Infinite groups and Hill's equation

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Any nth order linear differential equation with a discrete set of singular points on an arbitrary Riemann surface M has a monodromy group [10] G constructed as follows: A set $y_\nu(z)$ $(\nu=1, \ldots, n)$ of n linearly independent local solutions to the equation in a neighborhood of an ordinary point is analytically continued along a canonical set of cross cuts (i.e., closed loops) for the homotopy group [9] of $M'=M-\{\text{sing. of diff. eq.}\}$. The solutions $y_{\nu,\Lambda}$ obtained by analytically continuing each y_ν along a cross cut Λ are linear combinations of the y_ν and determine a matrix $A\in GL(n, \mathbf{C})$. G is generated by $A(\Lambda)$ where Λ ranges over all cross cuts mentioned. There is a natural homomorphism $\chi\colon \pi_1(M')\to G$. When $n=2$, $A(\Lambda)\in GL(2, \mathbf{C})$ for all $\Lambda\in\pi_1(M')$ and G is faithfully represented by the group G^* defined as the image of the composition of natural maps $G\overset{i}{\subset} GL(2, \mathbf{C})\to \text{Möb}$. G^* is isomorphic to G and can be regarded as the monodromy group when $n=2$. Monodromy groups have been studied extensively by Poincaré, Fuchs, Plemelj, Gunning, Deligne, Hejhal and others.

In this paper, we begin a classification of monodromy groups of the particular differential equation known as Hill's equation. The general Hill's equation [8] in $\mathbf{C}$ is a second order, linear, homogeneous differential equation of the form

$$y''+P(z)y=0 \tag{1}$$

with periodic coefficient $P(z)$, $z\in\mathbf{C}$. We consider only those equations for which $P(z)$ is a singly periodic, meromorphic function on $\mathbf{C}$ with real periods $2\pi n$ (for all $n\in\mathbf{Z}$) and with m double poles in every period strip for some $m\in\mathbf{Z}^+$. Such equations can be viewed as equations on the complex cylinder $M=\mathbf{C}/(z\to z+2\pi n$ for all $n\in Z)$ with m regular singular points.

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By substitution of suitable multi-valued functions $z=g(w)$ into the Euler equation

$$y''+\frac{\alpha}{z}y'+\frac{\beta}{z^2}y=0 \tag{2}$$

in the extended complex plane $\hat{\mathbf{C}}$ (see [1] for the properties of (2)), we obtain lifted equations which can be transformed to give Hill's equations restricted as specified in the previous paragraph. The generators for the monodromy group G^* of each resulting equation are found either by analytically continuing a ratio of its independent solutions along a generating set of loops for $\pi_1(M')$ or by analytically continuing a ratio of independent solutions to (2) along the images under $g(w)$ in $\mathbf{C}$ of a generating set of loops for $\pi_1(M')$ (see [5, 6] for a more detailed description of these techniques). Either method allows us to develop the results in Theorems 2 and 3. The abstract groups C_l, $(\bigtimes_{i=1}^{n} C_\infty)\times C_l$, $l\in\mathbf{Z}^+\cup\{+\infty\}$, $n\in\mathbf{Z}^+$ as well as Z_2-extensions of these groups are realized as monodromy groups of the lifted Hill's equations.

On the other hand, Theorem 4 and its corollary are developed by analytic continuation on M' of a ratio of solutions to a certain family of equations of type (1) depending on a complex parameter without lifting any Euler equation on $\hat{\mathbf{C}}$. The monodromy groups realized are two generator groups having certain commutator relators as well as relators arising from prime ideals in $\mathbf{Z}[\xi,\xi^{-1}]$ depending on the values of the parameter mentioned.

We now proceed to describe in detail our findings.

Theorem 1. *The substitutions* $z=t^{\lambda}(w)$, $\lambda\in\mathbf{C}^*$ *with* $t(w)$ *of form*

$$t(w)=e^{cw}\prod_{i=1}^{m}\sin^{s_i}\left(\frac{w-a_i}{2}\right),\quad c\in\mathbf{C} \tag{3}$$

or

$$t(w)=\prod_{i=1}^{m}\tan^{s_i}\left(\frac{w-a_i}{4}\right) \tag{4}$$

with $s_i\in\mathbf{C}^*$, $m>0$, $a_j\neq a_k+2n\pi$ *for all* $j,k=1,\dots,m$, $j\neq k$, *and for all* $n\in\mathbf{Z}$ *into any Euler equation* (2) *(with difference of indicial roots* $r=r_1-r_2$*) on* $\hat{\mathbf{C}}$ *produce lifted equations which can be transformed respectively into Hill's equations (with a period* 2π*)*

$$y''(w)+\frac{1}{2}\left[\frac{1-(\lambda r)^2}{2}\left(c+\sum_{j=1}^{m}\frac{s_j}{2}\cot\left(\frac{w-a_j}{2}\right)\right)^2+\theta_2 t(w)\right]y(w)=0 \tag{5}$$

or

$$y''(w)+\frac{1}{2}\left[\frac{1-(\lambda r)^2}{2}\left(\sum_{j=1}^{m}\frac{s_j}{2}\csc\left(\frac{w-a_j}{2}\right)\right)^2+\theta_2 t(w)\right]y(w)=0, \tag{6}$$

where θ_2 is the Schwarzian derivative operator [3]. *Equations* (5) *and* (6) *can be treated as equations on the complex cylinder* $\mathbf{C}/(z\to z+2\pi n$ *for all* $n\in \mathbf{Z})$ *and determine monodromy groups on this Riemann surface.*

Proof. The Euler equation (2) with difference of indicial roots $r=\sqrt{(\alpha-1)^2-4\beta}$ can be lifted by the map $z=t^\lambda(w)$ to $\mathbf{C}$ by a two step process as follows: Let $z=f\circ t(w)$ with $f(t)=t^\lambda$. First, lift (2) by $z=f(t)$ to a new Euler equation

$$y''(t)+\frac{\alpha'}{t}y'(t)+\frac{\beta'}{t^2}y(t)=0 \tag{7}$$

with $\alpha'=\lambda\alpha-\lambda+1$ and $\beta'=\beta\lambda^2$ and with difference of indicial roots

$$r'=\sqrt{(\alpha'-1)^2-4\beta'}=\lambda r. \tag{8}$$

Second, lift (7) to $\mathbf{C}$ by the substitution $t=t(w)$ to obtain

$$Y''(w)+P(w)Y'(w)+Q(w)Y(w)=0, \tag{9}$$

where $P(w)=\left(\frac{-t''}{t'}\right)+\alpha'\left(\frac{t'}{t}\right)$ and $Q(w)=\beta'\left(\frac{t'}{t}\right)^2$.

If $t(w)$ assumes form (3) or (4), observe that

$$\frac{t'}{t}=c+\sum_{j=1}^{n}\frac{s_j}{2}\cot\left(\frac{w-a_j}{2}\right) \tag{10}$$

or

$$\frac{t'}{t}=\sum_{j=1}^{n}\frac{s_j}{2}\csc\left(\frac{w-a_j}{2}\right) \tag{11}$$

respectively and that all singularities of $\frac{t'}{t}$ in (10) and (11) are simple poles. Furthermore, $\frac{t''}{t'}=\left[\left(\frac{t'}{t}\right)'+\left(\frac{t'}{t}\right)^2\right]\Big/\left(\frac{t'}{t}\right)$. Consequently, $\frac{t''}{t'}$ for (10) and (11) as well as $P(w)$ in (9) are meromorphic on $\mathbf{C}$ with simple poles as singularities. Therefore, the transformation [4]

$$Y(w)=e^{-\frac{1}{2}\int^w P(s)\,ds}y(w)$$

exists and can be used to transform (9) into

$$y''(w)+J(w)y(w)=0, \tag{12}$$

where

$$J(w)=Q(w)-\frac{1}{2}P'(w)-\frac{1}{4}P^2(w)$$

$$=\beta'\left(\frac{t'}{t}\right)^2-\frac{1}{2}\left(\frac{-t''}{t'}+\alpha'\frac{t'}{t}\right)'-\frac{1}{4}\left(\frac{-t''}{t'}+\alpha'\frac{t'}{t}\right)^2$$

$$=\left(\beta'+\frac{\alpha'}{2}-\frac{(\alpha')^2}{4}\right)\left(\frac{t'}{t}\right)^2+\frac{\theta_2 t(w)}{2}.$$

Elementary calculations using (8) produce

$$\frac{1-(\lambda r)^2}{4}=\beta'+\frac{\alpha'}{2}-\frac{(\alpha')^2}{4}$$

so that

$$J(w)=\frac{1}{2}\left[\frac{1-(\lambda r)^2}{2}\left(\frac{t'}{t}\right)^2+\theta_2 t(w)\right].$$

Equations (9) and (12) have the same ratio of linearly independent solutions. Eq. (12) is (5) or (6) for $t(w)$ of form (3) or (4) respectively.

If $t(w)$ assumes form (10) or (11), observe that $\frac{t'}{t}(w+2\pi)=\frac{t'}{t}(w)$ or $\frac{t'}{t}(w+2\pi)=\frac{-t'}{t}(w)$ respectively. Furthermore, observe that

$$\theta_2 t(w)=\frac{\left(\frac{t'}{t}\right)''}{\frac{t'}{t}}-\frac{\frac{3}{2}\left[\left(\frac{t'}{t}\right)'\right]^2}{\left(\frac{t'}{t}\right)^2}-\frac{1}{2}\left(\frac{t'}{t}\right)^2.$$

Hence, $\theta_2 t(w+2\pi)=\theta_2 t(w)$ for $t(w)$ of form (3) or (4). Also, $\left(\frac{t'}{t}\right)^2(w+2\pi)=\left(\frac{t'}{t}\right)^2(w)$. Thus, $J(w+2\pi)=J(w)$ in (12). It follows that (5) and (6) are Hill's equations with a period 2π.

From the periodicity of the coefficients of (5) and (6), we can conclude that these equations are defined on the complex cylinder $\mathbf{C}/(z\to z+2\pi n$ for all $n\in Z)$ and have monodromy groups there. □

Remark 1. The proof of Theorem 1 implies that if the substitutions $z=t^\lambda(w)$ and $z=t(w)$, $t(w)$ fixed of form (3) or (4), are made respectively into any two Euler equations with respectie differences of indicial roots r and $r'=r\lambda$, then the same transformed Hill's equation results.

We can now prove

Theorem 2. *Each equation of form* (5) *has monodromy group* G^* *of one of the following types;*

$$C_l, \quad (\mathsf{X}_{i=1}^n C_\infty)\times C_l, \quad l\in Z^+\cup\{+\infty\}, \quad n\in Z^+.$$

All of these groups (for all specified l *and* n*) are realized as* $r\lambda\in\mathbf{C}$ *and* $t(w)$ *of form* (3) *both vary.*

Proof. Remark 1 implies that there exists an equation (2) which lifts by map (3) to an equation which transforms into (5). Therefore, $\lambda=1$ can be assumed with no loss of generality. Let $u(z)$ be some ratio of linearly independent solutions to (2) and $h(w)=u\circ t(w)$ the corresponding ratio of linearly independent solutions to (5). Equation (5) has singularities at a_i $(i=1, \dots, m)$ and b_p $(p=1, \dots, n)$, the additional poles of $\theta_2 t(w)$, as well as at all translates $a_i+2\pi n$, $b_p+2\pi n$, $n\in Z$ where no translate of any b_p is a translate of any a_i. It can be assumed, without loss of generality, that all a_i $(i=1, \dots, m)$ and b_p $(p=1, \dots, n)$ lie in $D=\{z|c<\operatorname{Re} z<c+2\pi\}$ for some $c\in\operatorname{Re}$. Since, by Theorem 1, equation (5) can be viewed as an equation on the cylinder $\mathbf{C}/(z\to z+2\pi n$ for all $n\in Z)$, the group G^* is generated by the elements T_{a_i} and T_{b_j} corresponding to simple loops Λ_{a_i} and Λ_{b_j} in D about the points a_i and b_p in D as well as the element T_π corresponding to an arc $\Lambda_{2\pi}$ from some fixed base point w in D to $w+2\pi$. Here, all loops and arcs avoid singularities of (5).

Equation (2) has a ratio of linearly independent solutions given by

$$u(z)=\begin{cases} z^r & \text{if } r\in\mathbf{C}^* \\ \ln z & \text{if } r=0\end{cases}$$

so that

$$h(w)=\begin{cases} e^{crw}\prod_{i=1}^m \sin^{rs_i}\left(\dfrac{w-a_i}{2}\right) & \text{if } r\in\mathbf{C}^* \\ cw+\sum_{i=1}^m s_i\ln\sin\left(\dfrac{w-a_i}{2}\right) & \text{if } r=0\,.\end{cases}$$

Since $h(w)$ is locally single-valued in a neighborhood of b_p for all p, $T_{b_p}=\text{id}$. Furthermore, the generators T_{a_i} corresponding to the simple loops Λ_{a_i} are given by

$$T_{a_i}(z)=\begin{cases} e^{2\pi i r s_i}z & \text{if } r\in\mathbf{C}^* \\ 2\pi i s_i+z & \text{if } r=0\end{cases}, \quad i=1, \dots, m.$$

The generator T_π is obtained by determining the continuation $h(w+2\pi)$ along the arc $\Lambda_{2\pi}$. We obtain

$$h(w+2\pi)=\begin{cases} e^{cr2\pi}\prod_{i=1}^m e^{(2k_i+1)\pi i r s_i}h(w) & \text{if } r\in\mathbf{C}^* \\ \left[c2\pi+\sum_{i=1}^m (2k_i+1)\pi i s_i\right]+h(w) & \text{if } r=0,\end{cases}$$

where k_i $(i=1, \ldots, m)\in Z$ depend on the homotopy class $[\Lambda_{2\pi}]$ in $\mathbf{C}-\{a_i+2\pi n, n\in Z\}$. Therefore,

$$T_\pi(z) = \begin{cases} e^{cr2\pi} \prod_{i=1}^{m} e^{(2k_i+1)\pi i s_i} z & \text{if} \quad r\in\mathbf{C}^* \\ [c2\pi + \sum_{i=1}^{m} (2k_i+1)\pi i s_i] + z & \text{if} \quad r=0. \end{cases}$$

G^* is generated by T_{a_i} $(i=1, \ldots, m)$, T_π and is a group of affine mappings consisting entirely of multiplications if $r\in\mathbf{C}^*$ or of translations if $r=0$. Hence, G^* is Abelian and a direct product of at most $m+1$ cyclic groups [11].

We now show that G^* has at most one generator of finite order. If $r=0$, then clearly G^* has no generators or nontrivial elements of finite order. If $r\in\mathbf{C}^*$, then assume that

$$D_1(z) = e^{2\pi i \frac{I}{J}} z, \quad D_2(z) = e^{2\pi i \frac{K}{L}} z, \quad I, J, K, L\in Z^*$$

are generators of G^* having finite order. Define

$$C(z) = e^{\frac{2\pi i}{JL} gcd(IL, JK)}.$$

Number theory shows that the subgroups of G^* generated by D_i $(i=1, 2)$ and C are the same. We conclude that G^* has at most one generator of finite order. Therefore, G^* is one of the types claimed.

All of these types are realized as follows: Although G^* is generated by at most $m+1$ generators, it might have a minimal generating set with fewer elements. We will show, for fixed m and r in equation (5), that there are choices of s_i $(i=1, \ldots, m)$ and c for which a corresponding minimal generating set contains precisely $m+1$ $(m\geqq 1)$ elements of infinite order. Similar arguments are used to prove the existence of monodromy groups with one generator of finite order and with fewer than two generators of infinite order. Consequently, all groups listed will result as m and r are varied.

Suppose that, for fixed m, r and arbitrary s_i $(i=1, \ldots, m)$, c and a_i $(i=1, \ldots, m)$, there exists a minimal generating set having fewer than $m+1$ elements. This assumption leads to at least one relation of the form

$$T_\pi^{n_0} \circ \prod_{i=1}^{m} T_{a_i}^{n_i}(z) = z \quad \text{for some} \quad (n_0, n_1, \ldots, n_m)\in Z^{m+1} - \{\langle 0, 0, \ldots, 0\rangle\}.$$

Thus,

$$\begin{cases} e^{2\pi i\left(-icrn_0 + \sum_{i=1}^{m} \frac{2k_i+1}{2} r s_i n_0 + \sum_{i=1}^{m} r s_i n_i\right)} = 1 & \text{if} \quad r\in\mathbf{C}^* \\ -icn_0 + \sum_{i=1}^{m} \frac{2k_i+1}{2} s_i n_0 + \sum_{i=1}^{m} s_i n_i = 0 & \text{if} \quad r=0 \end{cases}$$

so that

$$\begin{cases} -icrn_0+\sum_{i=1}^{m}\frac{2k_i+1}{2}rs_i n_0+\sum_{i=1}^{m} rs_i n_i = N\in Z & \text{if} \quad r\in\mathbf{C}^* \\ -icn_0+\sum_{i=1}^{m}\frac{2k_i+1}{2}s_i n_0+\sum_{i=1}^{m} s_i n_i = 0 & \text{if} \quad r=0. \end{cases}$$

Equivalently,

$$\begin{cases} \vec{V}\cdot(-icr, rs_1, \ldots, rs_m) = N & \text{if} \quad r\in\mathbf{C}^* \\ \vec{V}\cdot(-ic, s_1, \ldots, s_m) = 0 & \text{if} \quad r=0, \end{cases}$$

where

$$\vec{V}=\left(n_0, n_1+\left(\frac{2k_1+1}{2}\right)n_0, \ldots, n_m+\left(\frac{2k_m+1}{2}\right)n_0\right)\in L^* = \frac{1}{2}Z^{m+1}-\{\langle 0, \ldots, 0\rangle\}.$$

It follows that

$$(-ic, s_1, \ldots, s_m)\in\begin{cases} \bigcup_{\vec{V}\in L^*} H\left(\vec{V}, \frac{N}{r}\right) & \text{if} \quad r\in\mathbf{C}^* \\ \bigcup_{\vec{V}\in L^*} H(\vec{V}, 0) & \text{if} \quad r=0, \end{cases}$$

where $H(\vec{V}, \tau)$ is the hyperplane in $\mathbf{C}^{m+1}$ with equation $\vec{V}\cdot(z_1, \ldots, z_{m+1})=\tau$. Since such countable unions of hyperplanes must be nowhere dense [12] in $\mathbf{C}^{m+1}$, there exist uncountably many choices of $(-ic, s_1, \ldots, s_m)$ in $\mathbf{C}^{m+1}$ which do not lie in the above countable unions thereby preventing the assumed existence of any relation among the generators T_π, T_{a_i} $(i=1, \ldots, m)$. Hence, $\bigtimes_{i=1}^{m+1} C_\infty$ for all $m\in Z^+$ appears as claimed.

The above construction allows the selection of $(-ic, s_1, \ldots, s_{m-1})$, $m>1$, with corresponding monodromy group $\bigtimes_{i=1}^{m} C_\infty$ for a class of equations having $m-1$ singularities. Let $r\in\mathbf{C}^*$ and choose s_m so that $rs_m=1/l$, $l\in Z^+$. Then, $(-ic, s_1, \ldots, s_{m-1}, s_m)$ corresponds to classes of equations having m singularities and monodromy groups $C_l\times\bigtimes_{i=1}^{m} C_\infty$, $m>1$.

Letting $r\in\mathbf{C}^*$ and $t^r(w)=\sin^{2/l}\left(\frac{w-a_i}{2}\right)$, $l\in Z^+$, produces an equation with one singularity and monodromy group C_l. Letting $r\in\mathbf{C}^*$ and $t^r(w)=e^w\sin^{2/l}\left(\frac{w-a_i}{2}\right)$, $l\in Z^+$, produces an equation with one singularity and monodromy group $C_l\times C_\infty$. Finally, letting $l=1$ produces the group C_∞. □

Theorem 3. *Each equation of form* (6) *has monodromy group* G^* *of one of the following types;*

$$\langle A, B_i \quad (i=1, \ldots, n); \quad A^2=1, \; B_jB_k=B_kB_j \;\; \textit{for all} \;\; j, k=1, \ldots, n,$$
$$AB_i=B_i^{-1}A \;\; \textit{for all} \;\; i=1, \ldots, n\rangle$$

or

$$\langle A, B_i\ (i=1,\dots,n);\ A^2=1,\ B_n^l=1,\ B_jB_k=B_kB_j$$
$$\textit{for all}\quad j,k=1,\dots,n,\ AB_i=B_i^{-1}A\quad \textit{for all}\quad i=1,\dots,n\rangle.$$

All of these groups (for all $l, n\in Z^+$*) are realized as* $r\lambda\in\mathbf{C}$ *and* $t(w)$ *of form* (4) *both vary.*

Remark 2. If $n=1$ in Theorem 3, then $G^*=D_\infty$ or D_l respectively.

Proof. As in the proof of Theorem 2, there exist generators

$$T_{a_i}(z)=\begin{cases} e^{2\pi i r s_i}z & \text{if}\quad r\in\mathbf{C}^* \\ 2\pi i s_i+z & \text{if}\quad r=0 \end{cases}\quad i=1,\dots,m$$

corresponding to analytic continuation of

$$h(w)=\begin{cases} \prod_{i=1}^m \tan^{rs_i}\left(\frac{w-a_i}{4}\right) & \text{if}\quad r\in\mathbf{C}^* \\ \sum_{i=1}^m s_i \ln\tan\left(\frac{w-a_i}{4}\right) & \text{if}\quad r=0 \end{cases}$$

along simple loops Λ_{a_i} about the singularities a_i $(i=1,\dots,m)$ of (6). Here, $\lambda=1$ has been assumed without loss of generality. Furthermore, the proof of Theorem 2 establishes that the subgroup of G^* generated by T_{a_i} $(i=1,\dots,m)$ is either

$$(13)\qquad \langle B_i\ (i=1,\dots,n\leqq m);\ B_jB_k=B_kB_j,\quad j,k=1,\dots,n\rangle$$

or

$$(14)\qquad \langle B_i\ (i=1,\dots,n\leqq m);\ B_jB_k=B_kB_j,\ j,k=1,\dots,n,\ B_n^l=1,\ l\in Z^+\rangle,$$

where each B_i $(i=1,\dots,n)$ is a word in the T_{a_i} $(i=1,\dots,m)$. Also, each of the generators T_{b_p} (defined as in the proof of Theorem 2) is the identity. The continuation of $h(w)$ along an arc $\Lambda_{2\pi}$ (avoiding a_i $(i=1,\dots,m)$) from some base point w to $w+2\pi$ determines the remaining generator

$$A(z)=\begin{cases} \dfrac{e^{\sum_{i=1}^m \pi i(2k_i+1)rs_i}}{z} & \text{if}\quad r\in\mathbf{C}^* \\ -z+\sum_{i=1}^m \pi i(2k_i+1)s_i & \text{if}\quad r=0, \end{cases}$$

where k_i $(i=1,\dots,m)\in Z$ depend on the homotopy class $[\Lambda_{2\pi}]$ in $\mathbf{C}-\{a_i+2\pi n;\ (i=1,\dots,m),\ n\in Z\}$. $AB_i=B_i^{-1}A$ for each B_i $(i=1,\dots,m)$ since each B_i is either a multiplicative (when $r\in\mathbf{C}^*$) or additive (when $r=0$) affine transformation. Clearly, $A^2=1$. It follows from the above relations that every word in the generators A and B_i $(i=1,\dots,m)$ can be reduced [7] to M or MA where M (possibly the identity) is a word in B_i. $MA\neq I$ since MA is an elliptic element of order 2 while $M=I$ only if M

is derivable from the relations in (13) and (14). Thus, G^* is as claimed. Finally, an analogous hyperplane construction proves that all groups listed must occur for appropriate choices of $r\lambda$ and $t(w)$ of form (4). □

Theorem 4. *The Hill's equation*

$$(15)\qquad y''(w)+\frac{1}{4}\left[\frac{1}{4}\tan^2\left(\frac{w}{2}\right)-\beta\tan\left(\frac{w}{2}\right)+\left(\frac{1}{2}-\beta^2\right)\right]y(w)=0$$

has monodromy group G^* *given by*

(A) $\langle T_0, T_\pi;\ X_1X_2=X_2X_1$ *where* X_i *range over* $T_\pi^N T_0 T_\pi^{-N}$ *for all* $N\in Z\rangle$

iff $e^{\left(\beta+\frac{i}{2}\right)2\pi}$ *is a transcendental number,*

(B) $\langle T_0, T;\ X_1X_2=X_2X_1$ *where* X_i *range over* $T_\pi^N T_0 T_\pi^{-N}$ *for all* $N\in Z$, $R_\lambda(T_\pi^N T_0 T_\pi^{-N}, N\in Z)=\text{id.}\rangle$

iff $e^{\left(\beta+\frac{i}{2}\right)2\pi}$ *is an algebraic number but not a root of unity,*

(C) $\langle T_0, T_\pi;\ T_\pi^K=\text{id.}$ *for fixed* $K\in Z^+-\{1\}$,
$X_1X_2=X_2X_1$ *where* X_i *range over* $T_\pi^N T_0 T_\pi^{-N}$
for all $N=0,1,\ldots,K-1$, $R_\lambda(T_\pi^N T_0 T_\pi^{-N}, N\in Z)=\text{id.}\rangle$

iff $e^{\left(\beta+\frac{i}{2}\right)2\pi}$ *is a primitive* Kth *root of unity but not* 1,

(D) C_∞ *iff* $e^{\left(\beta+\frac{i}{2}\right)2\pi}=1$.

Proof. Consider the equation

$$(16)\qquad \frac{t''(w)}{t'(w)}=\frac{1}{2}\tan\left(\frac{w}{2}\right)+\beta,\quad \beta\in\mathbf{C}.$$

Any solution $t(w)$ to (16) is a ratio of linearly independent solutions to

$$(17)\qquad y''(w)+\tfrac{1}{2}\,\theta_2 t(w)y(w)=0,$$

where

$$\theta_2 t(w)=\left(\frac{1}{2}\tan\left(\frac{w}{2}\right)+\beta\right)'-\frac{1}{2}\left[\frac{1}{2}\tan\left(\frac{w}{2}\right)+\beta\right]^2$$
$$=\frac{1}{8}\tan^2\left(\frac{w}{2}\right)-\frac{\beta}{2}\tan\left(\frac{w}{2}\right)+\frac{1}{2}\left(\frac{1}{2}-\beta^2\right).$$

(16) admits a solution

$$(18)\qquad t(w)=\int_0^w \sec\left(\frac{s}{2}\right)e^{\beta s}\,ds$$

multi-valued on $\mathbf{C}/(z \to z+2\pi n$ for all $n \in Z)$. The singularities of $\sec\left(\frac{s}{2}\right) e^{\beta s}$ are $w_l = (2l+1)\pi$ for all $l \in Z$ with corresponding residues $2(-1)^{l+1} e^{\beta(2l+1)\pi}$. Therefore, analytic continuation of (18) along simple loops Λ_l encircling w_l, $l \in Z$, determines monodromy generators

$$T_l(z) = z - 4\pi e^{\left(\beta+\frac{i}{2}\right)\pi} e^{\left(\beta+\frac{i}{2}\right)2\pi l}. \tag{19}$$

Furthermore, analytic continuation of (18) along an arc Λ_π on $\mathbf{C}$ corresponding to a simple non-contractible loop on $\mathbf{C}/(z \to z+2\pi n$ for all $n \in Z)$ gives

$$t(w+2\pi) = e^{\left(\beta+\frac{i}{2}\right)2\pi} t(w) + K_\beta,$$

$K_\beta \in \mathbf{C}$, with corresponding monodromy generator

$$T_\pi(z) = e^{\left(\beta+\frac{i}{2}\right)2\pi} z + K_\beta. \tag{20}$$

Here, $K_\beta = \int_{\Lambda_\pi} \sec\left(\frac{s}{2}\right) e^{\beta s}\, ds$ depends on the homotopy class $[\Lambda_\pi]$ with respect to the singularities w_l and is therefore known modulo the translations T_l for all $l \in Z$.

If $e^{\left(\beta+\frac{i}{2}\right)2\pi} = 1$, then contour integration over an appropriate choice of contour Λ_π gives $K_\beta = 0$, $T_\pi = \text{id.}$ and $T_l(z) = z \pm 4\pi$ for all $l \in Z$. Hence, $G^* = C_\infty$. Conversely, if $G^* = C_\infty$, then $e^{\left(\beta+\frac{i}{2}\right)2\pi} = 1$. Otherwise, G^* would have a minimal generating set consisting of two generators. Thus, Case (D) has been proved.

A calculation establishes that

$$T_{l+N}(z) = T_\pi^N \circ T_l \circ T_\pi^{-N}(z) \quad \text{for all} \quad l, N \in Z \tag{21}$$

so that T_0, T_π form a generating set for G^*. Furthermore, $X_1 X_2 = X_2 X_1$ for any choices of translations X_j $(j=1, 2)$ in $\{T_\pi^N \circ T_0 \circ T_\pi^{-N}$ for all $N \in Z\}$. Let

$$R_\lambda(T_0, T_\pi) = T_\pi^{l_1} \circ T_0^{j_1} \circ \ldots \circ T_\pi^{l_n} \circ T_0^{j_n} = \text{id.} \tag{22}$$

$$\text{where} \begin{cases} j_i & (i = 1, \ldots, n-1) \in Z^* \\ l_i & (i = 2, \ldots, n) \in Z^* \\ l_1, j_n \in Z \end{cases}$$

be an arbitrary relation in G^*. Observe that

$$R_\lambda(T_0, T_\pi) = (T_\pi^{l_1} \circ T_0 \circ T_\pi^{-l_1})^{j_1} \circ (T_\pi^{l_1+l_2} \circ T_0 \circ T_\pi^{-(l_1+l_2)})^{j_2} \circ \ldots$$
$$\ldots \circ (T_\pi^{\sum_{i=1}^n l_i} \circ T_0 \circ T_\pi^{-\sum_{i=1}^n l_i})^{j_n} \circ T_\pi^{\sum_{i=1}^n l_i} = \text{id.} \tag{23}$$

The relation in (23) shows that the multiplier $\left(e^{\left(\beta+\frac{i}{2}\right)2\pi}\right)^{\sum_{i=1}^{n} l_i}$ of the Möbius transformation $R_\lambda(T_0, T_\pi)$ in (23) is 1. Consequently, $T_\pi^{\sum_{i=1}^{n} l_i}=\text{id.}$ and $R_\lambda(T_0, T_\pi)$ is a word involving only the conjugates $T_\pi^N T_0 T_\pi^{-N}$, $N\in Z$. Let $e^{\left(\beta+\frac{i}{2}\right)2\pi}\neq 1$ so that $T_\pi\neq\text{id}$. Thus, every relation $T_\pi^k=\text{id.}$ can be obtained by free reduction (implying that $k=0$) iff $e^{\left(\beta+\frac{i}{2}\right)2\pi}$ is not a root of unity. Hence, Cases (A) and (B) have been distinguished from Case (C).

Let $e^{\left(\beta+\frac{i}{2}\right)2\pi}\neq 1$. If R_λ is any relation in (22) which cannot be derived from the previously discussed relations $X_1X_2=X_2X_1$, then R_λ can be written, using these relations, as

$$R_\lambda(T_0, T_\pi) = (T_\pi^{n_1}\circ T_0\circ T_\pi^{-n_1})^{j_1}\circ(T_\pi^{n_2}\circ T_0\circ T_\pi^{-n_2})^{j_2}\circ\ldots\circ(T_\pi^{n_k}\circ T_0\circ T_\pi^{-n_k})^{j_k} = \text{id.} \tag{24}$$

$$\text{where}\begin{cases} n_1>n_2>\ldots>n_k, & n_i\in Z\\ j_i\in Z^*, & i=1,\ldots,k\end{cases}$$

and conversely. Therefore,

$$R_\lambda(T_0, T_\pi)(z) = z-4\pi e^{\left(\beta+\frac{i}{2}\right)\pi}\tau^{n_k}[j_1\tau^{n_1-n_k}+j_2\tau^{n_2-n_k}+\ldots+j_k] = z, \tag{25}$$

$$\tau = e^{\left(\beta+\frac{i}{2}\right)2\pi}\neq 0, 1.$$

Hence, $P(\xi)=j_1\xi^{n_1-n_k}+j_2\xi^{n_2-n_k}+\ldots+j_k$ is a non-constant polynomial with integer coefficients having τ as a zero. Thus, τ is algebraic. Conversely, if τ is algebraic, then the reversal of the steps in the above argument constructs a non-trivial relator R_λ. Thus, Case (A) has been distinguished from Cases (B) and (C). This distinction together with the already known distinction of Cases (A) and (B) from Case (C) completes the proof. □

Remark 3. Observe that the map $\gamma_{G^*}\colon R_\lambda(T_0, T_\pi)\to\xi^{n_k}P(\xi)$ is an epimorphism from the group $\langle R_\lambda\rangle$ to the prime ideal (in the ring $Z[\xi,\xi^{-1}]$) consisting of all polynomials in ξ and ξ^{-1} having $\tau=e^{\left(\beta+\frac{i}{2}\right)2\pi}$ as a root. Here, Ker (γ_{G^*}) contains all of the relators $X_1X_2X_1^{-1}X_2^{-1}$.

Corollary. *The monodromy group G^* of equation* (15) *is Kleinian (in fact, elementary) iff $\tau=e^{\left(\beta+\frac{i}{2}\right)2\pi}$ satisfies $\tau^\nu=1$ for $\nu=1, 2, 3, 4, 6$.*

Proof. If $\tau\neq 1$, then G^* is generated by

$$T_0(z) = z-4\pi\tau^{1/2},\quad T_\pi(z) = \tau z+K_\beta.$$

The group $\bar{G}$ generated by

$$\bar{T}_0(z) = z+1,\quad \bar{T}_\pi(z) = \tau z$$

is conjugate to G^* in Möb. Hence, G^* is Kleinian iff $\bar{G}$ is Kleinian.

Now, the proof splits naturally into four cases.

Case 1. If $|\tau| \neq 1$, then the transformations

$$\bar{T}_\pi^n \circ \bar{T}_0 \circ \bar{T}_\pi^{-n} = z + \tau^n$$

include translations with $|\tau^n| < \varepsilon$ for any $\varepsilon > 0$ and suitable choices of $n \in Z$. Hence, $\bar{G}$ and G^* are not Kleinian.

Case 2. If $|\tau| = 1$ and τ is not a root of unity, then $\bar{T}_\pi$ is an elliptic element of infinite order. Hence, $\bar{G}$ and G^* are not Kleinian.

Case 3. If $|\tau| = 1$ and τ is a root of unity but not 1, then it is seen, using pp. 210—214 of [2], that $\bar{G}$ and G^* are Kleinian iff $\tau^\nu = 1$ for $\nu = 2, 3, 4, 6$. For these values of ν, G^* is an elementary Kleinian group.

Case 4. If $\tau = 1$, then the proof of Case (D) in Theorem 4 shows that G^* is generated by $T(z) = z + 4\pi$. Thus, G^* is an elementary Kleinian group. □

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