets in R. It will suffice to show that for each $\alpha>0$ there is a constant C such that the Helson constant of $f(E)$ is at most C if the Helson constant of E is at most α; here E ranges over compact Helson sets. Any compact Helson set has the form $E=\bigcup_{n=1}^{\infty}\bigcup_{j=1}^{16}E_{n,j}$ where $E_{0,j}=(-\infty,2]\cap E$ and $E_{n,j}=E\cap[2n+(j-1)/8,\ 2n+j/8]$ for $1\leqq j\leqq 16$ and $n=1,2,\ldots$. Application of the Saucer principle [GM, 11.4] shows that, for all j's and all $m\geqq 1$, the algebras $A\big(\bigcup_{n=1}^{m}[2n+(j-1)/8,\ 2n+j/8]\big)$ and $A\big([(j-1)/8, j/8]\times\{2,\ldots,2m\}\big)$ are isomorphic. For $1\leqq j\leqq 8$, f maps $[(j-1)8, j/8]+2n$ linearly to $[(j-1)/16, j/16]+2n$; for $8<j\leqq 16$, f maps $[(j-1)/8, j/8]+2n$ linearly to $[3(j-1)/16, 3j/16]+2n-1$. The Saucer Principle again applies: $A\big(\bigcup_1^m[a(j-1)/16,\ aj/16]+2n-b\big)$ and $A\big([a(j-1)/61, aj/16]\times\{2n+b\colon n=1,m,m\}\big)$ are isomorphic, where $a=1$, $b=0$ for $1<j\leqq 8$ and $a=3$, $b=1$ for $8<j\leqq 16$. Therefore for all $1\leqq j\leqq 16$, and $m\geqq 1$, $\alpha\big(f(\bigcup_1^m E_{n,j})\big)\leqq C_1\alpha\,\big(\bigcup_1^m E_{n,j}\big)$. By the union theorem for Helson sets, $\alpha\big(f(E)\big)\leqq C$.

3. Remarks. (i) This paper was stimulated by a question asked by R. S. Pierce at a seminar in Honolulu.

(ii) If the f of Theorem 1 is only assumed to preserve *compact* Sidon sets, then f is c.p.a. and the restrictions of f to each compact interval have but a finite number of slopes (the number may increase with the size of the interval).

(iii) A "compact" version of the example of Section 2 could be given if we could answer the following affirmatively.

Do there exist sequences $x_n\geqq 0$, $0\leqq a_n\leqq b_n$ with $\lim b_n=\lim x_n=0$ such that for every Sidon set $E_{n=1}^{\infty}(E\cap[a_n,b_n])+x_n$ is a Sidon set?

(iv) Our Theorem 1, combined with the Lemma of Beurling and Helson [BH, P. 121] yields immediately their Corollary:
if f: R→R preserves Fourier transforms, then f is affine.

(v) Our definition of "c.p.a." is a weak one. One might replace it with the following: f is c.p.a. if for all $L>0$, the sum of the lengths of the maximal intervals (a,b) in $[-L,L]$ such that f is affine in (a,b) equals $2L$. We do not know if Helson set preserving maps have that property.

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