Random coverings of thin sets

Svante Janson

1. Introduction

Let E be a bounded set in $\mathbf{R}^d$ and consider the problem of covering E by a random collection of small sets. For simplicity, we will consider in this paper only the case when the random small sets are cubes of a fixed size with edges parallel to the coordinate axes and with independent, uniformly distributed centers. (See below for precise formulations.) For sets E of positive d-dimensional Lebesgue measure $m_d(E)$ (and with $m_d(\partial E)=0$), this problem was studied in detail in [3]. Hence we are in this paper mainly interested in thin sets E such as fractal curves and Cantor sets. In order to avoid trivial complications, we assume that E is infinite.

Remark 1. When $m_d(E)>0$ and $m_d(\partial E)=0$, the results in [3] are more precise than the results presented here. On the other hand, the methods in the present paper are more elementary.

Remark 2. The results in [3] apply also to covering by spheres and, more generally, by sets of any given convex shape. Furthermore, the size, shape and orientation may be random. The results do not depend on the shape of the small sets except that a constant term changes. This strongly suggests that the results of the present paper are also valid e.g. for covering by spheres.

We will study two versions of the problem. In the first version we take a fixed set $V\subset\mathbf{R}^d$ with finite measure $m_d(V)$ such that $\bar{E}\subset\operatorname{int}(V)$. Denote the cube $[0, a]^d$ by Q_a, and let (for a fixed $a>0$) the random cubes be X_i+Q_a, $i=1, 2, \ldots$, where $\{X_i\}_1^\infty$ are independent, uniformly distributed points in V. Define

$$(1.1)\qquad N_a = \min\left\{n\colon \bigcup_1^n (X_i+Q_a) \supset E\right\}.$$

Thus the random variable N_a is the number of random cubes of size a required to cover E. Obviously

$$(1.2)\qquad P(N_a \leqq n) = P(\text{the first } n \text{ random cubes of size } a \text{ cover } E).$$

The set V is introduced because we want X_i to be uniformly distributed in some set, which then has to have finite, non-zero measure. Asymptotically (as $a\to 0$), the choice of V influences the distribution of N_a only through $m_d(V)$, which appears as a scale factor; see below.

In the second version we let ξ be a Poisson process of intensity λ on $\mathbf{R}^d$. We regard ξ as a random set of points $\xi=\{x_i\}_1^\infty$, and let the random cubes be $\{x_i+Q_a: x_i\in\xi\}$. We will denote this random set of cubes by $\Xi(\lambda, a)$, usually abbreviated to Ξ.

We will discuss the connection between these two versions in Section 2.

We will relate the random coverings to the most efficient non-random coverings. We define, for $a>0$, $n(a)$ to be the smallest number of sets of diameter at most a that cover E, i.e. $n(a)\leqq n$ iff there exist sets $E_1, \ldots, E_n$ with diameters $\leqq a$ and $\bigcup_1^n E_i\supset E$. (For our purposes, we may instead take the smallest number of cubes of size a, or balls of radius a, that cover E. These numbers are equivalent within constants depending on d.) We will also use a special covering by cubes. Let

$$F_a=\left\{\prod_1^d\,[m_k a, (m_k+1)a]\right\}_{m_1\ldots m_d=-\infty}^{\infty}$$

be a mesh of cubes of size a and define

$$n_1(a)=\#\{Q\in F_a\colon Q\cap E\neq\emptyset\}.$$

It is immediately seen that $n_1(a)\leqq 3^d n(a)$ and $n(a)\leqq n_1(a/\sqrt{d})$. Further, $n_1(a/m)\leqq m^d n_1(a)$ if m is an integer.

We will in this paper use C to denote different constants that depend on d only. We thus obtain

$$Cn(a)\leqq n_1(a)\leqq Cn(a) \tag{1.3}$$

and, if $0<\delta\leqq 1$,

$$n(a)\leqq n(\delta a)\leqq C\delta^{-d}n(a). \tag{1.4}$$

We define a dimension by

$$\alpha=\varlimsup_{a\to 0}\frac{\log n(a)}{\log(1/a)}. \tag{1.5}$$

This dimension is sometimes called the upper entropy dimension. It is never smaller than the Hausdorff dimension, but it may be larger. (See Example 1 in Section 5.) For references and other definitions of dimensions, see e.g. Mandelbrot [4].

We are particularly interested in sets E with the regularity property

$$0<\varliminf_{a\to 0} a^\alpha n(a)\leqq\varlimsup_{a\to 0} a^\alpha n(a)<\infty \tag{1.6}$$

for some $\alpha>0$ (necessarily the dimension given by (1.5)). We will give some of our results in two versions; one general and one special for sets satisfying (1.6).

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2. Results

In the Poisson case, we introduce a new parameter by

$$\mu = \lambda a^d. \tag{2.1}$$

Note that the number of cubes in $\Xi(\lambda, a)$ that contain a fixed point x is distributed as $Po(\mu)$, and in particular

$$P(x \text{ is covered by } \Xi) = 1-e^{-\mu}, \quad x\in\mathbf{R}^d. \tag{2.2}$$

The basic result of this paper is the following pair of estimates, proved in the next two sections.

Theorem. *If* $\mu\geqq 1$, *then, with notations as above,*

$$\exp\left(-Cn(a/\mu)e^{-\mu}\right) \leqq P\left(\Xi(\lambda, a) \text{ covers } E\right) \leqq \exp\left(-Cn(a/\mu)\, e^{-\mu}\right). \tag{2.3}$$

We obtain easily, using (1.4), the following corollary.

Corollary 1. *Let* $\mu=\mu(a)$ *be a function of* a. *Then, as* $a\to 0$,

$$P(\Xi \text{ covers } E) \to 1 \Leftrightarrow \mu-\log n(a/\mu) \to +\infty \Leftrightarrow \mu-\log n\left(a/\log n(a)\right)\to+\infty \tag{2.4}$$

and

$$P(\Xi \text{ covers } E) \to 0 \Leftrightarrow \mu -\log n(a/\mu) \to -\infty \Leftrightarrow \mu-\log n\left(a/\log n(a)\right) \to -\infty. \tag{2.5}$$

If (1.6) holds, then

$$\sup\left\{\left|\log n\left(a/\log n(a)\right)-\alpha\left(\log\frac{1}{a}+\log\log\frac{1}{a}\right)\right|:\ a \leqq \frac{1}{e}\right\} <\infty. \tag{2.6}$$

Hence

Corollary 2. *If* (1.6) *holds for some* $\alpha>0$, *then, as* $a\to 0$,

$$P(\Xi \text{ covers } E) \to 1 \Leftrightarrow \mu-\alpha\left(\log\frac{1}{a}+\log\log\frac{1}{a}\right) \to +\infty \tag{2.7}$$

$$P(\Xi \text{ covers } E) \to 0 \Leftrightarrow \mu-\alpha\left(\log\frac{1}{a}+\log\log\frac{1}{a}\right) \to -\infty. \tag{2.8}$$

We connect the two versions of our covering problem as follows. Let V be a set as in Section 1, and assume that $\sqrt{d}a<\inf\{|x-y|:\ x\in E,\ y\notin V\}$. Then only cubes x_i+Q_a with $x_i\in V$ meet E. The number of points in $\xi\cap V$ has the Poisson distribution $Po\left(\lambda m_d(V)\right)$ and the set $\xi\cap V$ is distributed as $\{X_i\}_1^M$, where $\{X_i\}_1^\infty$ is as in Section 1 and M has the distribution $Po\left(\lambda m_d(V)\right)$ and is independent of $\{X_i\}_1^\infty$. Hence

$$P\left(\Xi(\lambda, a) \text{ covers } E\right) = P\left(\{X_i+Q_a\}_1^M \text{ covers } E\right) = P(N_a \leqq M).$$

Now let $T_1, T_2, \ldots$ be independent of each other and of $\{X_i\}$, and with the common exponential distribution Exp (1), and let $S_n=\sum_1^n T_i$. We may take $M=\sup\{n: S_n \leqq \lambda m_d(V)\}$. Hence,

$$(2.9)\qquad P(\Xi \text{ covers } E) = P(N_a \leqq M) = P(S_{N_a} \leqq \lambda m_d(V)) = P(Z_a \leqq \mu),$$

where we put $Z_a=\dfrac{1}{m_d(V)}a^d S_{N_a}$.

Remark 3. This argument is closely related to a method by Holst [1], who introduces a time coordinate and a Poisson process in the space-time.

For simplicity we will henceforth assume that $m_d(V)=1$. In general, the results remain valid if N_a is divided by $m_d(V)$.

Now, assume that $n(a)\geqq 3$. By (1.4),

$$(2.10)\qquad \log n(a) \leqq \log n(a/\log n(a)) \leqq \log n(a)+d\log\log n(a)+C.$$

Take

$$\mu = \log n(a/\log n(a))+x.$$

If $x\geqq 0$ we obtain by (2.9), (2.3), (1.4) and (2.10),

$$(2.11)\qquad P(Z_a > \mu) \leqq 1-\exp(-Cn(a/\mu)e^{-\mu}) \leqq Cn(a/\mu)e^{-\mu} =$$
$$= Cn(a/\mu)(n(a/\log n(a)))^{-1}e^{-x} \leqq C(\mu/\log n(a))^d e^{-x} \leqq C(1+x/\log n(a))^d e^{-x}.$$

If $-\frac{1}{2}\log n(a)\leqq x\leqq 0$, then $\mu\geqq \log n(a)+x\geqq \frac{1}{2}\log n(a)$ and

$$(2.12)\qquad P(Z_a \leqq \mu) \leqq \exp(-Cn(a/\mu)e^{-\mu}) = \exp(-Cn(a/\mu)(n(a/\log n(a)))^{-1}e^{-x}) \leqq$$
$$\leqq \exp(-Ce^{-x}).$$

It follows from these estimates that, for $n(a_0)\geqq 3$,

$$\{Z_a-\log n(a/\log n(a))\}_{a\leqq a_0}$$

is a tight family of random variables, and further, as $a\to 0$,

$$(2.13)\qquad Z_a/\log n(a) \xrightarrow{P} 1$$

and

$$(2.14)\qquad EZ_a = \log n(a/\log n(a))+0(1) = \log n(a)+0(\log\log n(a)).$$

Since $S_{N_a}/N_a \xrightarrow{P} 1$ by the law of large numbers as $a\to 0$ and thus $N_a\to\infty$, and $ES_{N_a}=EN_a\cdot ET_1=EN_a$, we obtain

Corollary 3. *If* $m_d(V)=1$, *then, as* $a\to 0$,

$$(2.15)\qquad \frac{a^d N_a}{\log n(a)} \xrightarrow{P} 1,$$

$$(2.16)\qquad a^d EN_a = \log n(a)+\mathcal{O}(\log\log n(a)).$$

This yields the following expression for the dimension α defined in (1.5). We may here replace EN_a by the median or any other fixed percentile of the distribution of N_a.

Corollary 4. *If* $m_d(V)=1$, *then*

$$\overline{\lim_{a\to 0}} \frac{a^d EN_a}{\log(1/a)} = \alpha. \tag{2.17}$$

Furthermore

$$E(Z_a - a^d N_a)^2 = E\Big(a^d \sum_1^{N_a} (T_i-1)\Big)^2 = a^{2d} EN_a = \mathcal{O}\left(a^d \log\frac{1}{a}\right) \tag{2.18}$$

as $a\to 0$. Consequently

Corollary 5. *If* $m_d(V)=1$ *and* $n(a_0)\geqq 3$, *then*

$$\{a^d N_a - \log n(a/\log n(a))\}_{a\leqq a_0} \quad \textit{is tight.}$$

If further (1.6) *holds, then*

$$\left\{a^d N_a - \alpha\left(\log\frac{1}{a} + \log\log\frac{1}{a}\right)\right\}_{a\leqq 1/e} \quad \textit{is tight.}$$

If E is a subset of $\{(t_1, \dots, t_d)\colon t_{k+1}, \dots, t_d = 0\}$ for some $k\leqq d$, $m_k(E)>0$, then the dimension α defined by (1.5) equals k and (1.6) holds. Further, if $m_k(\partial E)=0$, where ∂E is the boundary of E regarded as a subset of $\mathbf{R}^k$, the results of [3] imply that

$$a^d N_a - k\left(\log\frac{1}{a} + \log\log\frac{1}{a}\right) \xrightarrow{d} k\log k + \log m_k(E) + U,$$

where $P(U\leqq u)=\exp(-e^{-u})$.

We further note that if a subsequence of $\{a^d N_a - \log n(a/\log n(a))\}$ converges in distribution to some W (as $a\to 0$), then, by (2.18), (2.9), (2.3), cf. (2.11), (2.12), W is stochastically larger than $U-C$ and smaller than $U+C$, where U is as above.

This might lead one to expect that $a^d N_a - \log n(a/\log n(a))$ (or $a^d N_a - \alpha\left(\log\frac{1}{a} + \log\log\frac{1}{a}\right)$ when (1.6) holds) converges in distribution as $a\to 0$. However, as we will see in Example 2 in Section 5, this is not true for the Cantor set. This suggests the following modification, at least for sufficiently regular sets E.

Conjecture. There exists a bounded function $\psi(a)$ (depending on E) such that (if $m_d(V)=1$)

$$a^d N_a - \log n(a/\log n(a)) - \psi(a) \xrightarrow{d} U \quad \text{as} \quad a\to 0.$$

(This is equivalent to the property that every limit distribution of a subsequence of $\{a^d N_a - \log n(a/\log n(a))\}$ is of the type $U+$constant.)

3. The lower bound

We will use the following correlation inequality for the Poisson process. For a more general version and a proof, see [2, Lemma 2.1].

Lemma 1. *Let $D_1, \dots, D_m$ be Borel sets in $\mathbf{R}^d$ and let A_j denote the event $\xi \cap D_j \neq \emptyset$. Then $P(\bigcap_1^m A_j) \geqq \prod_1^m P(A_j)$.*

Let $\varepsilon > 0$ and consider a covering $\{E_j\}_{j=1}^n$ of E consisting of sets of diameter at most εa, with $n = n(\varepsilon a)$. Thus, there exist points $y_j \in \mathbf{R}^d$ such that

$$E_j \subset y_j + Q_{\varepsilon a}, \quad j = 1, \dots, n.$$

Hence, if $\{x_i + Q_{a-\varepsilon a}\}$ covers $\{y_j\}_1^n$, then $\bigcup (x_i + Q_a)$ contains every $y_j + Q_{\varepsilon a}$ and thus every E_j, whence E is covered. Consequently,

$$(3.1) \qquad P\big(\Xi(\lambda, a) \text{ covers } E\big) \geqq P\big(y_j \in \bigcup \Xi(\lambda, a - \varepsilon a), \quad j = 1, \dots, n\big) \geqq$$
$$\geqq \prod_1^n P\big(y_j \in \bigcup \Xi(\lambda, a - \varepsilon a)\big),$$

where we have used Lemma 1. By (2.2), each factor in the last product equals

$$1 - \exp\big(-\lambda (a - \varepsilon a)^d\big) = 1 - \exp\big(-\mu (1-\varepsilon)^d\big) \geqq \exp\big(-C \exp(-\mu (1-\varepsilon)^d)\big)$$

provided $\mu \geqq 1$, $\varepsilon \leqq 1/2$, say. Consequently,

$$(3.2) \qquad P\big(\Xi(\lambda, a) \text{ covers } E\big) \geqq \exp\big(-Cn \exp(-\mu(1-\varepsilon)^d)\big).$$

Now take $\varepsilon = 1/(2d\mu)$. Then $\mu(1-\varepsilon)^d \geqq \mu(1 - d\varepsilon) \geqq \mu - 1$, whence

$$(3.3) \qquad P\big(\Xi(\lambda, a) \text{ covers } E\big) \geqq \exp\big(-Cn(\varepsilon a)\, e^{1-\mu}\big) \geqq \exp\big(-Cn(a/\mu)\, e^{-\mu}\big).$$

4. The upper bound

Lemma 2. *Let Q be a cube of size a and let $\{y_j\}_1^n$ be a set of points in Q such that if y_i and y_j are two points in the set and their coordinates are denoted by $y_{i,k}$ and $y_{j,k}$, $k = 1, \dots, d$, then $|y_{i,k} - y_{j,k}| \geqq a/\mu$ for at least one k. Then*

$$(4.1) \qquad P\big(\Xi \text{ covers } \{y_j\}_1^n\big) \leqq 1 - Cne^{-\mu} \leqq \exp(-Cne^{-\mu}).$$

Proof. By translation, we may assume that $Q = Q_a$. Let A_j denote the event $\{\Xi(\lambda, a)$ does not cover y_j, but every $t = (t_1, \dots, t_d) \in Q_a$ such that $t_k \geqq y_{j,k} + a/\mu$ for at least one k is covered$\}$. Then the events $\{A_j\}_1^n$ are disjoint, and

$$(4.2) \qquad P\big(\{y_j\}_1^n \text{ is covered}\big) \leqq 1 - P\big(\textstyle\bigcup_1^n A_j\big) = 1 - \sum_1^n P(A_j).$$

To estimate A_j, let, for each $k = 1, \dots, d$, $\{R_{j,k,l}\}_{l=1}^{2^{d-1}}$ denote the 2^{d-1} rectangular boxes $I_1 \times \ldots \times I_d$, where $I_k = (y_{j,k}, y_{j,k} + a/\mu)$ and either $I_i = (0, a/2)$ or $I_i = (-a/2, 0)$ for $i \neq k$.

Let $A_{j,k,l}$ denote the event $\xi \cap R_{j,k,l} \neq \emptyset$. Since $m_d(R_{j,k,l}) = \frac{a}{\mu}\left(\frac{a}{2}\right)^{d-1} = 2^{1-d}\frac{a^d}{\mu}$, we have

$$P(A_{j,k,l} \neq \emptyset) = 1 - \exp\left(-\lambda m_d(R_{j,k,l})\right) = 1 - \exp\left(-2^{1-d}\right).$$

It is easily seen that

$$A_j \supset \{y_j \text{ is not covered}\} \cap \bigcap_{k=1}^{d} \bigcap_{l=1}^{2d-1} A_{j,k,l}.$$

Since $\{y_j$ is not covered$\}$ and $\bigcap_k \bigcap_l A_{j,k,l}$ are independent, we obtain by Lemma 1,

$$\text{(4.3)} \qquad P(A_j) \geqq P\{y_j \text{ is not covered}\} P\left(\bigcap_k \bigcap_l A_{j,k,l}\right) \geqq$$

$$\geqq e^{-\mu} \bigcap_k \bigcap_l P(A_{j,k,l}) = \left(1 - \exp\left(-2^{1-d}\right)\right)^{d \cdot 2^{d-1}} e^{-\mu}.$$

The lemma follows by (4.2) and (4.3).

We now prove a special case of the sought inequality.

Lemma 3. *If $E \subset Q$, where Q is a cube of size a, then*

$$\text{(4.4)} \qquad P(\Xi \text{ covers } E) \leqq \exp\left(-C n_1(a/\mu) e^{-\mu}\right).$$

Proof. Let, for $m = (m_1, \ldots, m_d) \in \mathbf{Z}^d$,

$$Q'(m) = \prod_{k=1}^{d} [m_k(a/\mu), (m_k+1)(a/\mu)].$$

For any sequence $l = (l_1, \ldots, l_d) \in \{0, 1\}^d$, define $B_l = \{Q'(m) \colon Q'(m) \cap E \neq \emptyset$ and $m \equiv l \pmod 2\}$, where $m \equiv l$ means $m_k \equiv l_k \pmod 2$ for each $k = 1, \ldots, d$. By definition, $n_1(a/\mu) = \sum_l \# B_l$.

Fix l and choose one point in $Q'(m) \cap E$ for each $Q'(m) \in B_l$. This gives a set $\{y_j\}_1^n$, with $n = \# B_l$, that satisfies the condition of Lemma 2. Thus

$$P(\Xi \text{ covers } E) \leqq P(\Xi \text{ covers } \{y_j\}) \leqq \exp\left(-C \# B_l e^{-\mu}\right)$$

and, taking the product over all $l \in \{0, 1\}^d$,

$$P(\Xi \text{ covers } E)^{2^d} \leqq \prod_l \exp\left(-C \# B_l e^{-\mu}\right) = \exp\left(-C n_1(a/\mu) e^{-\mu}\right),$$

which yields (4.4) and proves the lemma.

We complete the proof of (2.3) by a similar argument. Define for $m \in \mathbf{Z}^d$, $Q(m) = \prod_1^d [m_k a, (m_k+1)a]$, and let $n_{1,m}$ denote $n_1(a/\mu)$ computed for the set $E \cap Q(m)$. Clearly $n_1(a/\mu) \leqq \sum_m n_{1,m}$.

Again let $l \in \{0, 1\}^d$. Note that the events $\{\Xi \text{ covers } E \cap Q(m)\}$ for m such that $m \equiv l \pmod 2$, are independent. Hence, using Lemma 3 on $E \cap Q(m)$,

$$P(\Xi \text{ covers } E) \leqq P(\Xi \text{ covers } E \cap Q(m) \text{ for all } m \equiv l)$$

$$= \prod_{m \equiv l} P(\Xi \text{ covers } E \cap Q(m)) \leqq \prod_{m \equiv l} \exp(-C n_{1,m} e^{-\mu}) = \exp\left(-C \sum_{m \equiv l} n_{1,m} e^{-\mu}\right).$$

Consequently, taking the product again,

$$P(\Xi \text{ covers } E)^{2^d} \leqq \exp\left(-C \sum_l \sum_{m \equiv l} n_{1,m} e^{-\mu}\right) \leqq \exp\left(-C n(a/\mu)\, e^{-\mu}\right).$$

The theorem is proved.

5. Examples

Example 1. Let $d=1$ and $E=\{k^{-\gamma}\}_{k=1}^{\infty}$ for some fixed $\gamma>0$. It is not difficult to show that the dimension defined by (1.5) equals $(1+\gamma)^{-1}$ and that (1.6) holds. Note that the dimension is positive although E is countable and its Hausdorff dimension thus equals zero. Corollaries 2—5 answer various questions about the random coverings.

Example 2. We will perform an explicit calculation for Cantor sets in $\mathbf{R}$ and certain values of a. First, let E be any closed subset of $[0, a]$ with $m_1(E)=0$, and let $\complement E=(-\infty, a_0)\cup(b_0, \infty)\cup\bigcup_1^\infty (a_i, b_i)$ where the intervals are disjoint. Denote the smallest positive element of ξ by Y_+, and the largest negative element by $-Y_-$. Then Y_+ and Y_- are independent Exp $(1/\lambda)$ random variables. Hence, since $P(Y_+ \in E)=0$,

$$(5.1) \qquad P(\Xi \text{ does not cover } E) = P(\Xi \text{ does not cover } E \text{ and } Y_+ < a_0)$$

$$+P(\Xi \text{ does not cover } E \text{ and } Y_+ > b_0)+\sum_i P(\Xi \text{ does not cover } E \text{ and } a_i < Y_+ < b_i)$$

$$= 0+P(Y_+ > b_0 \text{ and } -Y_- < b_0-a)+\sum_i P(a_i < Y_+ < b_i \text{ and } -Y_- < a_i-a)$$

$$= e^{-\lambda b_0} e^{-\lambda(a-b_0)} + \sum_i (e^{-\lambda a_i} - e^{-\lambda b_i}) e^{-\lambda(a-a_i)} = e^{-\lambda a}\left(1+\sum_i \left(1-e^{-\lambda(b_i-a_i)}\right)\right).$$

Now let $\delta \leqq 1/3$ be fixed and let $E_0=[0, 1]$, $E_1=[0, \delta]\cup[1-\delta, 1], \dots$ so that E_k consists of 2^k intervals of length δ^k separated by at least $(1-2\delta)\delta^{k-1}$. Let $E=\bigcap_1^\infty E_k$. It is easily seen that if $\delta^k \leqq a < \delta^{k-1}$, $k \geqq 0$, then $n(a)=2^k$. Consequently (1.6) is satisfied with the dimension $\alpha=\log 2/\log \delta^{-1}$, i.e. $\delta^\alpha=1/2$.

Suppose that

$$(5.2) \qquad \delta^k \leqq a \leqq (1-2\delta)\, \delta^{k-1}$$

for some $k \geqq 0$.

By (5.1), and a simple count of the intervals in $[0, \delta^k]\setminus E$,

$$P(\Xi \text{ covers } E\cap[0, \delta^k]) = 1-e^{-\mu}\big(1+\textstyle\sum_{j=0}^{\infty} 2^j(1-e^{-\lambda(1-2\delta)\delta^{k+j}})\big). \tag{5.3}$$

Since E consists of 2^k parts congruent to $E\cap[0, \delta^k]$ and these are covered independently of each other,

$$P(\Xi \text{ covers } E) = \big(1-e^{-\mu}\big(1+\textstyle\sum_{j=0}^{\infty} 2^j(1-e^{-\lambda(1-2\delta)\delta^{k+j}})\big)\big)^{2^k}. \tag{5.4}$$

Define

$$\varphi(x) = x^{-\alpha}\textstyle\sum_{-\infty}^{\infty} 2^l\big(1-\exp(-x(1-2\delta)\delta^l)\big), \quad x>0. \tag{5.5}$$

The series is uniformly convergent on compact sets and thus φ is continuous (and in fact analytic). Furthermore, $\varphi(x/\delta)=\delta^{\alpha}2\varphi(x)=\varphi(x)$. Hence φ has the multiplicative period δ^{-1}. Thus, since $\varphi(x)>0$, $\log\varphi(x)$ is bounded above and below. An elementary computation, which we omit, shows that the periodic function $\varphi\big((1/\delta)^{t/2\pi}\big)$ has Fourier coefficients

$$-(\log(1/\delta))^{-1}(1-2\delta)^{(\log 2+2\pi in)/\log(1/\delta)}\,\Gamma\big(-(\log 2+2\pi in)/\log(1/\delta)\big),$$

$n=\ldots-1, 0, 1, \ldots$. In particular, φ is not constant.

Now, let $a\to 0$ through a subsequence such that (5.2) holds (with k depending on a) and let $\mu=\lambda a\to\infty$. If $\delta^{-m}\leqq\lambda\delta^k<\delta^{-m-1}$ then $m\to\infty$ and

$$\begin{aligned}\textstyle\sum_{j=0}^{\infty} 2^j(1-e^{-\lambda(1-2\delta)\delta^{k+j}}) &= 2^m\textstyle\sum_{l=-m}^{\infty} 2^l(1-e^{-\lambda\delta^{k+m}(1-2\delta)\delta^l}) \\ &= 2^m(\lambda\delta^{k+m})^{\alpha}\big(\varphi(\lambda\delta^{k+m})+o(1)\big) = \lambda^{\alpha}2^{-k}\big(\varphi(\lambda)+o(1)\big).\end{aligned} \tag{5.6}$$

Consequently, since the last expression $\sim\lambda^{\alpha}\delta^{\alpha k}=(\lambda\delta^k)^{\alpha}\to\infty$, (5.4) yields

$$P(\Xi \text{ covers } E) = \big(1-2^{-k}\lambda^{\alpha}e^{-\mu}(\varphi(\lambda)+o(1))\big)^{2^k}. \tag{5.7}$$

Taking $\mu=\alpha\log\frac{1}{a}+\alpha\log\log\frac{1}{a}+\log\varphi\left(\alpha\frac{1}{a}\log\frac{1}{a}\right)+\alpha\log\alpha+x$, the right hand side of (5.7) converges to $\exp(-e^{-x})$ as $a\to 0$. Consequently, cf. (2.9),

$$Z_a-\alpha\left(\log\frac{1}{a}+\log\log\frac{1}{a}\right)-\alpha\log\alpha-\log\varphi\left(\alpha\frac{1}{a}\log\frac{1}{a}\right)\xrightarrow{d} U, \tag{5.8}$$

as $a\to 0$ through values satisfying (5.2). (We do not know whether (5.8) holds for unrestricted a.) By (2.18), we may here replace Z_a by a^dN_a (provided $m_1(V)=1$).

We repeat that this example suggested the conjecture at the end of Section 2.

References

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Svante Janson
Department of Mathematics
Uppsala University
Thunbergsvägen 3
752 38 Uppsala, Sweden