Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 3, 1994, 69–80

HEAT CONTENT ASYMPTOTICS OF NON-MINIMAL OPERATORS

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Dedicated to Jean Leray

0. Introduction

Let M be a compact smooth Riemannian manifold of dimension m with smooth boundary ∂M. Let V be a smooth unitary vector bundle over M and let P be a second order partial differential operator on $C^\infty(V)$ with positive definite leading symbol. We impose suitable boundary conditions $\mathcal{B}$ for P and assume $P_{\mathcal{B}}$ is strongly elliptic and self-adjoint. Let $f \in C^\infty(V)$. To study the short time behavior of the fundamental solution to the heat equation $e^{-tP_{\mathcal{B}}} f$, we introduce an auxiliary smooth test function $\widetilde{f} \in C^\infty(V)$ and define

$$\beta(f, \widetilde{f}, P, \mathcal{B})(t) := \int_M (e^{-tP_{\mathcal{B}}} f, \widetilde{f})\, dx. \tag{0.1}$$

Standard elliptic methods, see for example the discussion in [9, Lemma 1.3] show that as $t \downarrow 0^+$ there is an asymptotic series of the form

$$\beta(f, \widetilde{f}, P, \mathcal{B})(t) \sim \sum_{n=0}^{\infty} \beta_n(f, \widetilde{f}, P, \mathcal{B}) t^{n/2}. \tag{0.2}$$

[1]Research partially supported by the NSF (USA), by the MPIM (Bonn, FRG), and by the DaeWoo foundation (Korea).

We say that P is an operator of *Laplace type* if the leading symbol of P is scalar and is given by the metric tensor. This means that locally P has the form

$$P = -(g^{\nu\mu} I_V \partial_\mu \partial_\nu + a^\nu \partial_\nu + b) \tag{0.3}$$

where a^ν and b are endomorphisms of V; we adopt the Einstein convention and sum over repeated indices. The β_n have been computed for $n \leq 3$ with mixed boundary conditions [12], for $n \leq 4$ with Dirichlet boundary conditions [8], [9], [12], and for $n \leq 6$ with Neumann boundary conditions [8], [12]. We also refer to related work [5], [6], [7], [19].

These invariants have not been studied previously for operators which are not of Laplace type. In this paper, we will study an operator whose leading symbol is not scalar; such operators are often said to be "non-minimal" in the physics literature. Let $\Lambda^p M$ be the bundle of p-forms; $\Lambda^1 M = T^*M$. Let

$$\begin{aligned} &d_p : C^\infty \Lambda^p M \to C^\infty \Lambda^{p+1} M, \text{ and} \\ &\delta_p : C^\infty \Lambda^{p+1} M \to C^\infty \Lambda^p M \end{aligned} \tag{0.4}$$

be exterior differentiation d_p and the adjoint, interior differentiation δ_p. Let A and B be positive constants, let E be an endomorphism of T^*M, and let

$$D := A d_0 \delta_0 + B \delta_1 d_1 - E \qquad \text{on } C^\infty T^*M. \tag{0.5}$$

Then the leading symbol of D is positive definite. We impose absolute ($\mathcal{B}^a$), relative ($\mathcal{B}^r$), or Dirichlet ($\mathcal{B}^D$) boundary conditions; $D_\mathcal{B}$ is strongly elliptic and self-adjoint, see [15, S4.6] for details. We note that D is an operator of Laplace type only if $A = B$.

Operators of this form where E is a linear combination of the Ricci tensor ρ and its trace, the scalar curvature τ, arise in many contexts. In mathematical physics, they are used in the study of quantum gravity and gauge fields in curved space-time [3], [4], [13], [16], [20] and also in classical continuum mechanics [2], [11], [18]. In differential geometry, we refer to [10] for an application in conformal geometry, to [1] for an application in quasi-conformal geometry, and to [14] for an application related to conformal Killing vector fields.

In this paper, we will compute $\beta_n(D, \mathcal{B})$ for $n = 0, 1, 2$. Let $\omega, \widetilde{\omega} \in C^\infty T^*M$.

THEOREM 0.1 (absolute boundary conditions).

(a) $\beta_0(\omega, \widetilde{\omega}, D, \mathcal{B}^a) = (\omega \cdot \widetilde{\omega})[M]$.

(b) $\beta_1(\omega, \widetilde{\omega}, D, \mathcal{B}^a) = -2\pi^{-1/2} A^{1/2} (\omega_m \widetilde{\omega}_m)[\partial M]$.

$$\begin{aligned}\text{(c)}\ \beta_2(\omega,\widetilde{\omega},D,\mathcal{B}^a) = &-(A\delta_0\omega\cdot\delta_0\widetilde{\omega}+Bd_1\omega\cdot d_1\widetilde{\omega}-E\omega\cdot\widetilde{\omega})[M]\\ &+A(-\omega_m\widetilde{\omega}_{a:a}-\omega_{a:a}\widetilde{\omega}_m-\omega_{m;m}\widetilde{\omega}_m\\ &-\omega_m\widetilde{\omega}_{m;m}+\tfrac{3}{2}L_{aa}\omega_m\widetilde{\omega}_m)[\partial M].\end{aligned}$$

THEOREM 0.2 (relative boundary conditions).

(a) $\beta_0(\omega,\widetilde{\omega},D,\mathcal{B}^r)=(\omega\cdot\widetilde{\omega})[M]$.

(b) $\beta_1(\omega,\widetilde{\omega},D,\mathcal{B}^r)=-2\pi^{-1/2}B^{1/2}(\omega_a\widetilde{\omega}_a)[\partial M]$.

$$\begin{aligned}\text{(c)}\ \beta_2(\omega,\widetilde{\omega},D,\mathcal{B}^r) = &-(A\delta_0\omega\cdot\delta_0\widetilde{\omega}+Bd_1\omega\cdot d_1\widetilde{\omega}-E\omega\cdot\widetilde{\omega})[M]\\ &+B(-\omega_{a:a}\widetilde{\omega}_m-\omega_m\widetilde{\omega}_{a:a}-\omega_{a;m}\widetilde{\omega}_a\\ &-\omega_a\widetilde{\omega}_{a;m}+L_{ab}\omega_b\widetilde{\omega}_a+\tfrac{1}{2}L_{aa}\omega_b\widetilde{\omega}_b)[\partial M].\end{aligned}$$

Our results for Dirichlet boundary conditions are incomplete; there is one undetermined coefficient.

THEOREM 0.3 (Dirichlet boundary conditions).

(a) $\beta_0(\omega,\widetilde{\omega},D,\mathcal{B}^D)=(\omega\cdot\widetilde{\omega})[M]$.

(b) $\beta_1(\omega,\widetilde{\omega},D,\mathcal{B}^D)=-2\pi^{-1/2}(A^{1/2}\omega_m\widetilde{\omega}_m+B^{1/2}\omega_a\widetilde{\omega}_a)[\partial M]$.

$$\begin{aligned}\text{(c)}\ \beta_2(\omega,\widetilde{\omega},D,\mathcal{B}^D) = &-(A\delta_0\omega\cdot\delta_0\widetilde{\omega}+Bd_1\omega\cdot d_1\widetilde{\omega}-E\omega\cdot\widetilde{\omega})[M]\\ &+(c_4(A,B)(\omega_{a:a}\,\widetilde{\omega}_m+\omega_m\widetilde{\omega}_{a:a})\\ &-A(\omega_{m;m}\widetilde{\omega}_m+\omega_{m;m}\widetilde{\omega}_m)-B(\omega_{a;m}\widetilde{\omega}_a+\omega_a\widetilde{\omega}_{a;m})\\ &+\tfrac{3}{2}AL_{aa}\,\omega_m\widetilde{\omega}_m+BL_{ab}\omega_b\widetilde{\omega}_a+\tfrac{1}{2}BL_{aa}\omega_b\widetilde{\omega}_b)[\partial M].\end{aligned}$$

In Section 1, we establish notation and recall some previous results concerning operators of Laplace type. In Section 2, we derive the functorial properties of these invariants which we shall need and complete the proof. It is a pleasant task to thank M. van den Berg, S. Desjardins, and B. Orsted for many stimulating conversations on this subject.

1. Notational conventions and analytic results

We use Greek indices ν, μ to index local coordinate frames ∂_ν and dx^ν for the tangent TM and cotangent T^*M bundles. We use Roman indices i, j to index local orthonormal frames $\{e_i\}$ for these bundles. These indices range from 1 through m. Let ∇^g be the Levi-Civita connection on M. Let $\mathrm{ext}^l(\cdot)$ be left exterior multiplication and let $\mathrm{int}^l(\cdot)$ be the dual, left interior multiplication; these are covariant constant, i.e.

$$\nabla^g\mathrm{ext}^l=0\quad\text{and}\quad\nabla^g\mathrm{int}^l=0. \tag{1.1}$$

We express exterior differentiation d and interior differentiation δ, the formal adjoint, in the form:

$$d = \mathrm{ext}^l(e_i)\nabla^g_{e_i} \quad \text{and} \quad \delta = -\mathrm{int}^l(e_i)\nabla^g_{e_i}. \tag{1.2}$$

Near the boundary, let $x = (y, r)$ for $y = (y^1, \dots, y^{m-1})$ be local coordinates so that r is the geodesic distance to the boundary and so the curves $x(r) = (y, r)$ are unit speed geodesics which are perpendicular to the boundary when $r = 0$. Greek indices α, β range from 1 through $m-1$ and index the coordinate frames for $T\partial M$ and $T^*\partial M$. We choose an orthonormal frame $\{e_i\}$ so that $e_m = \partial_r$ is the inward unit geodesic normal; Roman indices a, b range from 1 through $m-1$ and index the corresponding frame for the tangent space of the boundary. The second fundamental form is defined on the boundary of M and measures the extent to which the boundary fails to be totally geodesic:

$$L_{ab} = g(\nabla^g_{e_a} e_b, e_m). \tag{1.3}$$

Let dx and dy be the Riemannian measures on M and ∂M. If $\psi \in C^\infty(M)$ and if $\phi \in C^\infty(\partial M)$, let

$$\psi[M] = \int_M \psi(x)\, dx \quad \text{and} \quad \phi[\partial M] = \int_{\partial M} \phi(y)\, dy. \tag{1.4}$$

Let ";" be multiple covariant differentiation with respect to the Levi-Civita connection of M. Similarly let ":" denote multiple tangential covariant differentiation on the boundary of M with respect to the Levi-Civita connection of the boundary. Since the normal index is treated differently, ":" and ";" differ by the second fundamental form.

The following technical lemma is immediate so we omit the proof.

LEMMA 1.1.

(a) *Let $\omega = \omega_i e_i \in C^\infty T^*M$. Then*

$$\omega_{b;a} = \omega_{b:a} - L_{ab}\omega_m \quad \textit{and} \quad \omega_{m;a} = \omega_{m:a} + L_{ab}\omega_b.$$

(b) *Let $\omega_p \in C^\infty \Lambda^p M$. Then*

$$(d_p\omega_p \cdot \omega_{p+1} - \omega_p \cdot \delta_p\omega_{p+1})[M] = -\{\omega_p \cdot \mathrm{int}^l(e_m)\omega_{p+1}\}[\partial M].$$

(c) *Let $\omega, \tilde\omega \in C^\infty T^*M$. Then*

$$\begin{aligned}-(\Delta_1\omega \cdot \tilde\omega)[M] = &- \{(\delta_0\omega \cdot \delta_0\tilde\omega) + (d_1\omega \cdot d_1\tilde\omega)\}[M] \\ &+ (-\omega_{a:a}\tilde\omega_m - \omega_m\tilde\omega_{a:a} - \omega_{m;m}\tilde\omega_m \\ &- \omega_{a;m}\tilde\omega_a + L_{aa}\omega_m\tilde\omega_m + L_{ab}\omega_b\tilde\omega_a)[\partial M].\end{aligned}$$

Let V be a smooth vector bundle over M which is equipped with a smooth pointwise positive definite inner product $\cdot$. We assume given an orthogonal splitting

(1.5) $$V|_{\partial M} = V^+ \oplus V^-.$$

Extend the splitting to be covariant constant on the normal geodesic rays near the boundary. Let $\Pi^\pm$ be the projections of V on $V^\pm$. Assume given an auxiliary endomorphism S of $V^+|_{\partial M}$; extend S to be zero on $V^-|_{\partial M}$. We define mixed boundary conditions $\mathcal{B}$ by

(1.6) $$\mathcal{B}f := \Pi^+(f_{;m} + Sf)|_{\partial M} \oplus \Pi^-(f)|_{\partial M}.$$

Let P be a second order partial differential operator on $C^\infty(V)$ with positive definite leading symbol. Let

(1.7) $$\mathrm{domain}(P_{\mathcal{B}}) := \{f \in C^\infty(M) : \mathcal{B}f = 0\}.$$

We assume $P_{\mathcal{B}}$ is self-adjoint and elliptic as discussed in [15, **S11**]; this is true for all the operators we shall be considering.

The heat content asymptotics have been computed for operators of Laplace type. Let $D := -(f_{;ii} + Ef)$ be a self-adjoint operator of Laplace type on $C^\infty(V)$.

THEOREM 1.2 (Desjardins-Gilkey [12]).

(a) $\beta_0(f, \tilde{f}, D, \mathcal{B}) = (f \cdot \tilde{f})[M]$.

(b) $\beta_1(f, \tilde{f}, D, \mathcal{B}) = -2\pi^{-1/2}(\Pi^- f \cdot \Pi^- \tilde{f})[\partial M]$.

(c) $\beta_2(f, \tilde{f}, D, \mathcal{B}) = -(Df \cdot \tilde{f})[M] + \{(\Pi^+ f_{;m} + S\Pi^+ f) \cdot \tilde{f}$
$+\frac{1}{2}L_{aa}\Pi^- f \cdot \Pi^- \tilde{f} - \Pi^- f \cdot \Pi^- \tilde{f}_{;m}\}[\partial M]$.

Absolute and relative boundary conditions appear in index theory. Let

(1.8) $$\omega = f_I dy^I + f_{I,m} dy^I \wedge dr \in C^\infty \Lambda M.$$

We take Neumann boundary conditions on the tangential component and Dirichlet boundary conditions on the normal component to define

(1.9) $$\mathcal{B}^a\omega = (\partial_r f_I) dy^I|_{\partial M} \oplus f_{I,m} dy^I|_{\partial M} \in C^\infty\Lambda(\partial M) \oplus C^\infty\Lambda(\partial M)$$

so these are boundary conditions of the sort we have been discussing. Let $\star$ be the Hodge operator. Relative boundary conditions are defined dually by

(1.10) $$\mathcal{B}^r := \star\mathcal{B}^a\star.$$

We denote the kernel and range of an operator by

(1.11) $$\mathfrak{N}(\cdot) \quad \text{and} \quad \mathfrak{R}(\cdot).$$

Let $H^p(\cdot)$ denote the singular cohomology groups. The Hodge-de Rham theorem gives isomorphisms

$$\mathfrak{N}(\Delta_{p,\mathcal{B}^a}) \simeq H^p(M) \quad \text{and} \quad \mathfrak{N}(\Delta_{p,\mathcal{B}^r}) \simeq H^p(M;dM) \tag{1.12}$$

so these boundary conditions are important topologically speaking. In this setting, the Hodge $\star$ operator defines Poincaré duality:

$$\star : \mathfrak{N}(\Delta_{p,\mathcal{B}^a}) \simeq \mathfrak{N}(\Delta_{m-p,\mathcal{B}^r}). \tag{1.13}$$

EXAMPLE 1.1. Absolute boundary conditions correspond to Neumann and relative boundary conditions correspond to Dirichlet boundary conditions on M. If we assume M is connected, then

$$\mathfrak{N}(\Delta_{0,\mathcal{B}^a}) = 1 \cdot \mathbf{C} \quad \text{and} \quad \mathfrak{N}(\Delta_{0,\mathcal{B}^r}) = 0. \tag{1.14}$$

We use Theorem 1.2 to compute the heat content asymptotics for Δ_1.

LEMMA 1.3.

(a) $\beta_1(\omega,\tilde{\omega},\Delta_1,\mathcal{B}^a) = -2\pi^{-1/2}(\omega_m\tilde{\omega}_m)[\partial M]$.

(b) $\beta_2(\omega,\tilde{\omega},\Delta_1,\mathcal{B}^a) = -(\delta_0\omega\cdot\delta_0\tilde{\omega} + d_1\omega\cdot d_1\tilde{\omega})[M]$
$\qquad + (-\omega_{a:a}\tilde{\omega}_m - \omega_m\tilde{\omega}_{a:a} - \omega_{m;m}\tilde{\omega}_m$
$\qquad - \omega_m\tilde{\omega}_{m;m} + \frac{3}{2}L_{aa}\omega_m\tilde{\omega}_m)[\partial M]$.

(c) $\beta_1(\omega,\tilde{\omega},\Delta_1,\mathcal{B}^r) = -2\pi^{-1/2}(\omega_a\tilde{\omega}_a)[\partial M]$.

(d) $\beta_2(\omega,\tilde{\omega},\Delta_1,\mathcal{B}^r) = -(\delta_0\omega\cdot\delta_0\tilde{\omega} + d_1\omega\cdot d_1\tilde{\omega})[M]$
$\qquad + (-\omega_{a:a}\tilde{\omega}_m - \omega_m\tilde{\omega}_{a:a} - \omega_{a;m}\tilde{\omega}_a$
$\qquad - \omega_a\tilde{\omega}_{a;m} + L_{ab}\omega_b\tilde{\omega}_a + \frac{1}{2}L_{aa}\omega_b\tilde{\omega}_b)[\partial M]$.

(e) $\beta_1(\omega,\tilde{\omega},\Delta_1,\mathcal{B}^D) = -2\pi^{-1/2}(\omega_a\tilde{\omega}_a + \omega_m\tilde{\omega}_m)[\partial M]$.

(f) $\beta_2(\omega,\tilde{\omega},\Delta_1,\mathcal{B}^D) = -(\delta_0\omega\cdot\delta_0\tilde{\omega} + d_1\omega\cdot d_1\tilde{\omega})[\partial M]$
$\qquad +(-\omega_{a:a}\tilde{\omega}_m - \omega_m\tilde{\omega}_{a:a} - \omega_{i;m}\tilde{\omega}_i - \omega_m\tilde{\omega}_{i;m}$
$\qquad + \frac{3}{2}L_{aa}\omega_m\tilde{\omega}_m + L_{ab}\omega_b\tilde{\omega}_a + \frac{1}{2}L_{aa}\omega_b\tilde{\omega}_b)[\partial M]$.

PROOF. We use Theorem 1.2. Let $\omega = \omega_a e_a + \omega_m e_m$. Then

$$\mathcal{B}^a\omega = \{(\omega_{a;m} - L_{ab}\omega_b)e_a + \omega_m e_m\}|_{\partial M} \tag{1.15}$$

so $\Pi^+\omega = \omega_a e_a$ and $Se_a = -L_{ab}e_b$. Then (a) is immediate and

$$\begin{aligned}\beta_2(\omega,\tilde{\omega},\Delta_1,\mathcal{B}^a) = &-(\Delta_1\omega\cdot\tilde{\omega})[M] + \{(\omega_{a;m} - L_{ab}\omega_b)\tilde{\omega}_a \\ &+ \tfrac{1}{2}L_{aa}\omega_m\tilde{\omega}_m - \omega_m\tilde{\omega}_{m;m})\}[\partial M];\end{aligned} \tag{1.16}$$

(b) follows from Lemma 1.1(c). Similarly,

$$\mathcal{B}^r\omega = \{\omega_a e_a + (\omega_{m;m} - L_{aa}\omega_m)e_m\}|_{\partial M} \tag{1.17}$$

so $\Pi^+(\omega) = \omega_m e_m$ and $Se_m = -L_{aa}e_m$. Then (c) is immediate and

$$\begin{aligned}\beta_2(\omega,\widetilde{\omega},\Delta_1,\mathcal{B}^r) = &- (\Delta_1\omega\cdot\widetilde{\omega})[M] + \{(\omega_{m;m} - L_{aa}\omega_m)\widetilde{\omega}_m \\ &+ \tfrac{1}{2}L_{aa}\omega_b\widetilde{\omega}_b - \omega_a\widetilde{\omega}_{a;m}\}[\partial M];\end{aligned} \tag{1.18}$$

(d) again follows from Lemma 1.1(c). Finally,

$$\mathcal{B}^D\omega = \{\omega_a e_a + \omega_m e_m\}|_{\partial M}, \tag{1.19}$$

so $\Pi^+ = 0$. Then (e) is immediate and

$$\beta_2(\omega,\widetilde{\omega},\Delta_1,\mathcal{B}^r) = -(\Delta_1\omega\cdot\widetilde{\omega})[M] + \{\tfrac{1}{2}L_{aa}\omega_i\widetilde{\omega}_i - \omega_i\widetilde{\omega}_{i;m}\}[\partial M]. \tag{1.20}$$

2. Functorial properties

We begin with some useful, if elementary, observations. Throughout this section, let

$$D := Ad_0\delta_0 + B\delta_1 d_1 - E \qquad \text{on } C^\infty T^*M. \tag{2.1}$$

Let $\omega = \omega_i e_i \in C^\infty T^*M$. Let $\mathcal{B} \in \{\mathcal{B}^D, \mathcal{B}^a, \mathcal{B}^r\}$.

LEMMA 2.1.

(a) $\beta_0(\omega,\widetilde{\omega},D,\mathcal{B}) = (\omega\cdot\widetilde{\omega})[M]$.

(b) $\beta_n(\omega,\widetilde{\omega},D,\mathcal{B}) = \beta_n(\widetilde{\omega},\omega,D,\mathcal{B})$.

(c) *If $\mathcal{B}\omega = 0$, then $\beta_n(\omega,\widetilde{\omega},D,\mathcal{B}) = -(2/n)\beta_{n-2}(D\omega,\widetilde{\omega},D,\mathcal{B})$.*

(d) *If $\partial M = \emptyset$, then $\beta_{2n-1} = 0$ and $\beta_{2n}(\omega,\widetilde{\omega},D,\mathcal{B}) = (-1)^n(D^n\omega\cdot\widetilde{\omega})[M]/n!$.*

PROOF. Let $\{\phi_\nu,\lambda_\nu\}$ be a spectral resolution of $D_{\mathcal{B}}$. Let

$$\gamma_\nu = (\omega\cdot\phi_\nu)[M] \quad \text{and} \quad \widetilde{\gamma}_\nu = (\widetilde{\omega}\cdot\phi_\nu)[M] \tag{2.2}$$

be the Fourier coefficients. Then

$$\beta(\omega,\widetilde{\omega},D,\mathcal{B})(t) = \sum_\nu e^{-t\lambda_\nu}\gamma_\nu\widetilde{\gamma}_\nu. \tag{2.3}$$

We set $t = 0$ to prove (a) by checking

$$\beta_0(\omega,\widetilde{\omega},D,\mathcal{B}) = \sum_\nu \gamma_\nu\widetilde{\gamma}_\nu = (\omega\cdot\widetilde{\omega})[M]. \tag{2.4}$$

Since (2.3) is symmetric, (b) follows. If ω satisfies the boundary conditions, then

$$(D\omega \cdot \phi_\nu)[M] = (\omega \cdot D\phi_\nu)[M] = \lambda_\nu(\omega \cdot \phi_\nu)[M]. \tag{2.5}$$

Consequently, $(D\omega \cdot \phi_\nu)[M] = \lambda_\nu\gamma_\nu$ so

$$\begin{aligned}\partial_t\beta(\omega,\widetilde{\omega},D,\mathcal{B})(t) &= -\sum_\nu \lambda_\nu e^{-t\lambda_\nu}\gamma_\nu\widetilde{\gamma}_\nu \\ &= -\beta(D\omega,\widetilde{\omega},D,\mathcal{B})(t).\end{aligned} \tag{2.6}$$

We equate terms in the asymptotic expansions to prove (c). If the boundary is empty, all functions satisfy the boundary conditions. We may therefore express

$$\beta_n(\omega,\widetilde{\omega},D,\mathcal{B}) = -(2/n)\beta_{n-2}(D\omega,\widetilde{\omega},D,\mathcal{B}). \tag{2.7}$$

(e) now follows by induction since $\beta_{-1} = 0$ and $\beta_0 = (\omega \cdot \widetilde{\omega})[M]$. $\square$

The following lemma will enable us to dimension shift. We make the following assumptions. Let $M = M_1 \times M_2$ be a Riemannian product where $\partial M_1 = \emptyset$. Decompose

$$T^*M = T^*M_1 \oplus T^*M_2. \tag{2.8}$$

Let $\omega, \widetilde{\omega} \in C^\infty T^*M_2$ be independent of the point in M_1. Let

$$\begin{aligned} E &= \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} && \text{for } E_i \in C^\infty \mathrm{End}(T^*M_i), \\ D_i &= Ad_0\delta_0 + B\delta_1 d_1 - E_i && \text{on } C^\infty T^*M_i, \\ D &= Ad_0\delta_0 + B\delta_1 d_1 - E && \text{on } C^\infty T^*M. \end{aligned} \tag{2.9}$$

LEMMA 2.2. $\beta_n(\omega,\widetilde{\omega},D,\mathcal{B}) = \mathrm{vol}(M_1)\beta_n(\omega,\widetilde{\omega},D_2,\mathcal{B})$.

PROOF. Let $\{\phi_\nu,\lambda_\nu\}$ be a spectral resolution of $D_{2,\mathcal{B}}$ on $C^\infty T^*M_2$. Then $\mathcal{B}\phi_\nu = 0$ and $D\phi_\nu = \lambda_\nu\phi_\nu$ since the auxiliary coordinates play no role. We expand $\omega = \sum_\nu \gamma_\nu\phi_\nu$ and $\widetilde{\omega} = \sum_\nu \widetilde{\gamma}_\nu\phi_\nu$. Then

$$e^{-tD_\mathcal{B}}\omega = e^{-tD_{2,\mathcal{B}}}\omega = \sum_\nu e^{-t\lambda_\nu}\gamma_\nu\phi_\nu. \tag{2.10}$$

We note $(\phi_\nu \cdot \phi_\mu)[M] = \mathrm{vol}(M_1)(\phi_\nu \cdot \phi_\mu)[M_2]$. This yields the identity

$$\beta(\omega,\widetilde{\omega},D,\mathcal{B})(t) = \mathrm{vol}(M_1)\sum_\nu e^{-t\lambda_\nu}c_\nu\widetilde{c}_\nu = \mathrm{vol}(M_1)\beta(\omega,\widetilde{\omega},D_2,\mathcal{B})(t). \tag{2.11}$$

$\square$

The interior integrands are not determined by Lemma 2.1(d) as we can always integrate by parts at the cost of introducing additional boundary terms. The boundary integrands are also not uniquely determined; we can integrate by parts tangentially to exchange tangential derivatives in the boundary integrands.

The local formulae for the boundary integrands are built universally and polynomially from the metric tensor, its inverse, the covariant derivatives of the ω_i, the covariant derivatives of the curvature tensor R, the covariant derivatives of the endomorphism E, and the tangential covariant derivatives of the second fundamental form L. By Weyl's work [21], these polynomials can be formed using only tensor products and contraction of tensor arguments (indices); this yields the Weyl spanning set. The structure group is the orthogonal group $O(m-1)$ and the normal direction plays a distinguished role.

LEMMA 2.3. *There exist constants $c_i = c_i(A, B, \mathcal{B})$ independent of m so that*

(a) $\beta_1(\omega, \widetilde{\omega}, D, \mathcal{B}) = (c_1\omega_a\widetilde{\omega}_a + c_2\omega_m\widetilde{\omega}_m)[\partial M]$.

(b)
$$\begin{aligned}\beta_2(\omega, \widetilde{\omega}, D, \mathcal{B}) = &-\{A\delta_0\omega\cdot\delta_0\widetilde{\omega} + Bd_1\omega\cdot d_1\widetilde{\omega} - E\omega\cdot\widetilde{\omega}\}[M]\\ &+\{c_3(\omega_{a;m}\widetilde{\omega}_a + \omega_a\widetilde{\omega}_{a;m}) + c_4(\omega_m\widetilde{\omega}_{a:a} + \omega_{a:a}\widetilde{\omega}_m)\\ &+c_5(\omega_{m;m}\widetilde{\omega}_m + \omega_m\widetilde{\omega}_{m;m}) + c_6L_{ab}\omega_a\widetilde{\omega}_b\\ &+c_7L_{aa}\omega_b\widetilde{\omega}_b + c_8L_{aa}\omega_m\widetilde{\omega}_m\}[\partial M].\end{aligned}$$

PROOF. We use Lemma 2.1(d) to see β_1 does not involve an interior integral and integrate by parts to express the interior integral for β_2 in the form given. We use dimensional analysis (see for example [9, Lemma 2.3]) to see that if $\mathcal{A}$ is a monomial appearing in the boundary integral for β_n of degree (k_R, k_E, k_L) in (R, E, L) and if k_∇ explicit covariant derivatives appear in $\mathcal{A}$, then

$$2k_R + 2k_E + k_L + k_\nabla = n - 1. \tag{2.12}$$

By Lemma 2.1(b), β_n is symmetric. We write down a suitable basis for the space of boundary invariants to complete the proof; we use Lemma 2.2 to see the constants involved are dimension free. □

We will complete the proof of Theorems 0.1, 0.2, and 0.3 by evaluating the constants c_i which appear in Lemma 2.3. Since these constants do not involve the endomorphism E, we set $E = 0$ henceforth. There are useful shuffle formulas if $\mathcal{B} \in \{\mathcal{B}^a, \mathcal{B}^r\}$; this is the feature of these boundary conditions that makes them so tractible and important.

LEMMA 2.4. *Let $D = Ad_0\delta_0 + B\delta_1 d_1$.*

(a) *Let $f \in C^\infty(M)$ and let $\omega \in C^\infty T^*M$. Then*

$$\beta_n(d_0 f, \omega, D, \mathcal{B}^a) = A^{n/2}\beta_n(d_0 f, \omega, \Delta_1, \mathcal{B}^a).$$

(b) *Let $\omega \in C^\infty T^*M$ and let $\Phi \in C^\infty\Lambda^2 M$. Then*

$$\beta_n(\omega, \delta_1\Phi, D, \mathcal{B}^r) = B^{n/2}\beta_n(\omega, \delta_1\Phi, \Delta_1, \mathcal{B}^r).$$

PROOF. We use the Hodge decomposition theorem to decompose

$$L^2T^*M = \mathfrak{R}_{\mathcal{B}}(d_0) \oplus \mathfrak{R}_{\mathcal{B}}(\delta_1) \oplus \mathfrak{N}(\Delta_{1,\mathcal{B}}) \tag{2.13}$$

with $\mathcal{B} \in \{\mathcal{B}^a, \mathcal{B}^r\}$; we are using at this point the observation that if ϕ is an eigenfunction of Δ_1 which satisfies absolute or relative boundary conditions, then both $d_0\delta_0\phi$ and $\delta_1 d_1\phi$ are eigenfunctions of Δ_1 satisfying the given boundary conditions; this fails for Dirichlet boundary conditions.

We use (2.13) to find a spectral resolution for $\Delta_{1,\mathcal{B}}$ of the form

$$\{\phi_i, \lambda_i\}_{1\le i<\infty}, \qquad \{\psi_j, \mu_j\}_{1\le j<\infty}, \qquad \{h_k, 0\}_{1\le k\le k_0} \tag{2.14}$$

where $k_0 = \dim \mathfrak{N}(\Delta_{1,\mathcal{B}})$ and:

$$0 < \lambda_1 \le \lambda_2 \le \dots, \quad d_0\delta_0\phi_i = \lambda_i\phi_i, \quad \text{and} \quad \mathcal{B}\phi_i = 0, \tag{2.15}$$

$$0 < \mu_1 \le \mu_2 \le \dots, \quad \delta_1 d_1\psi_j = \mu_j\psi_j, \quad \text{and} \quad \mathcal{B}\psi_j = 0, \tag{2.16}$$

$$d_1 h_k = 0, \quad \delta_0 h_k = 0, \quad \text{and} \quad \mathcal{B}h_k = 0. \tag{2.17}$$

Then (2.14) also gives the spectral resolution of $D_{\mathcal{B}}$ with appropriately modified eigenvalues $\{A\lambda_i,\ B\mu_j,\ 0\}$.

Suppose first $\mathcal{B} = \mathcal{B}^a$. Let $\psi \in \{\psi_j, h_k\}$ so $\delta_0\psi = 0$. Since $\mathcal{B}^a\psi = 0$, the normal component $\psi_m|_{\partial M} = 0$. By Lemma 1.1(b),

$$(d_0 f \cdot \psi)[M] = (f \cdot \delta_0\psi)[M] - (f \cdot \psi_m)[\partial M] = 0. \tag{2.18}$$

Consequently, only the Fourier coefficients of $d_0 f$ relative to the ϕ_i enter so that

$$\begin{aligned} &e^{-tD}(d_0 f) = \sum_i e^{-tA\lambda_i}\{(d_0 f \cdot \phi_i)[M]\}\phi_i, \\ &\beta(d_0 f, \omega, D, \mathcal{B}^a)(t) = \beta(d_0 f, \omega, \Delta_1, \mathcal{B}^a)(At), \qquad \text{and} \\ &\beta_n(d_0 f, \omega, D, \mathcal{B}^a) = A^{n/2}\beta(d_0 f, \omega, \Delta_1, \mathcal{B}^a). \end{aligned} \tag{2.19}$$

Next suppose $\mathcal{B} = \mathcal{B}^r$. Let $\phi \in \{\phi_i, h_k\}$ so that $d_1\phi = 0$. Since $\mathcal{B}^r\phi = 0$, the tangential component $\phi_a|_{\partial M} = 0$. Expand

$$\Phi = \tfrac{1}{2}\Phi_{ab} e_a \wedge e_b + \Phi_{am} e_a \wedge e_m; \qquad \mathrm{int}^l(e_m)\Phi = -\Phi_{am} e_a. \tag{2.20}$$

By Lemma 1.1(b),

$$(\phi \cdot \delta_1\Phi)[M] = (d_1\phi \cdot \Phi)[M] - (\phi_a \Phi_{am})[\partial M] = 0. \tag{2.21}$$

Consequently, only the Fourier coefficients of $\delta_1\Phi$ relative to the ψ_j enter so that

$$\begin{aligned} &e^{-tD}(\delta_1\Phi) = \sum_i e^{-tB\mu_j}\{(\delta_1\Phi \cdot \psi_j)[M]\}\psi_j, \\ &\beta(\delta_1\Phi, \omega, D, \mathcal{B}^r)(t) = \beta(\delta_1\Phi, \omega, \Delta_1, \mathcal{B}^r)(Bt), \\ &\beta_n(\delta_1\Phi, \omega, D, \mathcal{B}^r) = B^{n/2}\beta_n(\delta_1\Phi, \omega, \Delta_1, \mathcal{B}^r). \end{aligned} \tag{2.22}$$

□

If only the normal variable enters, the equations decouple. Let T be the $(m-1)$-dimensional torus with periodic parameters $(y^1, \dots, y^{m-1})$. Give $M := T \times [0, \pi]$ the warped product metric:

$$ds^2 := g_{\alpha\beta}(r) dy^\alpha \circ dy^\beta + dr \circ dr. \tag{2.23}$$

Let $\omega = \omega(r)$ and $\widetilde{\omega} = \widetilde{\omega}(r)$ only depend on the normal parameter. Decompose $\omega = \omega^T \oplus \omega^N$ into tangential and normal components where

$$\omega^T := \omega_a e_a \quad \text{and} \quad \omega^N := \omega_m e_m. \tag{2.24}$$

LEMMA 2.5. $\beta_n(\omega, \widetilde{\omega}, D, \mathcal{B}) = A^{n/2} \beta_n(\omega^N, \widetilde{\omega}^N, \Delta_1, \mathcal{B}) + B^{n/2} \beta_n(\omega^T, \widetilde{\omega}^T, \Delta_1, \mathcal{B})$.

PROOF. We decompose

$$D\omega = B\delta_1 d_1(\omega^T) \oplus A d_0 \delta_0(\omega^N), \qquad \mathcal{B}\omega = \mathcal{B}\omega^T \oplus \mathcal{B}\omega^N. \tag{2.25}$$

Since $\delta_1 d_1(\omega^T)$ has no normal component and $d_0 \delta_0 \omega^N$ has no tangential component, the action of $e^{-tD_\mathcal{B}}$ on ω decouples so that

$$e^{-tD_\mathcal{B}} = e^{-tB\Delta_1} \omega^T \oplus e^{-tA\Delta_1} \omega^N. \tag{2.26}$$

□

We can now complete the proof of Theorems 0.1, 0.2, and 0.3; we use Lemma 2.1 to compute β_0. Adopt the notation of Lemma 2.5. Since ω and $\widetilde{\omega}$ depend only on the normal coordinate, $\omega_{a:a}\widetilde{\omega}_m = \omega_m \widetilde{\omega}_{a:a} = 0$. This is the only additional relationship, however, which is imposed on the invariants of Lemma 2.3 and consequently c_i for $i \neq 4$ is determined by Lemmas 1.3 and 2.5.

We apply Lemma 2.4 to express $\beta_n(d_0 f, \widetilde{\omega}, D, \mathcal{B}^a)$ and $\beta_n(\delta_1 \Phi, \widetilde{\omega}, D, \mathcal{B}^r)$ in terms of $\beta_n(\cdot, \widetilde{\omega}, \Delta_1, \cdot)$. This determines the unknown coefficient c_4 for these two boundary conditions; there is no such argument available to determine c_4 in Theorem 0.3(c). This completes the proof of all the assertions in this paper.

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Manuscript received July 15, 1993

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