Cyclic elements under translation in weighted L^1 spaces on $\mathbf{R}^+$

Yngve Domar

0. Introduction

We shall be concerned with a closure problem for functions on $\mathbf{R}^+$. In order to illuminate the situation we start by presenting the corresponding problem for $\mathbf{Z}^+ \cup \{0\}$.

Let $w=(w_n)_0^\infty$ be a non-negative decreasing sequence, not identically vanishing, and satisfying $n^{-1} \log w_n \to -\infty$, as $n \to \infty$ (here $\log 0 = -\infty$). ℓ_w is the Banach space of complex-valued sequences $c=(c_n)_0^\infty$ with

$$\|c\|_w = \sum_0^\infty |c_n| w_n < \infty.$$

For every $m \in \mathbf{Z}^+ \cup \{0\}$, the translation operator T_m, defined by

$$(T_m c)_n = \begin{cases} 0, & 0 \leqq n < m, \\ c_{n-m}, & n \geqq m, \end{cases}$$

is a contraction in ℓ_w. A_w is the set of all $c \in \ell_w$ with $c_0 \neq 0$. B_w is the set of all $c \in \ell_w$ which are *cyclic* in the sense that the translates $T_m c$, $m \geqq 0$, span a dense subspace. Obviously $A_w \supseteq B_w$. Is $A_w = B_w$?

It is known that the answer to this question is yes if, for some constant $C > 0$, the sequence $(Cw_n)_0^\infty$ is submultiplicative on the additive semigroup $\mathbf{Z}^+ \cup \{0\}$. This is a direct consequence of the fact that ℓ_w is then a commutative unital Banach algebra under convolution, such that all closed translation invariant subspaces are ideals, and $\ell_w \setminus A_w$ is the only maximal ideal. In some other cases, too, it has been shown that $A_w = B_w$ (Styf [10]). On the other hand, there are weight sequences w, some of them very close to being of the above-mentioned submultiplicative type, and for which $A_w \neq B_w$ (Nikolskii [7], Styf [10]). Roughly speaking, equality holds if the decrease at infinity for w is sufficiently regular, whereas an irregular behavior can cause inequality.

We shall now formulate the analogous problem for $\mathbf{R}^+$. w is then a non-negative, bounded, decreasing function on $\mathbf{R}^+$, not identically vanishing, and satisfying $x^{-1}\log w(x)\to-\infty$, as $x\to\infty$. L_w is the Banach space of Lebesgue measurable complex-valued functions f on $\mathbf{R}^+$ with

$$\|f\|_w=\int_0^\infty |f(x)|\,w(x)\,dx<\infty.$$

A function w of this kind is called a *weight function.* For every $a\in\mathbf{R}^+\cup\{0\}$, the translation operator T_a, defined by

$$T_a f(x)=\begin{cases}0, & 0<x\leqq a,\\ f(x-a), & x>a,\end{cases}$$

is a contraction in L_w. A_w consists of every $f\in L_w$ with $0\in\operatorname{Supp}(f)$. B_w is the set of all $f\in L_w$ which are *cyclic* in the sense that the translations $T_a f$, $a\geqq 0$, span a dense subspace. Obviously $A_w\supseteqq B_w$. Is $A_w=B_w$?

The above-mentioned counter-examples of Nikolskii and Styf can be carried over to counter-examples for $\mathbf{R}^+$, simply by changing sequences to step-functions. Details of this are given in Dales and McClure [4], where also counter-examples of higher regularity are constructed.

Thus $A_w\neq B_w$ may occur. As for positive results it is tempting to conjecture, in analogy to the situation on $\mathbf{Z}^+\cup\{0\}$, that $A_w=B_w$ if, for some $C>0$, Cw is submultiplicative on the additive semigroup $\mathbf{R}^+$. Then L_w is a commutative Banach algebra under convolution. But this time the algebra is radical, and elementary Banach algebra theory does not suffice to provide a confirmation of the conjecture. As a matter of fact, for no strictly positive w of this submultiplicative type do we know whether or not $A_w=B_w$. Perhaps we have inequality for every w, or at least for some w. If w vanishes somewhere, then since w is decreasing it vanishes for all larger values of the variable and $A_w=B_w$ is an immediate consequence of Titchmarsh's convolution theorem (Titchmarsh [11], Boas [3]) and elementary functional analysis.

Our results are thus very incomplete. We present different sets of conditions on the function $f\in A_w$ which imply that $f\in B_w$. In Theorem 1, a corollary of results of Nyman [8], we demand that f is not too large at infinity. In Theorems 2 and 3, we make instead assumptions which prevent f from being too small at 0. In these last theorems, it is necessary to assume additional regularity and growth conditions on w.

There is some overlap with the paper [1], which presents a similar approach, and which has been taken into consideration in the final draft of our paper. Other papers, dealing with the conjecture $A_w=B_w$, and giving interesting information on the problem, are Bade and Dales [2], and Rubel [9]. A summary of the present paper was given in [5].

1. From now on we restrict ourselves to strictly positive weight functions w. L_w^* is the dual of L_w, identified with the Banach space of complex-valued functions φ on $-\mathbf{R}^+$ with $\varphi/\check{w}\in L^\infty(-\mathbf{R}^+)$, $\|\varphi\|_w^*=\|\varphi/\check{w}\|_\infty$. Here $\check{w}(x)=w(-x)$, $x\in-\mathbf{R}^+$. Thus

$$\langle\varphi, f\rangle = \int_0^\infty \varphi(-x)f(x)\,dx = \varphi * f(0),$$

for every $f\in L_w$. Convolution of functions, defined on subsets of $\mathbf{R}$, is defined (whenever definable) by first giving the functions the value 0 on the complement of their sets of definition. (n) in the exponent denotes n-fold convolution.

Theorem 1. *Let $f\in A_w$ and $\int_0^\infty |f(x)|e^{-bx}\,dx<\infty$, for some $b\in\mathbf{R}$. Then $f\in B_w$.*

Proof. If $f\notin B_w$, Hahn—Banach's theorem gives a non-zero element $\varphi\in L_w^*$, such that

$$\varphi * f(x) = \varphi * T_{-x}f(0) = \langle\varphi, T_{-x}f\rangle = 0, \quad x\in-\mathbf{R}^+.$$

Putting $f(x)e^{-bx}=g(x)$, $\varphi(-x)e^{bx}=\psi(x)$, we obtain

$$\int_0^\infty \psi(y+t)g(t)\,dt = 0, \quad t\in\mathbf{R}^+,$$

where $\psi\in L^\infty(\mathbf{R}^+)$, $g\in L^1(\mathbf{R}^+)$. A theory for convolution equations of this type has been developed by Nyman [8], and we can obtain a contradiction directly from his results. By Titchmarsh's convolution theorem, $f\in A_w$ implies that the support of ψ is non-compact. Hence, by Theorem 1 in [8], the spectrum Λ_ψ of ψ is non-empty (spectrum is defined in § 8 of [8]). By Theorem 2 in [8], Λ_ψ coincides with the set of singularities of the analytic continuation to $\mathbf{C}$ of the Laplace transform of ψ. But in our case this continuation is entire. Hence Λ_ψ is empty, and we have a contradiction.

Remarks. The paper [8] is not easily accessible. An alternative reference, containing the needed results, is Gurarii [6]. In [1] a simple proof is given, which avoids Nyman's theory.

For the remaining theorems we need the following lemma. In [1] there is a similar result (Lemma 5), which is applicable to more general weight functions. We shall from now on assume that $\log w$ is concave. This implies that w is of sub-multiplicative type, thus L_w is a Banach algebra under convolution.

Lemma. *Let w be a weight function such that $\log w$ is concave on $\mathbf{R}^+$, and $x^{-1}\log w(x)\to-\infty$, as $x\to\infty$. Let $f\in L_w$ and let $\varphi\in L_w^*$ be in the annihilator of the subspace of L_w spanned by f and its translates. If $f_1\in L_w$ coincides with f on $]0,\varepsilon]$, $\varepsilon>0$, then*

$$\|\varphi * f_1^{(n)}\|_w^* \leqq \frac{w(n\varepsilon)}{w(\varepsilon)^n}\|\varphi\|_w^*\|f-f_1\|_w^n, \quad n\in\mathbf{Z}^+.$$

Proof. Since $\log w$ is concave,

$$w(x_1+x_2+\ldots+x_n)\,w(x_1)^{-1}\,w(x_2)^{-1}\cdot\ldots\cdot w(x_n)^{-1}$$

decreases in each variable individually, if all $x_i>0$. Hence

$$w(x_1+x_2+\ldots+x_n) \leqq \frac{w(n\varepsilon)}{w(\varepsilon)^n}\, w(x_1)\,w(x_2)\cdot\ldots\cdot w(x_n), \tag{1}$$

if $x_i \geqq \varepsilon$, $i=1, 2, \ldots, n$. We put $f_1-f=f_2$. Then $f_2\in L_w$, and

$$f_1^{(n)}-f_2^{(n)}=fP(f_1, f),$$

where $P(f_1,f)$ is a polynomial in f_1 and f under the convolution operation. Hence $f_1^{(n)}-f_2^{(n)}$ is included in the ideal in L_w which is generated by f. Elementary considerations show that every translate of $f_1^{(n)}-f_2^{(n)}$ then has to be contained in the closed subspace, generated by f and its translates. Hence

$$\varphi*(f_1^{(n)}-f_2^{(n)}) = 0,$$

on $-\mathbf{R}^+$. This gives

$$\|\varphi*f_1^{(n)}\|_w^* = \|\varphi*f_2^{(n)}\|_w^*$$

$$= \sup_{x\in\mathbf{R}^+} w(x)^{-1}\int_{\mathbf{R}^{+n}} |\varphi(-x-x_1-x_2-\ldots-x_n)|\,|f_2(x_1)|\,|f_2(x_2)|\cdot\ldots\cdot|f(x_n)|\,dx_1\,dx_2\cdot\ldots\cdot dx_n$$

$$\leqq \|\varphi\|_w^*\int_{\mathbf{R}^{+n}} w(x_1+x_2+\ldots+x_n)\,|f_2(x_1)|\,|f_2(x_2)|\cdot\ldots\cdot|f_2(x_n)|\,dx_1\,dx_2\cdot\ldots\cdot dx_n.$$

By (1), the right hand member is majorized by

$$\|\varphi\|_w^*\, w(n\varepsilon)\,w(\varepsilon)^{-n}\,\|f_2\|_w^n,$$

and the lemma is proved.

2. In the sequel, we use the convention that, for a complex-valued function g, defined on a set $E\subset\mathbf{R}$, the Fourier transform G is defined by

$$G(\zeta) = \int_E g(x)e^{i\zeta x}\,dx,$$

for all $\zeta\in\mathbf{C}$ which give absolute convergence. This means in particular, that if φ and f_1 are as in the lemma, with $\operatorname{Supp}(f_1)$ compact, then their Fourier transforms Φ and F_1 are entire functions, and the Fourier transform of $\varphi*f_1^{(n)}$ is ΦF_1^n.

Theorem 2. *Let w be a weight function with* $\log w$ *concave, and such that*

$$x^{-2}\log w(x) \to -\infty, \tag{2}$$

as $x\to\infty$. For sufficiently large $\eta>0$ we define

$$M(\eta) = w^{-1}\Big(\Big[\int_0^\infty e^{\eta x} w(x)\,dx\Big]^{-1}\Big), \tag{3}$$

where w^{-1} *is the inverse of* w. *We assume that* f *and* f_1 *are as in the lemma, with* Supp (f_1) *compact, and that there is a constant* $C>0$ *such that*

$$|F_1(i\eta)| \geqq \exp\{-C\eta/M(\eta)\}, \tag{4}$$

for sufficiently large positive η. *Then* $f\in B_w$.

Proof. Simple estimates show that (2) implies

$$\eta^{-2}\log\left[\int_0^\infty e^{\eta x} w(x)\,dx\right] \to 0, \tag{5}$$

as $\eta\to\infty$. It follows from (3) and (5) that

$$\eta^{-2}\log w\big(M(\eta)\big) \to 0,$$

as $\eta\to\infty$, and this and (2) imply that

$$M(\eta)/\eta \to 0, \tag{6}$$

as $\eta\to\infty$, $\eta\in\mathbf{R}^+$.

Let φ be an arbitrary element in L_w^*, annihilating f and its translates. It suffices to prove that φ vanishes almost everywhere. We are of course free to assume that $\|\varphi\|_w^*\leqq 1$, and that $|\varphi(x)|\leqq 1$, $x\in -\mathbf{R}^+$. Then, for $x\in\mathbf{R}^+$,

$$|\varphi * f_1^{(n)}(x)| \leqq \int_0^\infty |\varphi(x-y)|\,|f_1^{(n)}(y)|\,dy \leqq \Big(\int_0^\infty |f_1(y)|\,dy\Big)^n. \tag{7}$$

For $x\in -\mathbf{R}^+$, the lemma gives

$$|\varphi * f_1^{(n)}(x)| \leqq w(-x)\,w(n\varepsilon)\,D^n, \tag{8}$$

for some constant D, independent of n and x. (7) and (8) give the following estimate of the Fourier transform of $\varphi * f_1^{(n)}$, for $\zeta = i\eta$, η positive and large,

$$|\Phi(i\eta)\,F_1(i\eta)^n| \leqq \int_{-\infty}^0 w(-x)\,w(n\varepsilon)\,D^n e^{-\eta x}\,dx \tag{9}$$

$$+\Big(\int_0^\infty |f_1(y)|\,dy\Big)^n \int_0^\infty e^{-\eta x}\,dx = \frac{w(n\varepsilon)}{w\big(M(\eta)\big)}D^n + \frac{1}{\eta}\Big(\int_0^\infty |f_1(y)|\,dy\Big)^n,$$

where the last inequality follows from (3).

For every sufficiently large $\eta\in\mathbf{R}^+$ we choose $n=n(\eta)$ as the smallest positive integer, such that $n\varepsilon\geqq M(\eta)$. Then, for large η,

$$M(\eta) \leqq n\varepsilon \leqq 2M(\eta). \tag{10}$$

Then there is a constant $E>0$, such that the right hand member of (9) is $\leqq E^n$, if η is large. Thus (9) and (4) give, for large η,

$$|\Phi(i\eta)| \leqq E^n |F_1(i\eta)|^{-n} \leqq E^n \exp\left\{\frac{Cn\eta}{M(\eta)}\right\}. \tag{11}$$

(6) and (10) show that (11) implies that

$$|\Phi(i\eta)| \leqq e^{C_0\eta}, \tag{12}$$

for some constant C_0, if $\eta\in\mathbf{R}^+$ is large enough.

Now (5) shows that Φ is of order 2, type 0, and obviously Φ is bounded in the lower half-plane. Therefore (12) can be used in a standard application of the Phragmén—Lindelöf principle to the upper quadrants to show that Φ is of exponential type. It is well known (see for instance Boas [3]), that this implies that Supp (φ) is compact. Returning to the relation $\varphi*f=0$ in $-\mathbf{R}^+$, Titchmarsh's theorem shows that we have two alternatives, $\varphi=0$ almost everywhere or $f\notin A_w$. In the second case, there exists a positive constant D such that

$$|F_1(i\eta)| \leqq e^{-D\eta}, \tag{13}$$

for large positive η. But $M(\eta)\to\infty$, as $\eta\to\infty$, and hence (4) and (13) are contradictory. Thus the first case holds, and the theorem is proved.

Example. Theorem 2 is valid if $\log w(x)=-x^p$, where $p>2$. Then $M(\eta)\sim C\eta^\alpha$, where $\alpha=1/(p-1)$, and C is a constant, and therefore (4) has the form

$$|F_1(i\eta)| \geqq \exp\{-D\eta^{\frac{p-2}{p-1}}\},$$

for some constant D. A condition of this type holds for instance if

$$f(x) > \exp\{-Ex^{-(p-2)}\},$$

near 0, for some constant E.

3. The following theorem is applicable to a larger class of weight functions than Theorem 2. On the other hand the conditions on f are rather restrictive.

Theorem 3. *Let w be a weight function with* $\log w$ *concave, and such that*

$$(x\log x)^{-1}\log w(x) \to -\infty, \tag{14}$$

as $x\to\infty$. We assume that $f\in A_w$ and that f_1 coincides with f near 0. Furthermore we assume that $f_1\in L^2(\mathbf{R}^+)$ and that Supp (f_1) *is compact. If the values on $\mathbf{R}$ of the Fourier transform F_1 of f_1 are included in a closed sector of $\mathbf{C}$ with vertex at 0 and opening angle $<2\pi$, then $f\in B_w$.*

Proof. We assume that $\varphi\in L_w^*$ satisfies $\varphi*f=0$, $x\in-\mathbf{R}^+$, and shall show that φ is equivalent to 0.

Fix an arbitrary $x\in-\mathbf{R}^+\cup\{0\}$ and put

$$a_n=\varphi*f_1^{(n)}(x), \quad n\in\mathbf{Z}^+.$$

By the lemma, there exists a constant C, independent of n, such that

$$|a_n| \leqq C^n w(n\varepsilon), \quad n \in \mathbf{Z}^+.$$

(14) shows that

$$(n \log n)^{-1} \log |a_n| \to -\infty,$$

as $n \to \infty$. Hence there exists, for every $d \in \mathbf{Z}_+$, a constant $K(d)$ such that

$$\sum_1^\infty |a_n| |\zeta|^n \leqq K_d \sum_1^\infty \frac{|\zeta|^n}{(dn)!},$$

for every $\zeta \in \mathbf{C}$. But the right hand member is dominated by $K_d \exp(|\zeta|^{1/d})$. Hence

$$G(\zeta) = \sum_1^\infty a_n \zeta^{n-1},$$

$\zeta \in \mathbf{C}$, defines an entire function of order 0. We shall show that $G(\zeta) \to 0$, as $\zeta \to \infty$ along some ray from $\zeta = 0$. By Phragmén—Lindelöf's principle this implies that $G \equiv 0$. In particular $a_1 = 0$. Hence $\varphi * f_1 = 0$ on $-\mathbf{R}_+$. Since $f \in A_w$, we have $f_1 \in A_w$, and it follows from Theorem 1 that φ is equivalent to 0.

Without loss of generality we can assume that F_1 does not take any values w which are $\neq 0$ and satisfy $|\operatorname{Arg} w| < \varepsilon$, for some $\varepsilon > 0$. We shall then show that $G(\zeta) \to 0$, as $\zeta \to \infty$ along the positive axis, and this proves the theorem.

By our assumptions, both f_1 and φ are included in $L^1(\mathbf{R}^+) \cap L^2(\mathbf{R}^+)$, and we obtain from absolute convergence, if $|\zeta|$ is small enough,

$$\begin{aligned} G(\zeta) &= \sum_1^\infty \zeta^{n-1} \varphi * f_1^{(n)}(x) \\ &= (2\pi)^{-1} \sum_1^\infty \zeta^{n-1} \int_{\mathbf{R}} \Phi(t) F_1(t)^n e^{-itx}\, dt \\ &= (2\pi)^{-1} \int_{\mathbf{R}} \frac{F_1(t)}{1 - \zeta F_1(t)} \Phi(t) e^{itx}\, dt. \end{aligned}$$

By the assumption on $\operatorname{Arg} F_1(t)$, $(1 - \zeta F_1(t))^{-1}$ is uniformly bounded if $|\operatorname{Arg} \zeta| < \varepsilon/2$, and since $F_1 \Phi \in L^1(\mathbf{R})$, the right hand member is analytic in this region, and thus, by analytic continuation, equals $G(\zeta)$. If ζ is real and $\to \infty$, $(1 - \zeta F_1(t))^{-1} \to 0$ except at the denumerably many zeros of the analytic function F_1. Hence, by Lebesgue's dominated convergence theorem, $G(\zeta) \to 0$.

Remark. Theorem 3 is applicable if f is of bounded variation near 0, with $f(+0) = 0$. For it is easy to find a function φ on $\mathbf{R}$, with support in [0, 1], coinciding with 1 in some interval $[0, \delta]$, absolutely continuous except for the jump at 0, and such that its Fourier transform does not take values in some closed sector of $\mathbf{C}$ with vertex at 0. Defining φ_ε by $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)$, we find that $f_1 = f\varphi_\varepsilon$ satisfies the conditions of Theorem 3, if $\varepsilon > 0$ is small enough. A different method to prove Theorem 3 in this case has been given in [1] (the proof of the corollary of Theorem 4).

References

1. ALLAN, G. R., Ideals of rapidly growing functions. Manuscript. (Leeds University, 1977)
2. BADE, W. G. and H. G. DALES, Norms and ideals in radical convolution algebras. *J. Functional analysis* (To appear)
3. BOAS, R. P., *Entire functions.* New York, 1954.
4. DALES H. G. and J. P. MCCLURE, Nonstandard ideals in radical convolution algebras on a half-line. Manuscript. (Leeds University)
5. DOMAR, Y., A closure problem for functions on $\mathbf{R}_+$, *Zap. Naučn. Sem. LOMI* **81** (1978), 160—162.
6. GUARII, V. P., Spectral synthesis of bounded functions on the half-axis. *Funkcional Anal. i Priložen* 3 (1969), 34—48.
7. NIKOLSKII, N. K., Invariant subspaces of weighted shift operators. *Mat. Sbornik* **74** (116), 1967, 171—190.
8. NYMAN, B., On the one-dimensional translation group and semi-group in certain function spaces. Thesis Uppsala 1950.
9. RUBEL, L. A., Convolution cut-down in some radical convolution algebras. *Pacific J. Math.* **82** (1979), 523—525.
10. STYF, B., Closed translation invariant subspaces in a Banach space of sequences, summable with wrights. *Uppsala University, Department of Mathematics,* Report 1977:3.
11. TITCHMARSH, K. C., *An introduction to the theory of Fourier integrals.* Oxford, 1937.

Received February 15, 1980

Yngve Domar
Department of Mathematics
University of Uppsala
Thunbergsvägen 3
752 38 Uppsala
Sweden