A counterexample in conformal welding concerning chord-arc curves

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1. Introduction

Suppose that Γ is an oriented Jordan curve in the plane which passes through ∞ and separates the plane into two complementary regions Ω_+ and Ω_-. Let Φ_+ and Φ_- be conformal mappings of the upper and lower halfplanes U and L onto Ω_+ and Ω_-, respectively, each taking ∞ to itself. These two mappings extend homeomorphically to the boundary, and hence $\Phi_+^{-1}\circ\Phi_-$ determines an increasing homeomorphism h of the line onto itself.

The reverse process through which the curve Γ is obtained from h is called conformal welding. There are three basic questions concerning this correspondence: given h, when does Γ exist, when is it unique, and how are the geometrical properties of Γ reflected in h?

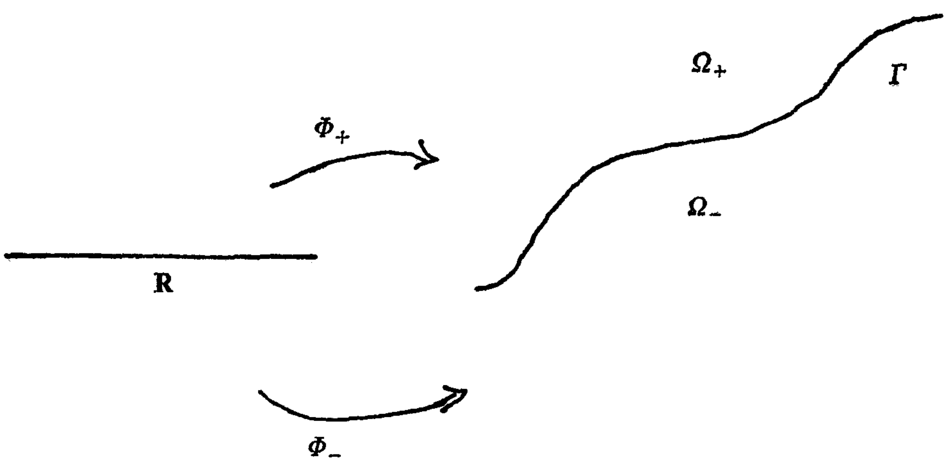

Figure 1

The theory of quasiconformal mappings contains the following result. (See [1, 2].) Suppose that there is a $C>0$ such that $\frac{1}{C}\leqq\frac{h(x+t)-h(x)}{h(x)-h(x-t)}\leqq C$ for any $x, t\in\mathbf{R}$. Then Γ exists and is unique (up to conformal equivalence), and Γ must be a quasicircle, that is, the image of the real line under a quasiconformal mapping of the plane onto itself. Conversely, if h corresponds to a quasicircle and some choice of conformal mappings, then h must satisfy a doubling condition like the one above.

There is an analogous question for chord-arc curves which was asked by Jerison and Kenig [5]. A chord-arc curve is a rectifiable curve for which there is a $C>0$ such that $|s-t|\leqq C|z(s)-z(t)|$ for all $s, t\in\mathbf{R}$, where $z(\cdot)$ is the arc length parameterization of Γ. If Γ is a chord-arc curve, then $|\Phi'_+(t)|$ is locally integrable on $\mathbf{R}$ (because Γ is rectifiable), and in fact $|\Phi'_+(t)|$ belongs to the Muckenhoupt class A_∞ of weights. This last statement is a theorem of Lavrentiev, and a proof can be found in [5]. Because $|\Phi'_-(t)|\in A_\infty$ also, it follows from the basic properties of A_∞ weights (i.e., the A_∞ condition is an equivalence relation — see [3]) that h is locally absolutely continuous and $h'\in A_\infty$.

The question asked in [5] is whether the converse holds: if $h'\in A_\infty$, must Γ be a chord-arc curve? This is known to be true if $\log h'$ has small BMO norm, and in fact that implies that the chord-arc constant of Γ is close to 1. (See the proof of Theorem 2 in [4].) There is also a VMO analogue of this last result which is true. (See [6].) The answer to the question, however, is no, even if $\log h'\in L^\infty$.

Theorem. *There is a non-locally rectifiable Jordan curve Γ with a corresponding pair of conformal mappings Φ_+, Φ_- such that $h=\Phi_+^{-1}\circ\Phi_-$ satisfies*

$$\frac{1}{c}\leqq\left|\frac{h(x)-h(y)}{x-y}\right|\leqq c \tag{1.1}$$

for $x, y\in\mathbf{R}$.

The curve Γ will be obtained as the limit of an iterative construction, and the estimate (1.1) will come from comparing the harmonic measure of arcs of Γ with respect to the two complementary domains. The sort of curves that we consider come from the standard method of constructing nonrectifiable curves, i.e., one starts with the real line and puts teeth on it, and then teeth on the teeth, etc. (See Figure 2). Unfortunately, if one does this naively by, say, alternating the directions the teeth are put on at each stage of the construction, then it is not clear how to keep harmonic measure from building up on one side. This problem is avoided by not constructing the curve in the naive way, but by recursively constructing a sequence of curves $\{\Gamma_n\}$ in which Γ_{n+1} is obtained by putting teeth on Γ_n in a way that depends on what Γ_n looks like, and not according to some prearranged plan. These $(n+1)^{\text{th}}$ order teeth are chosen according to the size of the ratio of the upper and lower harmonic measures on Γ_n at a given location (where the tooth is being built). This part of the construction (dealing with the harmonic measure estimates) is carried out in Section 4, and the basic rules for building the Γ_n's (e.g., concerning convergence to a limit curve and nonrectifiability) are described in Section 3. In Section 2 we discuss preliminary harmonic measure estimates needed in Section 4.

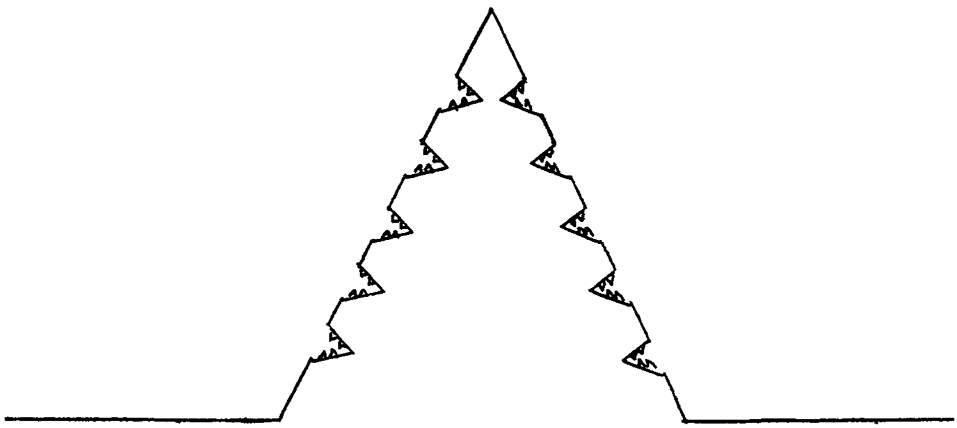

Figure 2

This is a picture of the naive approach, where the directions that the teeth point (in or out) alternates at each stage. In our construction this dental direction depends on what the preceeding stage of the construction looks like.

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2. Some harmonic measure estimates

Let θ_1 and θ_2 be given, $0<\theta_1<\theta_2$, and suppose that γ is any Jordan arc which connects $-\theta_1$ to θ_1 and which is contained in the rectangle $\{z\colon -1/2\leqq y\leqq 1/2, -\theta_1\leqq x\leqq\theta_1\}$. We define two new arcs γ^1 and γ^2 by adding $\left[-\frac{\theta_1+\theta_2}{2}, -\theta_2\right]\cup\left[\theta_1, \frac{\theta_1+\theta_2}{2}\right]$ and $[-\theta_2, -\theta_1]\cup[\theta_1, \theta_2]$ to γ, respectively.

Let R be the boundary of $\{-3/2\leqq y\leqq 3/2, \ -\theta_2\leqq x\leqq\theta_2\}$. The real line divides R into two arcs R_1 and R_2, the top and bottom halves of R. We can join R_1 and R_2

to γ^2 to obtain two new Jordan curves Γ_1 and Γ_2. Let D_1 be the interior of Γ_1 and D_2 be the exterior of Γ_2, so that $i\in D_1\subseteqq D_2$.

Suppose that Γ is any Jordan curve which passes through ∞ and which agrees with γ^2 inside of R. Denote by Ω_+ and Ω_- the complementary regions of Γ, where $i\in\Omega_+$, so that $D_1\subseteqq\Omega_+\subseteqq D_2$. Define Φ_+ and Φ_- to be the conformal mappings of the upper and lower half-planes U and L onto Ω_+ and Ω_- which fix $-\theta_1$, θ_1, and ∞. These mappings extend homeomorphically to the boundary, and we can use them to pull Lebesgue measure on the line back to two positive measures μ and λ on Γ.

For any domain D we let $\omega(E, z, D)$ denote the harmonic measure of $E\subseteqq\partial D$ with respect to $z\in D$.

Lemma 2.1. *There exists* $k_1(\theta_1, \theta_2)>1$ *which is independent of* γ *and* Γ *such that*

$$\frac{1}{k_1(\theta_1, \theta_2)}\mu(E) \leqq \omega(E, i, \Omega_+) \leqq k_1(\theta_1, \theta_2)\mu(E)$$

for any measurable subset E *of* γ.

We shall refer to such an inequality by saying that the two measures are equivalent on γ, with bound $k_1(\theta_1, \theta_2)$.

Lemma 2.1 is proved by showing that $\Phi_+(i)$ stays away from the real line and ∞, so that $\omega(\cdot, \Phi_+(i), U)$ (U=the upper half-plane) and Lebesgue measure are equivalent measures on $[-\theta_1, \theta_1]$. The position of $\Phi_+(i)$ is controlled by the fact that $\omega(E, i, \Phi_+)\geqq\varepsilon$ for $E=\gamma$, $[\theta_1, \theta_2]$, or $[-\theta_2, -\theta_1]$, where $\varepsilon>0$ depends only on θ_1 and θ_2. We omit the details.

Lemma 2.2. *There exists* $k_2(\theta_1, \theta_2)>1$ *independent of* γ *such that*

$$\omega(E, i, D_1) \leqq \omega(E, i, D_2) \leqq k_2(\theta_1, \theta_2)\omega(i, E, D_1)$$

for all measurable subsets E *of* γ.

We can map D_2 conformally onto the unit disk in such a way that i is carried to the origin, γ, γ^1, and γ^2 are taken to circular arcs I, I^1, and I^2, and R_1 is mapped onto a Jordan arc A_1 with endpoints on the unit circle. Lemma 2.2 is proved by controlling the relative position of I, I^1, and A_1 (with harmonic measure estimates) and using Harnack's inequality and the conformal invariance of harmonic measure. We omit the details.

Suppose now that $\theta_1=1$ and $\theta_3=3$, and let $\theta_3>0$ be given. Suppose also that c and d are real numbers, $-3/2<c<d<3/2$, such that c and d lie on Γ and $\theta_3(d-c)\leqq 1/2$. For $z, w\in\Gamma$ let $A(z, w)$ denote the arc of Γ which joins z and w.

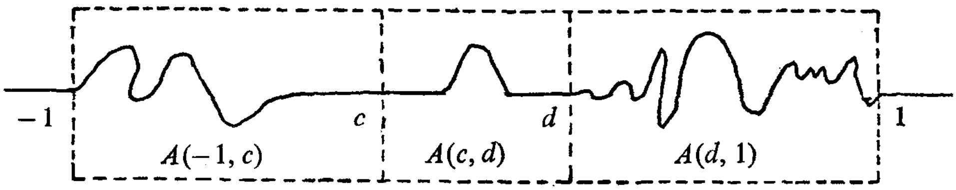
Lemma 2.3. *There is a constant* $k_3(\theta_3)>1$ *independent of* c *and* d *such that if* $A(x_0, x_1)\subseteqq\{z\colon x_0\leqq x\leqq x_1,\ -\theta_3(d-c)\leqq y\leqq\theta_3(d-c)\}$ *when* $(x_0, x_1)=(-1, c), (c, d),$

or $(d, 1)$, *then*

$$\frac{1}{k_3(\theta_3)}(d-c) \leqq \mu(A(c, d)) \leqq k_3(\theta_3)(d-c)$$

and

$$\frac{1}{k_3(\theta_3)}(d-c) \leqq \lambda(A(c, d)) \leqq k_3(\theta_3)(d-c).$$



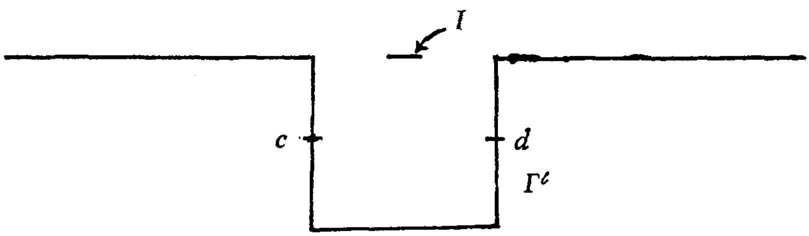


Figure 3

By symmetry it is enough to consider only the estimate for μ, and by Lemma 2.1 it is enough to estimate $\omega(A(c, d), i, \Omega_+)$ instead. Consider the Jordan curve Γ^l obtained by adding together the segments $\{x+\theta_3(d-c)i\colon -\infty<x\leqq c\}$, $\{c+iy\colon -\theta_3(d-c)\leqq y\leqq \theta_3(d-c)\}$, $\{x-\theta_3(d-c)i\colon c\leqq x\leqq d\}$, $\{d+yi\colon -\theta_3(d-c)\leqq y\leqq \theta_3(d-c)\}$, and $\{x+\theta_3(d-c)i\colon d\leqq x<\infty\}$. (See Figure 3.) By the maximum principle $\omega(A^l(c, d), i, \Omega^l_+)\leqq\omega(A(c, d), i, \Omega_+)$. Let $u(z)=\omega(A^l(c, d), z, \Omega^l_+)$ for $z\in\Omega^l_+$. There exists $\eta(\theta_3)>0$ such that $u(z)\geqq\eta(\theta_3)$ if $z\in I=\left\{w\colon \operatorname{Im} w=\theta_3(d-c), \left|\operatorname{Re} w-\frac{c+d}{2}\right|\leqq(d-c)/10\right\}$. Thus $u(i)\geqq\eta(\theta_3)\omega(I, i, U')$, where $U'=\{w\colon \operatorname{Im} w> \theta_3(d-c)\}$. It follows that $u(i)\geqq\frac{1}{k_3(\theta_3)}(d-c)$ for some $k_3(\theta_3)>1$.

Similarly, let

$$\Gamma^u = \{x-\theta_3(d-c)i\colon -\infty < x \leqq c\}\cup\{c+yi\colon -\theta_3(d-c) \leqq y \leqq \theta_3(d-c)\}\cup$$
$$\{x+\theta_3(d-c)i\colon c \leqq x \leqq d\}\cup\{d+yi\colon -\theta_3(d-c) \leqq y \leqq \theta_3(d-c)\}\cup$$
$$\{x = \theta_3(d-c)i\colon d \leqq x < \infty\},$$ so that $\omega(A(c, d), i, \Omega_+) \leqq \omega(A^u(c, d), i, \Omega^u_+)$.

Let

$$J=\left\{w\colon \operatorname{Im} w=-\theta_3(d-c),\ \left|\operatorname{Re} w-\frac{c+d}{2}\right|\leqq(d-c)\right\},\quad U''=\{w\colon \operatorname{Im} w>-\theta_3(d-c)\},$$

and define $v(z)$ and $\tilde{v}(z)$ by $v(z)=\omega(A^u(c,d),z,\Omega^u_+)$ and $\tilde{v}(z)=\omega(J,z,U'')$ for $z\in\Omega^u_+$ and $z\in U''$. Then $\tilde{v}(z)\geqq\eta'(\theta_3)$ for some $\eta'(\theta_3)>0$ and all $z\in A^u(c,d)$, and hence $\tilde{v}(z)\geqq\eta'(\theta_3)v(z)$ for all $z\in\Omega^u_+$, by the maximum principle. Thus $v(i)\leqq k_3(\theta_3)(d-c)$ for some $k_3(\theta_3)>1$.

Let $\alpha, 0<\alpha<\frac{\pi}{2}$, be given (think of α as small) and let T_α be the two legs of the isosceles triangle having $[-1,1]$ as its base and with α as the angle at the third vertex v_α, which lies on the positive y-axis. Consider the Jordan curve J_α formed by connecting T_α to the real axis and throwing away $(-1,1)$, and let U_α and L_α be the upper and lower domains. Let Φ_α and Ψ_α be the conformal mappings of the upper and lower half-planes onto U_α and L_α such that $-\theta_4|v_\alpha|$, $\theta_4|v_\alpha|$, and ∞ are fixed for a given $\theta_4>0$. By symmetry, $\Phi_\alpha(0)=v_\alpha=\Psi_\alpha(0)$.

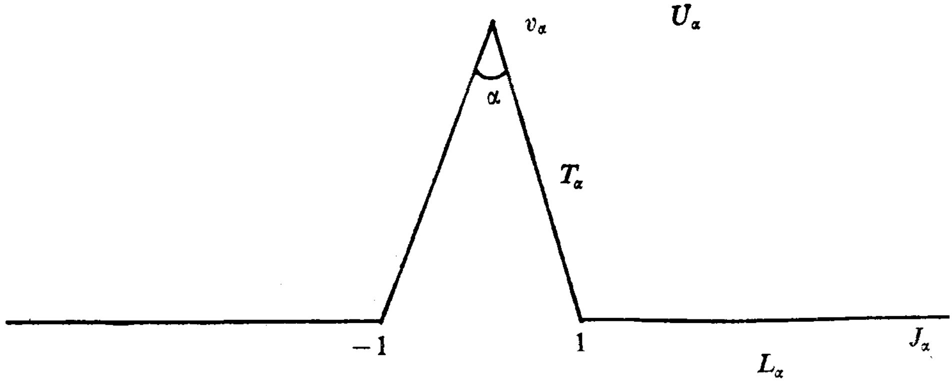


Figure 4

Define positive measures μ_α and λ_α on J_α as the pull-backs of Lebesgue measure on the line using the mappings Φ_α and Ψ_α. These measures are both mutually absolutely continuous with respect to the arclength measure $|dz|$, and near v_α, μ_α gets large while λ_α becomes small.

Lemma 2.4. *For any* $\eta>0$, $\theta_4>0$, *and* $M>0$ *there is an* $\alpha_0>0$ *so that if* $0<\alpha\leqq\alpha_0$ *then* $d\mu_\alpha/d\lambda_\alpha\geqq M$ *for all* $z\in(T_\alpha)_\eta=J_\alpha\cap\{w\colon \operatorname{Im} w\geqq\eta|v_\alpha|\}$.

We shall show that $d\mu_\alpha\geqq c(\eta,\theta_4)|dz|$ and $d\lambda_\alpha\leqq\varepsilon(\eta,\theta_4,\alpha)|dz|$ on $(T_\alpha)_\eta$, where $c(\eta,\theta_4)>0$ and $\varepsilon(\eta,\theta_4,\alpha)\to 0$ as $\alpha\to 0$.

From Lemma 2.1 and rescaling it follows that $d\mu_\alpha \geqq c(\theta_4)\frac{1}{\alpha}\,d\omega(\cdot, 2|v_\alpha|i, U_\alpha)$ on T_α, since $|v_\alpha| \sim \frac{1}{\alpha}$. Let r_+ and r_- be the two rays with endpoint v which pass through ± 1, and let R_α be their union, a Jordan curve. Let A_α denote the angular region (with angle $2\pi-\alpha$) which lies above R_α. Then $\omega(\cdot, 2|v_\alpha|i, A_\alpha) \leqq C(\eta)\omega(\cdot, 2|v_\alpha|i, U_\alpha)$ on $(T_\alpha)_\eta$. (This is simply a variation of Lemma 2.2, and it can be proved using conformal mapping.) By conformal mapping it is easy to show that $\omega(\cdot, 2|v_\alpha|i, A_\alpha) \geqq C\alpha|dz|$ on T_α. Thus $d\mu_\alpha \geqq c(\eta, \theta_4)|dz|$ on $(T_\alpha)_\eta$.

Let $\Psi_\alpha(1)=a_\alpha$, so that $\Psi_\alpha(-1)=-a_\alpha$ and $\frac{1}{a(\theta_4)} \leqq a_\alpha \leqq a(\theta_4)$ for some $a(\theta_4)>1$, since $2a_\alpha=\lambda_\alpha(T_\alpha) \sim \frac{1}{\alpha}\omega(T_\alpha, -|v_\alpha|i, L_\alpha) \sim 1$, by Lemma 2.1. Define $b_\alpha>0$ by $\Psi_\alpha(-i)=-b_\alpha i$, so that $\frac{1}{b(\theta_4)} \leqq b_\alpha \leqq b(\theta_4)$, since $\omega([-a_\alpha, a_\alpha], -b_\alpha i, L)=\omega(T_\alpha, -i, L_\alpha) \sim 1$. Because $\omega(\cdot, -b_\alpha i, L)$ and Lebesgue measure are equivalent measures on $[-a_\alpha, a_\alpha]$, with bounds depending only on θ_4, the same is true of λ_α and $\omega(\cdot, -i, L_\alpha)$ on T_α. If A'_α is the angular region lying below R_α (and determined by T_α), then $\omega(\cdot, -i, L_\alpha) \leqq C(\eta)\omega(\cdot, -i, A'_\alpha)$ on $(T_\alpha)_\eta$ (another variation of Lemma 2.2). An elementary calculation shows that $\omega(\cdot, -i, A'_\alpha) \leqq \varepsilon(\eta, \alpha)|dz|$ on $(T_\alpha)_\eta$, where $\varepsilon(\eta, \alpha) \to 0$ as $\alpha \to 0$ for any fixed $\eta>0$. Thus $d\lambda_\alpha \leqq \varepsilon(\eta, \theta_4, \alpha)|dz|$ on $(T_\alpha)_\eta$, and the lemma is proved.

Because of the singularities at the three vertices, the curve J_α is not quite suitable for our purposes. Thus for each α we consider a smoothed-up version of J_α which we call SJ_α, and which we may assume has the following properties:

(a) SJ_α is a smooth Jordan curve;

(b) $SJ_\alpha = ST_\alpha \cup (-\infty, -1) \cup (1, \infty)$, where ST_α is a Jordan arc connecting -1 and 1;

(c) ST_α is contained in the union of T_α and its interior;

(d) ST_α agrees with T_α outside of very small neighborhoods of the three vertices (with radii of 10^{-12}, say);

(e) the length of ST_α is about the same as that of T_α (e.g., $|T_\alpha| \leqq |ST_\alpha| \leqq 1.01|T_\alpha|$);

(f) if $s\mu_\alpha$ and $s\lambda_\alpha$ are the corresponding measures on SJ_α (for a fixed θ_4, which will turn out to be $\frac{2}{3}\frac{1}{1440}\Big)$, then $1/2 \leqq d(s\mu_\alpha)/d\lambda_\alpha \leqq 2$ and $1/2 \leqq d(s\lambda_\alpha)/d\mu_\alpha \leqq 2$ on $\{z \in T_\alpha\colon 10^{-10}|v_\alpha| \leqq y \leqq (1-10^{-10})|v_\alpha|\}$. For each α we choose such an SJ_α and fix it for the rest of the paper.

Let $N=|v_\alpha|/1440$, so that $N \geqq 100$ if α is small enough, and let P be any subset of the integers. Define a Jordan curve SJ_α^P by constructing an upwards-pointing copy of ST_α on $[\ell N-1, \ell N+1]$ exactly when $\ell \in P$, and by letting SJ_α^P agree with the real line outside these intervals. Let U_α^P and L_α^P be the upper and lower complementary domains of SJ_α^P, and let $\omega_+^{\alpha,P}(\cdot)=\omega(\cdot, 2|v_\alpha|i, U_\alpha^P)$ and $\omega_-^{\alpha,P}(\cdot)=\omega(\cdot, -2|v_\alpha|i, L_\alpha^P)$. Let $s\mu_\alpha^P$ and $s\lambda_\alpha^P$ be the measures on SJ_α^P obtained by pulling Lebesgue measure on the line back to SJ_α^P using the conformal mappings of U and L onto U_α^P and L_α^P which fix $-\frac{2}{3}N$, $\frac{2}{3}N$, and ∞.

Lemma 2.5. *There is a constant $X(\alpha)>1$ which is independent of P such that $\frac{1}{X(\alpha)}\leqq d\omega_+^{\alpha,P}/d\omega_-^{\alpha,P}\leqq X(\alpha)$ and $\frac{1}{X(\alpha)}\leqq d(s\mu_\alpha^P)/d(s\lambda_\alpha^P)\leqq X(\alpha)$ everywhere on J_α^P.*

We shall show that $d(s\mu_\alpha^P)$ and $d(s\lambda_\alpha^P)$ are each comparable to $|dz|$, and $d\omega_+^{\alpha,P}$ and $d\omega_-^{\alpha,P}$ are each comparable to $(1+|z|^2)^{-1}|dz|$, with bounds that depend only on α. We shall do this only for μ_α^P and $\omega_+^{\alpha,P}$, since λ_α^P and $\omega_-^{\alpha,P}$ can be treated similarly. Also, the estimate for $\omega_+^{\alpha,P}$ implies the estimate for μ_α^P, and so we need only consider the former.

To estimate ω_+ we trap SJ_α between two other curves. For each integer j let $A_j=A_\alpha^P\left(\left(j-\frac{2}{3}N\right), \left(j+\frac{2}{3}\right)N\right)$ and $A_j^1=A^P\left(\left(j-\frac{3}{4}\right)N, \left(j+\frac{3}{4}\right)N\right)$, where $A_\alpha^P(z, w)$ is the arc of SJ_α^P which connects z and w, and let I_j and I_j^1 denote the intervals of the real line with the same endpoints. Let k be a fixed integer, and define Γ_k^ℓ by replacing A_j with I_j for every $j\neq k$. Then Γ_k^ℓ is either the real line or it is SJ_α translated over kN steps.

For the curve Γ_k^u we start off with the line $y=|v_\alpha|$ for $x\leqq\left(k-\frac{7}{8}\right)N$, come down smoothly to the line $y=0$ and connect up to A_k^1, leave A_k^1 at $x=\left(k+\frac{3}{4}\right)N$ and go smoothly back up to the line $y=|v_\alpha|$ at $x=\left(k+\frac{7}{8}\right)N$, and go out to ∞ on that line. We require that the smooth connections between $y=|v_\alpha|$ and A_k^1 are done the same way for each k. There are two possiblities for what Γ_k^u looks like, depending on whether or not $k\in P$.

Thus the shape of Γ_k^ℓ and Γ_k^u depends only on whether or not $k\in P$; if k_1 and k_2 both do or do not lie in P, then $\Gamma_{k_1}^\ell$ and $\Gamma_{k_1}^u$ are translations of $\Gamma_{k_2}^\ell$ and $\Gamma_{k_2}^u$. This implies that the conformal mappings of the upper half-plane onto $\Omega_+^{k,\ell}$ and $\Omega_+^{k,u}$ are smooth, even at ∞, with estimates that do not depend on k or P. Thus $\omega(\cdot, 2|v_\alpha|i, \Omega_+^{k,u})$ and $(1+|z|^2)^{-1}|dz|$ are equivalent measures on Γ_k^u, with bounds that depend only on α, and similarly for Γ_u^ℓ. Thus there is a $C(\alpha)>0$ such that for any measurable $E\subseteq A^1$,

$$\frac{1}{C(\alpha)}(|k|N+1)^{-2}|E| \leqq \omega(E, 2|v_\alpha|i, \Omega_+^{k,u})$$

$$\leqq \omega_+^{\alpha,P}(E) \leqq \omega(E, 2|v_\alpha|i, \Omega_+^{k,\ell}) \leqq C(\alpha)(|k|N+1)^{-2}|E|.$$

This proves that $(1+|z|^2)^{-1}|dz|$ and $d\omega_+^{\alpha,P}$ are comparable on A_k^1 for all k, and hence on SJ_α^P, with a bound that depends only on α. This proves Lemma 2.5.

Suppose now that $0\in P$, and let ST_α be as before.

Lemma 2.6. *For any* $M>0$ *there is an* $\alpha_1>0$ *(depending only on M) such that* $d(s\mu_\alpha^P)/d(s\lambda_\alpha^P)\geqq M$ *on* $\{z\in ST_\alpha\colon \frac{1}{2}|v_\alpha|\leqq y\leqq(1-10^{-10})|v_\alpha|\}$ *if* $0<\alpha\leqq\alpha_1$.

If $P=\{0\}$ then $SJ_\alpha^P=SJ_\alpha$ and this lemma follows from Lemma 2.4 (with $\eta=\frac{1}{2}$ and $\theta_4=\frac{2}{3}\left(\frac{1}{1440}\right)$) and the definition of SJ_α. For general P, $0\in P$, SJ_α^P and SJ_α are the same inside the strip $\{z\colon |x|\leqq 3N/4\}$, and one can reduce to the $P=\{0\}$ case using Lemmas 2.1 and 2.2.

3. The construction, part 1

We are going to obtain a curve Γ_∞ satisfying the properties of the theorem in Section 1 as the limit of a sequence $\{\Gamma_n\}$ of smooth curves which will be defined recursively. In this section we shall describe the preliminary rules governing this construction, which will imply that $\{\Gamma_n\}$ converges to a nonrectifiable quasicircle Γ_∞. The final steps of the construction will be carried out in Section 4.

To construct the Γ_n's we shall take a tooth and put teeth on it, and then put teeth on the teeth, etc. By a tooth we mean anything that is obtained from ST_α (defined in Section 2) by translating, dilating, or rotating. The parameter α should be thought of as small and fixed throughout the construction, although we shall not choose α until the end of Section 4. We do require now that α be small enough so that $|v_\alpha|\geqq 144.000$.

We shall call v_α the node of the tooth ST_α, and the points ± 1 will be called its base vertices. The interval $[-1, 1]$ is the base of the tooth. The rectangle that circumscribes T_α will be called the fence of ST_α. The interior of the fence will be called the yard of the tooth. The nodal neighborhood $N(ST_\alpha)$ is defined to be $ST_\alpha\cap\{z\colon |v_\alpha|/8\leqq |z-v_\alpha|\leqq|v_\alpha|/4\}$, which is the union of two intervals of ST_α. (Here "interval" will mean an arc of a curve which is also a line segment.) These definitions will also be applied to the corresponding objects of a general tooth t.

For the first curve Γ_1 in the sequence $\{\Gamma_n\}$ we take simply the curve SJ_α. The second curve is obtained by putting new teeth on the nodal neighborhood of ST_α. More precisely, we shall specify in Section 4 a subset CS_1 ("construction site") of $N(ST_\alpha)$ which is made up of finitely many small intervals and which satisfies $|CS_1|\geqq 160|N(ST_\alpha)|/|V_\alpha|$. (Here $|E|$ denotes the arclength of E.) The new curve Γ_2 is obtained by replacing the intervals of CS_1 by teeth. The directions that these teeth point in will be independent of each other and will be specified in Section 4.

We may take the intervals of CS_1 to be as small as we like, but they do have to be small. This keeps the new teeth from crashing into each other, and it is also needed to make the sequence of curves converge. This will also keep the new teeth and their associated yards contained in the yard of ST_α. (It would perhaps be useful at this point to look again at Figure 2 in Section 1.)

The general construction follows the same pattern. Suppose that Γ_m has been constructed, $m\geqq 1$, and let B_m be the union of all the m^{th}-order teeth on Γ_m. Let $G_m=\Gamma_m\setminus B_m$, and let C_m be the union of the nodal neighborhoods of all the m^{th}-order teeth, so that $C_m\subseteq B_m$.

In the next section we shall specify a certain subset CS_m of C_m which is a union of finitely many small intervals. The next curve Γ_{m+1} is obtained by constructing teeth on these intervals, so that the old interval is the base of the new $(m+1)^{\text{th}}$-order tooth. The direction in which the tooth points will also be specified in Section 4. Note that Γ_{m+1} agrees with Γ_m off C_m.

There are two requirements that we place on CS_m. The first is that $|CS_m|\geqq 160|C_m|/|V_\alpha|$. The second requirement is that each of the intervals of CS_m should have small length, less than ε_m for a certain sequence $\{\varepsilon_m\}$ of positive numbers which tend to 0. We shall specify how small ε_m is as we go along in this section. We should emphasize that ε_m will depend on what Γ_m looks like, but not on Γ_n for any $n\geqq m$.

How small the intervals of CS_m are controls how far the $(m+1)^{\text{th}}$-order teeth can go from the parent tooth on which they were built. In particular, if ε_m is small enough, then for any m^{th}-order tooth t the $(m+1)^{\text{th}}$-order teeth constructed on t, along with their fences and yards, will lie inside the yard of t. Hence all of the Γ_m's agree with the real line outside $\{z\colon -1\leqq x\leqq 1, 0\leqq y\leqq |v_\alpha|\}$, the yard of ST_α.

Under these conditions it will be true that each Γ_m is a smooth Jordan curve, and that the sequence $\{\Gamma_m\}$ will converge to a nonrectifiable quasicircle Γ_∞. Let us verify these facts.

It is clear that each Γ_m is smooth, and each Γ_m will be a Jordan curve if the ε_m's are small enough. If ε_m goes to 0 sufficiently quickly, then Γ_m will converge to a Jordan curve Γ_∞ in any reasonable sense, e.g., there will exist a sequence of parameterizations of the Γ_m's which converges uniformly. We should also point out that $G_\infty=\cup G_m$ is a dense open subset of Γ_∞, and $G_m\subseteq G_n\subseteq \Gamma_n$ whenever $m\leqq n$.

To show that Γ_∞ is a quasicircle we use Ahlfors's three point condition [1, 2]: a Jordan curve Γ which passes through ∞ is a quasicircle if and only if there is a constant $c>0$ such that if $z_1, z_2\in\Gamma$ and z_3 lies on the arc joining them then $|z_1-z_3|\leqq c|z_1-z_2|$. This condition will be satisfied by each Γ_m with a constant independent of m if the ε_m's are sufficiently small, and so Γ_∞ will also satisfy this condition.

The arc A on Γ_∞ that joins -1 to 1 has infinite length. Indeed, $B_m\setminus C_m\subseteq G_m$ is contained in this arc for each $m\geqq 1$, and hence $|A|\geqq |B_m|/2$, since $|C_m|\leqq |B_m|/2$ (which can be verified tooth by tooth). The length of each tooth is at least twice its height, which is $|v_\alpha|/2$ times the length of the base of the tooth. The union of all the m^{th}-order teeth is precisely CS_{m-1}, and $|CS_{m-1}|\geqq\frac{160}{|v_\alpha|}|C_{m-1}|\geqq\frac{160}{|v_\alpha|}\left(\frac{1}{16}|B_{m-1}|\right)$. Thus $|B_m|\geqq 2(\frac{1}{2}|v_\alpha|)|CS_{m-1}|\geqq 10|B_{m-1}|$, and hence $|B_m|\geqq 10^{m-1}|B_1|$. Because this is true for all $m\geqq 1$, A must have infinite length, and so Γ_∞ is not locally rectifiable.

We can also say something about the convergence of harmonic measure. Let Ω_+^m and Ω_-^m be the complementary domains of Γ_m (including $m=\infty$), so that $2|v_\alpha|i\in\Omega_+^m$ and $-2|v_\alpha|i\in\Omega_-^m$. Consider the measures $\omega_+^m(\cdot)=\omega(\cdot, 2|v_\alpha|i, \Omega_+^m)$ and $\omega_-^m(\cdot)=\omega(\cdot, -2|v_\alpha|i, \Omega_-^m)$. For $z_1, z_2\in\Gamma_m$ let $A^m(z_1, z_2)$ denote the arc of Γ_m which joins z_1 and z_2. If the ε_m's are sufficiently small, and if $z_1, z_2\in G_\infty$ (so that $z_1, z_2\in\Gamma_n$ for n large enough), then

$$\lim_{m\to\infty}\omega_+^m(A^m(z_1, z_2)) = \omega_+^\infty(A^\infty(z_1, z_2)), \tag{3.1}$$

and similarly for ω_-^m.

In fact, the following is true. If Γ_m has been constructed, if Γ_n for $n>m$ are constructed according to the above rules, and if ε_m is small enough, then

$$(1+2^{-m})^{-1}\leqq\omega_+^\infty(A^\infty(z_1, z_2))/\omega_+^m(A^m(z_1, z_2))\leqq 1+2^{-m} \tag{3.2}$$

for any $z_1, z_2\in G_m$.

We know that for all $n\geqq m$ (including $n=\infty$) Γ_n must contain $\Gamma_m\setminus C_m$, and that the rest of Γ_n must lie within a distance of $\varepsilon_m|v_\alpha|$ of C_m. Indeed, for the $(m+1)^{\text{th}}$-order teeth this comes from the fact that the intervals of $CS_m\subseteqq C_m$ have length at most ε_m. For the higher order teeth this is a consequence of the requirement that a tooth and its yard lie inside the yard of its parent.

Thus $\Gamma_\infty\setminus\Gamma_m$ must lie as close as we want to C_m. On the other hand, $z\in G_m=\Gamma_m\setminus B_m$ must be a positive distance away from C_m. Thus $\omega_+^\infty(A^\infty(z_1, z_2))/\omega_+^m(A^m(z_1, z_2))$ will be uniformly close to 1 for $z_1, z_2\in G_m$ if ε_m is small enough.

To prove the theorem stated in Section 1 we must estimate the homeomorphism h corresponding to Γ_∞. This can be restated in terms of measures. For each $m\geqq 1$, including $m=\infty$, let Φ_+^m and Φ_-^m be the conformal mappings of the upper and lower half-planes onto Ω_+^m and Ω_-^m, respectively, that fix $-|v_\alpha|$, $|v_\alpha|$, and ∞. Let μ_m and λ_m be the measures obtained by pulling Lebesgue measure on the line back to Γ_m using these mappings. What we must show is that there is a $C>0$ such that $\frac{1}{C}\mu_\infty(A)\leqq\lambda_\infty(A)\leqq C\mu_\infty(A)$ for any arc A on Γ_∞. It is enough to prove this only for small arcs.

If A lies outside $\{z\colon |z|\leqq 3|v_\alpha|\}$ then there is no problem, because Γ_∞ looks like the real line there. On $\Gamma_\infty\cap\{z\colon |z|\leqq 3|v_\alpha|\}$ μ_∞ and ω_+^∞ are equivalent, as are λ_∞

and ω_-^∞, by Lemma 2.1. Thus we are reduced to proving that $\frac{1}{C}\omega_+^\infty(A)\leqq\omega_-^\infty(A)\leqq C\omega_+^\infty(A)$ for any arc A of Γ_∞ contained in $\{z\colon |z|\leqq 3|v_\alpha|\}$.

It is enough to consider arcs with their endpoints in the dense subset G_∞. Because of (3.1) we are now reduced to proving that there is a $C>0$ such that for any arc A^m of Γ_m ($m\geqq 1$ arbitrary),

$$\frac{1}{C}\,\omega_+^m(A^m) \leqq \omega_-^m(A^m) \leqq C\omega_+^m(A^m). \tag{3.3}$$

One advantage of this is that the curves Γ_m are smooth, so that $d\omega_+^m/d\omega_-^m$ is smooth.

In the next section we shall show that (3.3) holds if $\alpha>0$ is small enough and if the CS_m's and the directions that the teeth point in are chosen properly.

4. The construction, part 2

We are going to construct the Γ_m's recursively, in such a way that for α small enough, the following properties hold, in addition to the requirements of Section 3:

(i) on C_m it will be true that

$$X_m^{-1} \leqq d\omega_+^m/d\omega_-^m \leqq X_m, \tag{4.1}$$

where $X_m=B_1X(\alpha)\prod_{j=1}^m(1+2^{-j})$, $B_1>1$ is an absolute constant, and $X(\alpha)$ is from Lemma 2.5;

(ii) on all of Γ_m it will be true that

$$Y_m^{-1} \leqq d\omega_+^m/d\omega_-^m \leqq Y_m, \tag{4.2}$$

where $Y_m=B_1B_2X(\alpha)^2\prod_{j=1}^m(1+2^{-j})^2$ and $B_2>1$ is another absolute constant. The precise values of B_1 and B_2 will come out of the rest of the construction; $B_1=B_2=10^3k_1^8k_2^2k_3^2$ will do, where k_1, k_2, and k_3 denote the maximum of the constants from Lemmas 2.1, 2.2, and 2.3 with $10^{-10}\leqq\theta_1,\theta_2,\theta_3\leqq 10^{10}$. The important thing is that B_1 and B_2 are independent of α.

If we can construct such a sequence of Γ_m's we will be finished, because the Y_m's are bounded and so (3.3) will be satisfied.

The plan of the construction is as follows. Suppose that Γ_m is constructed. When we construct teeth on C_m we can make them small and keep them away from the boundary of C_m so that the estimate (4.2) is only disturbed slightly, by at most a factor of $1+2^{-m-1}$, on $\Gamma_m\setminus C_m$. On the $(m+1)^{\text{th}}$-order teeth we can get the estimate (4.2) for $m+1$ from (4.1) for Γ_m and Lemma 2.5. Finally, on the nodal neighborhoods of the $(m+1)^{\text{th}}$-order teeth we can preserve the estimate (4.1) by using Lemma 2.6 and by choosing correctly the directions that the teeth point in.

The curve $\Gamma_1(=SJ_\alpha)$ of Section 3 satisfies (4.1) and (4.2) by the definition of $X(\alpha)$. Now suppose that Γ_n has been constructed which satisfies (4.1) and (4.2).

To construct Γ_{n+1} we must specify the subset CS_n of C_n and the directions that the teeth point in. Recall that C_n is the union of the nodal neighborhoods of all (finitely many) n^{th}-order teeth, and that each nodal neighborhood is made up of two symmetrically placed intervals around the node of the tooth.

Let I be any one of these intervals and let I_0 be its middle third. We shall not put any of $I \setminus I_0$ into CS_n. Partition I_0 into a large number of small intervals of equal size. How small depends on the following two considerations. First, the requirements of Section 3 state that the intervals of CS_n must be "sufficiently small", that is, of length at most ε_n. We require that the partitioning of I_0 yields intervals of length at most ε_n. Second, we require that the partition be so fine that if H is any one of the resulting subintervals of I_0, then

$$\max\left\{\frac{d\omega_+^n}{d\omega_-^n}(z)\colon\ z\in H\right\} \leqq (1+2^{-10n})\min\left\{\frac{d\omega_+^n}{d\omega_-^n}(z)\colon\ z\in H\right\}. \tag{4.3}$$

This is possible, since $d\omega_+^n/d\omega_-^n$ is smooth.

If we apply this procedure to all such intervals I of C_n we get a family $\mathscr{F}_n$ consisting of finitely many disjoint intervals contained in C_n such that $|\bigcup\{H\colon\ H\in\mathscr{F}_n\}|=\frac{1}{3}C_n$. Let $H_1, \ldots, H_\ell$ be an enumeration of this family (with no significance in the ordering), and let H_j^0 denote the middle third of H_j. The set CS_n will be a subset of $\bigcup H_j$ obtained in the following way. Partition each H_j^0 into a large number of subintervals of equal size. This number will depend on j and will be chosen in a few moments, but we do require that it is an integer multiple of $100N$, where $N=|v_\alpha|/1440$. (We may assume that N is an integer, since this will be true for some arbitrarily small α's.) For each j we take every N^{th} interval in this partition of H_j^0 and put it into CS_n, but we put nothing else in CS_n. Then $|CS_n|=160|C_n|/|v_\alpha|$, and so CS_n satisfies the requirements of Section 3.

We still must specify how finely each H_j^0 is partitioned and which direction the teeth on H_j^0 point, and we do this one j at a time. Let $\Gamma_{n,j}$ be the curve obtained from Γ_n by constructing teeth on $H_1^0, \ldots, H_j^0$, but not on $H_{j+1}^0, \ldots, H_\ell^0$, so that $\Gamma_{n,0}=\Gamma_n$ and $\Gamma_{n,\ell}$ will be Γ_{n+1}. Define $\Omega^{n,j}$ accordingly, and let $\omega_\pm^{n,j}(\cdot)=\omega(\cdot,\ \pm 2|v_\alpha|i,\ \Omega_\pm^{n,j})$.

The arc that you get by replacing subintervals of H_{j+1}^0 will be denoted by K_{j+1}^0, and we let $K_{j+1}=K_{j+1}^0\cup(H_{j+1}\setminus H_{j+1}^0)$, so that $\Gamma_{n,j+1}=(\Gamma_{n,j}\setminus H_{j+1}^0)\cup K_{j+1}^0=(\Gamma_{n,j}\setminus H_{j+1})\cup K_{j+1}$. Let $\overline{H_{j+1}^0}$ be the double of H_{j+1}^0 and $\overline{K_{j+1}^0}=K_{j+1}^0\cup(\overline{H_{j+1}^0}\setminus H_{j+1}^0)$, so that $H_{j+1}=3H_{j+1}^0=\frac{3}{2}\overline{H_{j+1}^0}$ and $\Gamma_{n,j+1}=(\Gamma_{n,j}\setminus\overline{H_{j+1}^0})\cup\overline{K_{j+1}^0}$.

The curves $\Gamma_{n,j}$ will be constructed according to the following three conditions.

First, let $\{\beta_j\}_{j=0}^{\ell}$ be a fixed sequence of positive numbers (depending on n) such that each $\beta_j > 1$ and $\prod_{j=0}^{\ell} \beta_j \leqq 1 + 2^{-10n}$. Then we require that

$$\frac{1}{\beta_{j+1}} \frac{d\omega_+^{n,j+1}}{d\omega_-^{n,j+1}}(z) \leqq \frac{d\omega_+^{n,j}}{d\omega_-^{n,j}}(z) \leqq \beta_{j+1} \frac{d\omega_+^{n,j+1}}{d\omega_-^{n,j+1}}(z) \tag{4.4}$$

for all $z \in \Gamma_{n,j} \setminus \overline{H_{j+1}^0}$.

Second, if $z \in \overline{K_{j+1}^0}$, then

$$\frac{1}{Y_n} \leqq \frac{d\omega_+^{n,j+1}}{d\omega_-^{n,j+1}}(z) \leqq Y_n, \tag{4.5}$$

where Y_n is as in (4.2) above.

Finally, to the $(n+1)^{\text{th}}$-order teeth being constructed on Γ_n there correspond $(n+1)^{\text{th}}$-order nodal neighborhoods and the set C_{n+1}, the union of these neighborhoods. In particular, $C_{n+1} \cap K_{j+1}^0$ is the union of the nodal neighborhoods of the teeth put on H_{j+1}^0. Our last requirement is that

$$\frac{1}{X_n} \leqq \frac{d\omega_+^{n,j+1}}{d\omega_-^{n,j+1}}(z) \leqq X_n \tag{4.6}$$

for all $z \in C_{n+1} \cap K_{j+1}^0$, where X_n is as in (4.1).

Thus, if the $\Gamma_{n,j}$'s are constructed according to these three conditions, then $\Gamma_{n+1} = \Gamma_{n,\ell}$ will satisfy (4.1) and (4.2) with $m = n+1$, and we are ready.

The base case of $\Gamma_{n,0} = \Gamma_n$ is already given, and so we assume that $\Gamma_{n,j}$ is given and we construct $\Gamma_{n,j+1}$ now. If the partition of H_{j+1}^0 is fine enough, then the teeth on H_{j+1}^0 will be very small, and K_{j+1}^0 will be very close to being the same as H_{j+1}^0; outside $\overline{H_{j+1}^0}$, harmonic measure will be changed only very slightly. In particular, if the partition of H_{j+1}^0 is sufficiently fine, then (4.4) will be satisfied.

Let $Q = \omega_+^{n,j}(\overline{H_{j+1}^0}) / \omega_+^{n,j}(\overline{H_{j+1}^0})$. From (4.1), (4.3), and (4.4) we know that

$$(1+2^{-10n})^{-2} X_n^{-1} \leqq Q \leqq (1+2^{-10n})^2 X_n. \tag{4.7}$$

Without loss of generality we may suppose that $Q \leqq 1$, so that harmonic measure for $\Omega_-^{n,j}$ is too big. We now choose to have the teeth of H_{j+1}^0 point upwards, into $\Omega_+^{n,j}$.

If the partition of H_{j+1}^0 is sufficiently fine then $\overline{K_{j+1}^0}$ is very close to $\overline{H_{j+1}^0}$. Because the curves $\Gamma_{n,j}$ and $\Gamma_{n,j+1}$ agree except for these two arcs, this implies that $\omega_+^{n,j}(\overline{H_{j+1}^0})$ and $\omega_+^{n,j+1}(\overline{K_{j+1}^0})$ must be very close, and the same for ω_-. In particular we may suppose that

(4.8)

$$1/2 \leqq \omega_+^{n,j+1}(\overline{K_{j+1}^0}) / \omega_+^{n,j}(\overline{H_{j+1}^0}) \leqq 2 \quad \text{and} \quad 1/2 \leqq \omega_-^{n,j+1}(\overline{K_{j+1}^0}) / \omega_-^{n,j}(\overline{H_{j+1}^0}) \leqq 2.$$

We now fix the partition of H^0_{j+1} so that these inequalities hold and so that all our ealier requirements are valid. Thus we have specified how the teeth on H^0_{j+1} are built, and so $\Gamma_{n,j+1}$ is constructed. It remains to verify (4.5) and (4.6).

Let z_{j+1} be the center of H_{j+1}, and let ℓ_{j+1} be the length of H_{j+1}. Let S_{j+1} be the square with center z_{j+1}, side length ℓ_{j+1}, and sides parallel and perpendicular to H_{j+1}, and let D_{j+1} be its interior. Because H_{j+1} was chosen so that $\ell_{j+1} \leqq \varepsilon_n$, we know from Section 3 that the two yards corresponding to H_{j+1} do not intersect $\Gamma_{n,j}$. This implies that the square intersects $\Gamma_{n,j}$ only at the endpoints of H_{j+1} and that $D_{j+1} \cap \Gamma_{n,j} = H_{j+1}$.

It will be convenient for us to make a renormalization. By making a linear change of variables we can map the curve $\Gamma_{n,j+1}$ to a new curve Γ such that z_{j+1} is mapped to 0, H^0_{j+1}, $\overline{H^0_{j+1}}$, and H_{j+1} get mapped to $[-1, 1]$, $[-2, 2]$, and $[-3, 3]$, S_{j+1} goes to the square with side length 6 and sides parallel to the axes, and $\Omega_+^{n,j+1}$ and $\Omega_+^{n,j+1}$ become the domains Ω_+ and Ω_- complementary to Γ and satisfying $2i \in \Omega_+$ and $-2i \in \Omega_-$. The arcs K^0_{j+1}, $\overline{K^0_{j+1}}$, and K_{j+1} will be taken to arcs A^0, $\overline{A^0}$, and A. Let μ and λ denote the measures on Γ which are associated to the conformal mappings onto Ω_+ and Ω_- that fix $-2, 2$, and ∞.

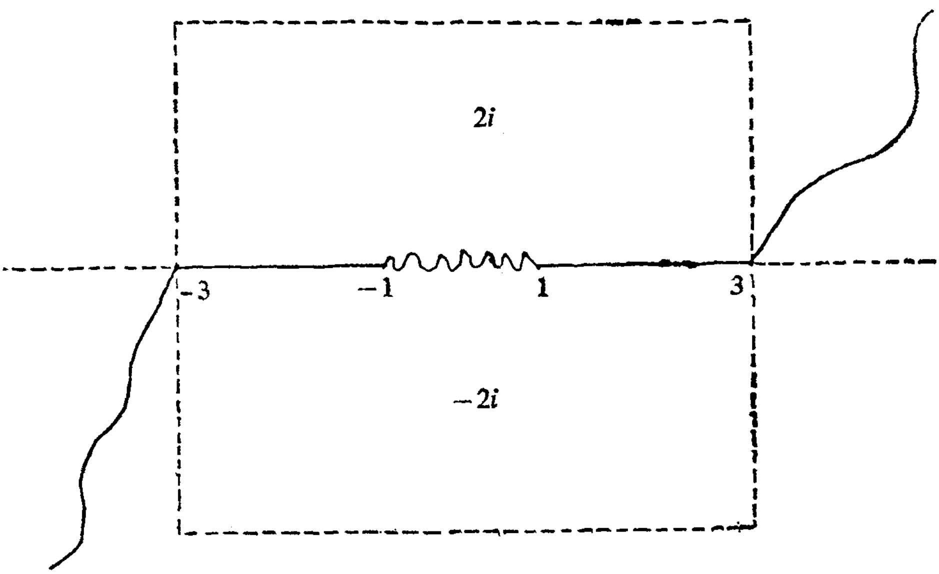


Figure 5

We need to compare $d\mu/d\lambda$ to $d\omega_+^{n,j+1}/d\omega_-^{n,j+1}$. Let $\mu_{n,j+1}$ and $\lambda_{n,j+1}$ be the measures on $\Gamma_{n,j+1}$ which are associated to the conformal mappings onto $\Omega_+^{n,j+1}$ and $\Omega_-^{n,j+1}$ which fix $-|v_\alpha|$, $|v_\alpha|$, and ∞. By rescaling and then applying Lemma 2.1

(with $\theta_1=1/2$, $\theta_2=1/2$),

$$\text{(4.9)}\qquad \begin{aligned} \frac{1}{k_1}\mu_{n,j+1} &\leqq 2|v_\alpha|\omega_+^{n,j+1} \leqq k_1\mu_{n,j+1} \\ \frac{1}{k_1}\lambda_{n,j+1} &\leqq 2|v_\alpha|\omega_-^{n,j+1} \leqq k_1\lambda_{n,j+1} \end{aligned}$$

on the arc of $\Gamma_{n,j+1}$ which joins $-|v_\alpha|$ to $|v_\alpha|$, and hence on K_{j+1}.

Let ν and ϱ denote the measures on $\Gamma_{n,j+1}$ obtained by pulling μ and λ on Γ over to $\Gamma_{n,j+1}$ by the linear mapping which takes $\Gamma_{n,j+1}$ to Γ. Then $\mu_{n,j+1}=c_1\nu$ for some $c_1>0$, because both measures are induced by conformal mappings of the upper half-plane onto $\Omega_+^{n,j+1}$ which fix ∞, and similarly $\lambda^{n,j+1}=c_2\varrho$. Clearly $c_1=\mu_{n,j+1}(\overline{K_{j+1}^0})/\nu(\overline{K_{j+1}^0})$ and $c_2=\lambda_{n,j+1}(\overline{K_{j+1}^0})/\varrho(\overline{K_{j+1}^0})$. Because $\nu(\overline{K_{j+1}^0})=\varrho(\overline{K_{j+1}^0})=4$, we have that

$$\text{(4.10)}\qquad d\mu_{n,j+1}/d\lambda_{n,j+1} = Q'd\nu/d\varrho,$$

where $Q'=\mu_{n,j+1}(\overline{K_{j+1}^0})/\lambda_{n,j+1}(\overline{K_{j+1}^0})$, and hence

$$\text{(4.11)}\qquad (1+2^{-10n})^{-2}X_n^{-1}(4k_1^2)^{-1} \leqq Q' \leqq 4k_1^2$$

by (4.7), (4.8), (4.9), and the assumption that $Q\leqq 1$.

This reduces estimates for $\Gamma_{n,j+1}$ to estimates for Γ, and we shall reduce further to a simpler curve $\hat\Gamma$. This smooth Jordan curve is obtained by taking the arc A and adding $(-\infty,-3)\cup(3,\infty)$ to it. Thus $\hat\Gamma$ looks like Γ inside $\{z\colon |x|\leqq 3,\ |y|\leqq 3\}$, and it looks like the real line outside of that region. Let $\hat\Omega_+$ and $\hat\Omega_-$ be the complementary regions of $\hat\Gamma$, and let $\hat\mu$ and $\hat\lambda$ be the measures on $\hat\Gamma$ associated to the conformal mappings onto $\hat\Omega_\pm$ which fix $-2,2$, and ∞. On $\overline{A^0}$

$$\text{(4.12)}\qquad \begin{aligned} \frac{1}{k_1^2k_2}\mu &\leqq \hat\mu \leqq k_1^2k_2\mu \\ \frac{1}{k_1^2k_2}\lambda &\leqq \hat\lambda \leqq k_1^2k_2\lambda \end{aligned}$$

by Lemmas 2.1 and 2.2, with $\theta_1=1$ and $\theta_2=3/2$.

Let t be any tooth in A with base vertices a and b, $-1<a<b<1$. Let $x_0=(a+b)/2$, $c=x_0-2N(b-a)/3$, and $d=x_0+2N(b-a)/3$. (Recall that $N=|v_\alpha|/1440$ is the number of steps of length $(b-a)$ taken between teeth.) Denote by $\hat\mu^{cd}$ and $\hat\lambda^{cd}$ the measures associated to the conformal mappings onto $\hat\Omega_\pm$ which fix c, d, and ∞. Then $\hat\mu^{cd}$ and $\hat\lambda^{cd}$ are constant multiples of $\hat\mu$ and $\hat\lambda$, and hence

$$\hat\mu = \frac{\hat\mu(\hat A(c,d))}{\hat\mu^{cd}(\hat A(c,d))}\hat\mu^{cd} \quad\text{and}\quad \hat\lambda = \frac{\hat\lambda(\hat A(c,d))}{\hat\lambda^{cd}(\hat A(c,d))}\hat\lambda^{cd},$$

where $\hat{A}(c, d)$ denotes the arc of Γ which joins c and d. By definition $\hat{\lambda}^{cd}(\hat{A}(c, d)) = \hat{\mu}^{cd}(\hat{A}(c, d))$, and

$$k_3^{-2} \leqq \frac{\hat{\mu}(\hat{A}(c, d))}{\hat{\lambda}(\hat{A}(c, d))} \leqq k_3^2$$

by Lemma 2.3, with $\theta_3 = \frac{4}{3}\left(\frac{1}{1440}\right)$, and a rescaling. Therefore

$$d\hat{\mu}/d\hat{\lambda} = r\, d\hat{\mu}^{cd}/d\hat{\lambda}^{cd}, \tag{4.13}$$

where $k_3^{-2} \leqq r \leqq k_3^2$.

On the other hand $X(\alpha)^{-1} \leqq d\hat{\mu}^{cd}/d\hat{\lambda}^{cd} \leqq X(\alpha)$, by Lemma 2.5 and a linear change of variables. Thus, by (4.12) and (4.13),

$$(k_1^4 k_2^2 k_3^2 X(\alpha))^{-1} \leqq d\mu/d\lambda \leqq k_1^4 k_2^2 k_3^2 X(\alpha)$$

on $\overline{A^0}$, which is equivalent to

$$(k_1^4 k_2^2 k_3^2 X(\alpha))^{-1} \leqq d\nu/d\varrho \leqq k_1^4 k_2^2 k_3^2 X(\alpha)$$

on $\overline{K_{j+1}^0}$. From (4.10) and (4.11) it follows that

$$(10 k_1^6 k_2^2 k_3^2 X(\alpha) X_n)^{-1} \leqq d\mu_{n, j+1}/d\lambda_{n, j+1} \leqq 4 k_1^6 k_2^2 k_3^2 X(\alpha) \tag{4.14}$$

on $\overline{K_{j+1}^0}$, and hence (4.9) implies that

$$(10 k_1^8 k_2^2 k_3^2 X(\alpha) X_n)^{-1} \leqq d\omega_+^{n, j+1}/d\omega_-^{n, j+1} \leqq 4 k_1^8 k_2^2 k_3^2 X(\alpha) \tag{4.15}$$

on $\overline{K_{j+1}^0}$. Thus (4.5) and the right side of (4.6) are valid on $\overline{K_{j+0}^0}$.

Thus we are left with showing that the left side of (4.6) holds on $C_{n+1} \cap K_{j+1}^0$, i.e., on the union of the nodal neighborhoods of the teeth in K_{j+1}^0. Let t_{j+1} be an arbitrary tooth in K_{j+1}^0, and let t denote the corresponding tooth in A. Let a, b, c, and d have the same meaning (with respect to t) as before.

If we make a linear change of variables such that a goes to -1, b goes to 1, and the upper half-plane is preserved, then $\hat{\Gamma}$ will be transformed into a curve to which Lemma 2.6 applies. Thus $d\hat{\mu}^{cd}/d\hat{\lambda}^{cd} \geqq 10 k_1^8 k_2^2 k_3^2$ on $N(t)$ if α is small enough. By (4.12) and (4.13), $d\mu/d\lambda \geqq 10 k_1^4$ on $N(t)$, or equivalently, $d\nu/d\varrho \geqq 10 k_1^4$ on $N(t_{j+1})$. Hence

$$d\omega_+^{n, j+1}/d\omega_-^{n, j+1} \geqq X_n^{-1} \quad \text{on} \quad N(t_{j+1}), \tag{4.16}$$

by (4.10) and (4.9). Because this is true for any tooth t_{j+1} on K_{j+1}^0, (4.6) is valid on all of $C_{n+1} \cap K_{j+1}^0$.

Thus $\Gamma_{n,j}$ satisfies all three of the desired conditions. This recursive construction yields the curve $\Gamma_{n,\ell} = \Gamma_{n+1}$ with the desired properties, and so we have also finished the recursive construction of the sequence $\{\Gamma_m\}$. This completes the proof of the theorem stated in Section 1.

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