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ON NIRENBERG'S PROBLEM AND RELATED TOPICS

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Dedicated to Jean Leray

0. Introduction

Let $(\mathbf{S}^n, g_0)$ be the standard n-sphere. The following question was raised by Professor L. Nirenberg. Which function $K(x)$ on $\mathbf{S}^2$ is the Gauss curvature of a metric g on $\mathbf{S}^2$ conformally equivalent to g_0? Naturally, one may ask a similar question in higher dimensional cases, namely, which function $K(x)$ on $\mathbf{S}^n$ is the scalar curvature of a metric g on $\mathbf{S}^n$ conformally equivalent to g_0? In [20]–[22] we have given some existence results on the Nirenberg problem for $n \geq 4$ which are quite natural extensions of previous results of A. Chang & P. Yang ([7]) and A. Bahri & J. M. Coron ([2]) for $n = 2, 3$. A related critical exponent equation in $\mathbb{R}^n$ has also been studied and some existence results have been given in [17]–[22]. In this note we summarize the main results in [17]–[22] and outline the proofs. For $n = 2$, if we write $g = e^{2v} g_0$, the Nirenberg problem is equivalent to finding a function v on $\mathbf{S}^2$ which satisfies the equation

$$-\Delta_{g_0} v + 1 = K(x) e^{2v}, \tag{0.1}$$

where Δ_{g_0} denotes the Laplace-Beltrami operator associated with the metric g_0. For $n \geq 3$, if we write $g = v^{4/(n-2)} g_0$, the problem is equivalent to finding a

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positive function v on $\mathbf{S}^n$ which satisfies the equation

$$-\Delta_{g_0} v + c(n) R_0 v = c(n) K(x) v^{(n+2)/(n-2)}, \tag{0.2}$$

where $c(n) = (n-2)/(4(n-1))$, $R_0 = n(n-1)$. We observe that a necessary condition for solving the problem is that K has to be positive somewhere. For $n = 2$ this follows from the Gauss-Bonnet theorem, while for $n \geq 3$, it follows from multiplying (0.2) by v and integrating by parts on $\mathbf{S}^n$. It turns out that there is at least one more obstruction (Kazdan-Warner obstruction, see [15]) to solving the problem which is obtained by exploiting the centered dilation conformal transformations of $\mathbf{S}^n$. See also [3] for more obstructions in the same spirit. In particular, the problem is not solvable if $\mathbf{S}^n$ is embedded as usual in $\mathbb{R}^{n+1}$ and $K \in C^1(\mathbf{S}^n)$ is strictly monotone in one direction. For instance, there is no solution for $K(x) = x^{n+1} + 2$. There has been much work devoted to the existence and multiplicity results, trying to understand under what conditions (0.2) is solvable. For details, see the introduction and references in [21]. We will only recall a few results in the following:

THEOREM (J. Moser [23]). *Let $K \in C^1(\mathbf{S}^2)$ be positive somewhere and $K(-x) = K(x)$ for all $x \in \mathbf{S}^2$. Then, there is at least one solution to* (0.1).

THEOREM (A. Chang & P. Yang [7] for $n = 2$, A. Bahri & J. M. Coron [2] for $n = 3$). *For $n = 2, 3$, let $K \in C^2(\mathbf{S}^n)$ be positive and have only nondegenerate critical points $x_1, \ldots, x_m$ of Morse index $k_1, \ldots, k_m$. Assume further that*

$$-\Delta_{g_0} K(x_i) \neq 0, \qquad \forall i = 1, \ldots, m.$$

If $\sum_{-\Delta_{g_0} K(x_i) > 0} (-1)^{k_i} \neq (-1)^n$, then there is at least one solution to the Nirenberg problem.

THEOREM (A. Chang & P. Yang [8]; we only state a weaker form). *For $n \geq 4$, let $K \in C^2(\mathbf{S}^n)$ be positive and have only nondegenerate critical points $x_1, \ldots, x_m$ of Morse index $k_1, \ldots, k_m$. Assume further that*

$$-\Delta_{g_0} K(x_i) \neq 0, \qquad \forall i = 1, \ldots, m.$$

If $\sum_{-\Delta_{g_0} K(x_i) > 0} (-1)^{k_i} \neq (-1)^n$, and $\|K - 1\|_{L^\infty(\mathbf{S}^n)} < \varepsilon(n)$ (where $\varepsilon(n)$ is some small positive number depending only on n), then there is at least one positive solution to (0.2).

THEOREM (J. F. Escobar & R. Schoen [13]; we only state a special case). *For $n \geq 3$, let $K \in C^{n-2}(\mathbf{S}^n)$ be positive somewhere and $K(-x) = K(x)$ for all $x \in \mathbf{S}^n$. Assume further that there is at least one maximum point of K at which all partial derivatives of K of order less than or equal to $n-2$ vanish. Then there is at least one positive solution to* (0.2).

The following question is very natural and has attracted much attention.

QUESTION. Are the results of A. Chang & P. Yang and of A. Bahri & J. M. Coron for $n = 2, 3$ true also for $n \geq 4$?

The answer to the above question is still unknown. In [21] we have established the following result. (More general results are given in [21]–[22].)

THEOREM 0.1 ([21]). *For $n \geq 3$, suppose that $K \in C^1(\mathbf{S}^n)$ is some positive function such that for any critical point q_0 of K, there exists some real number $\beta = \beta(q_0) \in (n-2, n)$ such that in some geodesic normal coordinate system centered at q_0,*

$$K(y) = K(0) + \sum_{j=1}^{n} a_j |y_j|^\beta + R(y),$$

where $a_j = a_j(q_0) \neq 0$, $\sum_{j=1}^n a_j \neq 0$, $R(y)$ is $C^{[\beta]-1,1}$ near 0 and satisfies

$$\lim_{|y|\to 0} \sum_{0 \leq |\alpha| \leq [\beta]} |\nabla^\alpha R(y)||y|^{-\beta+|\alpha|} = 0.$$

Suppose also that

$$\sum_{\nabla_{g_0} K(q_0)=0, \sum_{j=1}^n a_j(q_0)<0} (-1)^{i(q_0)} \neq (-1)^n,$$

where

$$i(q_0) = \#\{a_j(q_0) \mid a_j(q_0) < 0, 1 \leq j \leq n\}.$$

Then there is at least one positive solution to (0.2).

A key step in establishing Theorem 0.1 is the following a priori estimates in L^∞-norm for positive solutions of (0.2).

THEOREM 0.2 ([21]). *Under the hypotheses of Theorem 0.1, for any $\varepsilon > 0$, there exists some positive constant $C(K, n, \varepsilon)$ such that for all $\varepsilon \leq \mu \leq 1$, any positive solution v of* (0.2) *with K replaced by $K_\mu = \mu K + (1-\mu)R_0$ satisfies*

$$C(K,n,\varepsilon)^{-1} < v(q) < C(K,n,\varepsilon), \qquad \forall q \in \mathbf{S}^n.$$

We have also established in [19] and [21] the following result:

THEOREM 0.3 ([19], [21]). *For $n \geq 3$, suppose that $K \in C^0(\mathbf{S}^n)$ and satisfies $K(x_0) > 0$ for some $x_0 \in \mathbf{S}^n$. Then, for any $\varepsilon > 0$ and any integers $k \geq 1$ and $m \geq 2$, there exists $K_{\varepsilon,k,m} \in C^0(\mathbf{S}^n)$ with $\|K_{\varepsilon,k,m} - K\|_{C^0(\mathbf{S}^n)} < \varepsilon$, $K_{\varepsilon,k,m} \equiv K$ in $\mathbf{S}^n \setminus B_\varepsilon(x_0)$, such that, for each $2 \leq l \leq m$, the equation*

$$-\Delta_{g_0} v + c(n) R_0 v = c(n) K_{\varepsilon,k,m}(x) v^{(n+2)/(n-2)}$$

has at least k positive solutions with l bumps.

COROLLARY 0.1. *The scalar curvature functions of metrics conformal to g_0 are dense in $C^0(\mathbf{S}^n)$.*

REMARK 0.1. See [19] for the precise meaning of "l bumps" in the statement of Theorem 0.3. Roughly speaking, we say that a solution has l bumps if most of its mass is concentrated in l disjoint small balls.

Next, we look at a related problem,

$$\begin{cases} -\Delta u = K(x) u^{(n+2)/(n-2)} & \text{in } \mathbb{R}^n, \\ u > 0. \end{cases} \tag{0.3}$$

There has been much work on (0.3). See the introduction and references in [21]. We point out that if both K and u behave well at infinity, then (0.3) is equivalent to (0.2) after making a stereographic projection. Let E be the closure of $C_c^\infty(\mathbb{R}^n)$ ($n \geq 3$) (set of all smooth functions with compact support) under the norm $\|u\|_E = \left(\int_{\mathbb{R}^n} |\nabla u|^2\right)^{1/2}$. E is clearly a Hilbert space. In [17] we established the following result:

THEOREM 0.4 ([17]). *Suppose that $K \in C^1(\mathbb{R}^3)$ and satisfies*

(i) *For some positive constant $T > 0$, $K(x_1 + \ell T, x_2, x_3) = K(x_1, x_2, x_3)$ for all integers ℓ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.*
(ii) *$K_{\max} \equiv \max_{x \in \mathbb{R}^n} K(x) > 0$ is achieved and*

$$K^{-1}(K_{\max}) \equiv \{x \in \mathbb{R}^n \mid K(x) = K_{\max}\}$$

has at least one bounded connected component, denoted by $\mathcal{C}$.

Then (0.3) has infinitely many solutions in E modulo translations by T in the x_1 variable.

Theorem 0.4 has been extended to higher dimensional cases under some additional flatness hypothesis on K near $\mathcal{C}$.

THEOREM 0.5 ([19], [21]). *For $n \geq 4$, we suppose that $K \in C^{n-2}(\mathbb{R}^n)$ and satisfies*

(i) *For some positive constant $T > 0$, $K(x_1 + \ell T, x_2, \dots, x_n) = K(x_1, x_2, \dots, x_n)$ for all integers ℓ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.*

(ii) *$K_{\max} \equiv \max_{x \in \mathbb{R}^n} K(x) > 0$ is achieved and $K^{-1}(K_{\max}) \equiv \{x \in \mathbb{R}^n \mid K(x) = K_{\max}\}$ has at least one bounded connected component, denoted by $\mathcal{C}$.*

Further assume that for some constants $\beta > n - 2$ and C,

$$|\partial^\alpha K(y)| \leq C |\nabla K(y)|^{(\beta - |\alpha|)/(\beta - 1)}, \quad \text{for } y \text{ near } \mathcal{C} \text{ and } 2 \leq |\alpha| \leq n-2. \tag{0.4}$$

Then (0.3) has infinitely many solutions in E modulo translations by T in the x_1 variable.

COROLLARY 0.2. *Let K be a C^1 function in $\mathbb{R}^3$ which is periodic in x_1 and positive somewhere. Suppose that the maximum of K is achieved at least at one isolated point. Then (0.3) has infinitely many solutions in E modulo translations by the period of K in the x_1 variable.*

COROLLARY 0.3. *For $n \geq 4$, let $2m > n-2$ be some integer. Suppose that K is a C^{2m} function in $\mathbb{R}^n$ which is periodic in x_1 with $K(0) = \max_{x \in \mathbb{R}^n} K(x) > 0$ and, for some $\lambda_j > 0$ $(1 \leq j \leq n)$, $K(x) = K(0) - \sum_{j=1}^n \lambda_j x_j^{2m} + o(|x|^{2m})$ for x close to 0. Then (0.3) has infinitely many solutions in E modulo translations by the period of K in the x_1 variable.*

REMARK 0.2. We tend to believe that Theorem 0.5 still holds without assuming (0.4). However, we have not been able to prove it yet.

1. Outline of the proof of Theorem 0.2

The following result is established by R. Schoen [26]:

THEOREM (R. Schoen). *Let $\{K_i\}$ be a sequence of functions bounded in $C^2(\mathbf{S}^3)$ norm such that for some positive constant A_1,*

$$K_i(q) \geq 1/A_1, \qquad \text{for all } q \in \mathbf{S}^3.$$

Let $\{v_i\}$ be solutions of

$$\begin{cases} -\Delta_{g_0} v_i + \frac{3}{4} v_i - \frac{1}{8} K_i(x) v_i^{p_i} = 0 & \text{on } \mathbf{S}^3, \\ v_i > 0 & \text{on } \mathbf{S}^3, \end{cases}$$

where $p_i \leq 5$, $\lim_{i \to \infty} p_i = 5$. Then, after passing to a subsequence, either

$$v_i(q) \leq C \qquad \text{for all } i \text{ and } q \in \mathbf{S}^3, \tag{1.1}$$

or

(1.2) $\{v_i\}$ *has precisely one isolated simple blow-up point.*

REMARK 1.1. The notion of isolated simple blow-up was introduced by R. Schoen ([26]), see also Definition 0.3 in [21]. As a consequence, if (1.2) occurs, then for i large v_i satisfies

$$\max_{q\in\mathbf{S}^3}|q-q_i|^{2/(p_i-1)}v_i(q)\le C,$$

and

$$\max_{q\in\mathbf{S}^3}v_i(q_i)|q-q_i|v_i(q)\le C,$$

where $q_i\in\mathbf{S}^3$ is the unique maximum point of v_i.

Following the approach of R. Schoen, we have established in [21] the following result:

THEOREM 1.1. *For $n\ge 4$, let $\{K_i\}$ be a sequence of functions bounded in $C^{n-2}(\mathbf{S}^n)$ norm such that for some positive constants A_1, C and $\beta>n-2$,*

$$K_i(q)\ge 1/A_1,\qquad \text{for all } q\in\mathbf{S}^n,$$

$$|\nabla^\alpha K_i(q)|\le C|\nabla K_i(q)|^{(\beta-|\alpha|)/(\beta-1)},\qquad \text{for all } q\in\mathbf{S}^n,\ 2\le|\alpha|\le n-2.$$

Let $\{v_i\}$ be solutions of

$$\begin{cases} -\Delta_{g_0}v_i+c(n)R_0v_i-c(n)K_i(x)v_i^{p_i}=0 & \text{on } \mathbf{S}^n,\\ v_i>0, & \text{on } \mathbf{S}^n,\end{cases}$$

where $p_i\le(n+2)/(n-2)$, $\lim_{i\to\infty}p_i=(n+2)/(n-2)$. Then after passing to a subsequence, either

(1.3) $$v_i(q)\le C\qquad \text{for all } i \text{ and } q\in\mathbf{S}^n,$$

or

(1.4) $\{v_i\}$ *has precisely one isolated simple blow-up point.*

REMARK 1.2. As a consequence, if (1.4) occurs then for i large v_i satisfies

$$\max_{q\in\mathbf{S}^n}|q-q_i|^{2/(p_i-1)}v_i(q)\le C,$$

and

$$\max_{q\in\mathbf{S}^n}v_i(q_i)|q-q_i|^{n-2}v_i(q)\le C,$$

where $q_i\in\mathbf{S}^n$ is the unique maximum point of v_i.

By using a contradiction argument, Theorem 0.2 can be derived from Theorem 1.1 and an application of the Kazdan-Warner identity. See [21] for the details.

2. Outline of the proof of Theorem 0.1

Set

$$X = \left\{ u \in H^1(\mathbf{S}^n) \,\middle|\, |\mathbf{S}^n|^{-1} \int_{\mathbf{S}^n} |u|^{2n/(n-2)} = 1 \right\},$$
$$\mathcal{S}_0 = \left\{ u \in X \,\middle|\, \int_{\mathbf{S}^n} x|u|^{2n/(n-2)} = 0 \right\}.$$

For $P \in \mathbf{S}^n$, $1 \le t < \infty$, we define a conformal transformation by

$$\varphi_{P,t} : \mathbf{S}^n \to \mathbf{S}^n, \qquad y \to ty,$$

where y is the stereographic projection coordinates of points on $\mathbf{S}^n$ while the stereographic projection is performed with P as the north pole to the equatorial plane of $\mathbf{S}^n$. For a conformal transformation $\varphi : \mathbf{S}^n \to \mathbf{S}^n$, we define $T_\varphi : X \to X$ by

$$T_\varphi u = u \circ \varphi |\det d\varphi|^{(n-2)/2n},$$

where $|\det d\varphi|$ denotes the Jacobian of φ satisfying

$$\varphi^* g_0 = |\det d\varphi|^{2/n} g_0,$$

and g_0 is the pull back metric of the flat metric $\sum_i dx_i^2$ of $\mathbb{R}^{n+1}$ to $\mathbf{S}^n$. It is well known that the scalar curvature of g_0 is $R_0 = n(n-1)$.

Let B denote the open unit ball in $\mathbb{R}^{n+1}$, $\partial B = \mathbf{S}^n$. We define

$$\pi : \mathcal{S}_0 \times B \to X$$

by

$$u = \pi(w, \xi) = T^{-1}_{\varphi_{P,t}} w,$$

where $w \in \mathcal{S}_0$, $\xi = sP$, $0 \le s < 1$, $P \in \mathbf{S}^n$, $s = (t-1)/t$, $1 \le t < \infty$. It is easy to see that

$$\begin{cases} T^{-1}_{\varphi_{P,t}} w = T_{\varphi^{-1}_{P,t}} w, \\ \varphi^{-1}_{P,t} = \varphi_{-P,t} = \varphi_{P,t^{-1}}, \\ \varphi_{P,1} = \text{identity} \qquad \text{for all } P \in \mathbf{S}^n. \end{cases}$$

It follows that

$$\begin{cases} u = \pi(w, \xi) = T_{\varphi_{-P,t}} w, \\ w = T_{\varphi_{P,t}} u. \end{cases}$$

The following result is established in [21]:

THEOREM 2.1. *$\pi : \mathcal{S}_0 \times B \to X$ is a C^2 diffeomorphism.*

REMARK 2.1. The analog of Theorem 2.1 for dimension $n = 2$ was derived earlier by K. C. Chang and J. Liu in [6].

Let K satisfy the hypotheses of Theorem 0.1 and set $K_\mu = \mu K + (1-\mu)R_0$ for $0 \le \mu \le 1$. For $0 < \alpha < 1$, we define $T_\mu : C^{2,\alpha}(\mathbf{S}^n) \to C^{2,\alpha}(\mathbf{S}^n)$ by

$$T_\mu : v \mapsto (-\Delta_{g_0} + c(n)R_0)^{-1}(K_\mu |v|^{4/(n-2)} v).$$

It follows from Theorem 0.2 and some standard elliptic regularity theorem that for any $0 < \varepsilon < 1$, there exists some positive constant $C(K, n, \varepsilon)$ such that

$$\begin{aligned}(2.1)\quad \{v \in C^{2,\alpha}(\mathbf{S}^n) \,|\, v(q) > 0 \ \forall q \in \mathbf{S}^n,\\ (I - T_\mu)v = 0 \text{ for some } \varepsilon \le \mu \le 1\} \subset \mathcal{O}_\varepsilon,\end{aligned}$$

where

$$\mathcal{O}_\varepsilon = \left\{v \in C^{2,\alpha}(\mathbf{S}^n) \,\middle|\, C(K,n,\varepsilon)^{-1} < v < C(K,n,\varepsilon),\ \|v\|_{C^{2,\alpha}(\mathbf{S}^n)} < C(K,n,\varepsilon)\right\}.$$

It is obvious that $T_\mu : \mathcal{O}_\varepsilon \to C^{2,\alpha}(\mathbf{S}^n)$ is compact. We set

$$i_\mu = \deg(I - T_\mu, \mathcal{O}_\varepsilon, 0),$$

where deg denotes the Leray-Schauder degree of the map. It follows from (2.1) and the homotopy invariance of the Leray-Schauder degree (see [16] and [24]) that for any $0 < \varepsilon < 1$, $i_1 = i_\varepsilon$. To establish Theorem 0.1, we only have to show that $i_\varepsilon \ne 0$ for $\varepsilon > 0$ small enough. We observe that $\|K_\varepsilon - R_0\|_{L^\infty(\mathbf{S}^n)} \le C\varepsilon$. When the scalar curvature function is close to R_0 in L^∞ norm, the problem has been investigated by Chang, Yang and Gursky ([8], [5]). Following Chang-Yang ([8]) we have given in [21] a seemingly more transparent and simpler presentation of the results. It follows that there exists $\delta > 0$ such that for $\varepsilon > 0$ small we have

$$\deg(I - T_\varepsilon, \mathcal{O}_\varepsilon \cap \mathcal{U}_\delta, 0) = \sum_{\nabla_{g_0} K(q_0) = 0,\, \sum_{j=1}^n a_j(q_0) < 0} (-1)^{i(q_0)} - (-1)^n,$$

where

$$\mathcal{U}_\delta = \{\pi(w,\xi) \,|\, w \in \mathcal{S}_0,\ \|w - 1\| < \delta,\ \xi \in B\}$$

is a tubular neighborhood of $\{\pi(1,\xi) \,|\, \xi \in B\}$ in X. On the other hand, from Theorem 0.2 follow the uniqueness theorem (up to the conformal group of $\mathbf{S}^n$) of M. Obata ([25]), and some standard compactness arguments that for $\varepsilon > 0$ small enough all positive solutions $v \in C^{2,\alpha}(\mathbf{S}^n)$ of $(I - T_\varepsilon)v = 0$ belong to $\mathcal{U}_\delta$. It follows immediately that

$$i_\varepsilon = \sum_{\nabla_{g_0} K(q_0) = 0,\, \sum_{j=1}^n a_j(q_0) < 0} (-1)^{i(q_0)} - (-1)^n.$$

This concludes the proof of Theorem 0.1.

3. Outline of the proof of Theorems 0.4 and 0.5

What we established in [19] and [21] is stronger than Theorems 0.4 and 0.5. For example, we know that under the hypotheses of Theorems 0.4 and 0.5, either (0.3) has a sequence of one-bump solutions, or for any $m \geq 2$ there are infinitely many m-bump solutions. Here we only outline a proof of Theorems 0.4 and 0.5. This can be achieved by a contradiction argument. In the following, we always assume the contrary of Theorems 0.4 and 0.5. We will then construct infinitely many two-bump solutions, which leads to a contradiction. We first consider a compactified problem,

$$(3.1) \qquad \begin{cases} -\Delta u = K(x)(2/(1+|x|^2))^{(n-2)\tau/2} u^{(n+2)/(n-2)-\tau} & \text{in } \mathbb{R}^n, \\ u > 0, \end{cases}$$

where $\tau > 0$ is small. We refer to (3.1) as a compactified problem in the sense that for $\tau > 0$ small, if $u_i \rightharpoonup u$ in E, we have

$$\int_{\mathbb{R}^n} \left(\frac{2}{1+|x|^2}\right)^{(n-2)\tau/2} |u_i - u|^{2n/(n-2)-\tau} \to 0.$$

The first step in establishing Theorems 0.4 and 0.5 is to construct two-bump solutions of (3.1) for small $\tau > 0$. Along with the construction of solutions, we obtain good control of the distribution of the mass. Roughly speaking, the mass will concentrate near two small balls far apart and the centers of the two balls will stay uniformly bounded for all small $\tau > 0$. Condition (ii) in Theorems 0.4 and 0.5 is essential in obtaining such a control. This step has been achieved in [19] (see Theorem 3.1 there). There are two main ingredients in this step. One is an adaptation of the variational gluing technique developed by E. Séré ([27]), Coti Zelati and P. H. Rabinowitz ([10]–[11]) for periodic ode's and subcritical pde's. We follow more closely [10]–[11]. The other is some a priori estimates established in [17] and [19] (Props. 2.1–2.3 in [17], Props. 1.1–1.4 in [19]) which are essential to make the variational gluing technique applicable.

The second step is to show that, along a subsequence, u_τ converges in C^2_{loc} norm to a positive solution of (0.3). A priori $\{u_\tau\}$ might blow up and, as a result, weakly converge to 0. We have to rule out this possibility. Under the hypotheses of Theorems 0.4 and 0.5, we can apply Theorem 0.2 (or rather the proof, since the hypotheses of Theorem 0.2 are not exactly satisfied here) to conclude that $\{u_\tau\}$ can only have isolated simple blow up points and cannot blow-up at more than one point. Using the Harnack inequality and the fact that the only blow-up point is isolated simple, we conclude that $\{u_\tau\}$ converges strongly to 0 in C^0_{loc}

norm away from the blow-up point. This violates the structure of $\{u_\tau\}$ since, according to our construction, $\{u_\tau\}$ has two bumps.

Let us give a more detailed description. It follows from condition (ii) of Theorems 0.4 and 0.5 that there exists some bounded open neighborhood O of C such that

$$\max_{x \in \partial O} K(x) \le K_{\max} - \delta_1,$$

where $\delta_1 > 0$ is some positive number. For large positive integer ℓ, we set

$$O_\ell^{(1)} = \{x \mid x + (\ell T, 0, \dots, 0) \in O\}, \qquad O_\ell^{(2)} = \{x \mid x - (\ell T, 0, \dots, 0) \in O\}.$$

For $\varepsilon > 0$ small we define $V_\ell(2, \varepsilon) \subset E$ by declaring $u \in V_\ell(2,\varepsilon)$ if $u \in E$ and there exist $\alpha_1, \alpha_2, \lambda_1, \ \lambda_2 \in \mathbb{R}$ and $x = (x_1, x_2) \in O_\ell^{(1)} \times O_\ell^{(2)}$ such that

$$\lambda_1, \lambda_2 > 1/\varepsilon,$$

$$|\alpha_i - (K_{\max})^{(2-n)/4}| < \varepsilon, \qquad i = 1, 2,$$

$$\|u - \alpha_1 \delta(x_1, \lambda_1) - \alpha_2 \delta(x_2, \lambda_2)\|_E < \varepsilon,$$

where

$$\delta(x_i, \lambda_i)(y) = (n(n-2))^{(n+2)/(n-2)} \left(\frac{\lambda_i}{1 + \lambda_i^2 |y - x_i|^2} \right)^{(n-2)/2}.$$

Similarly, for $\delta > 0$ small, we define $V(1, \delta) \subset E$ by letting $u \in V(1,\delta)$ if $u \in E$ and there exist $\alpha_1, \lambda_1 \in \mathbb{R}$ and $x_1 \in O$ such that

$$\lambda_1 > 1/\delta,$$

$$|\alpha_1 - (K_{\max})^{(2-n)/4}| < \delta,$$

$$\|u - \alpha_1 \delta(x_1, \lambda_1)\|_E < \delta.$$

It is well known that for any $x_i \in \mathbb{R}^n$ and $\lambda_i > 0$,

$$\delta(x_i, \lambda_i) \in E,$$

$$-\Delta \delta(x_i, \lambda_i) = \delta(x_i, \lambda_i)^{(n+2)/(n-2)} \qquad \text{in } \mathbb{R}^n.$$

By adapting the variational gluing technique and some a priori estimates mentioned earlier, the following result is established in [19]. In fact, it is a corollary of Theorem 3.1 in [19].

THEOREM 3.1. *Under the hypotheses of Theorems* 0.4 *and* 0.5, *either*

$$\{u \in E \,|\, u \text{ is a positive solution of } (0.3)\} \cap V(1,\delta) = \emptyset, \quad \text{for all } \delta > 0, \tag{3.2}$$

or, for any $\varepsilon > 0$, *there exists* $\bar{\ell}_\varepsilon > 0$ *such that, for all* $\ell \geq \bar{\ell}_\varepsilon$, *there exists* $\bar{\tau}_\ell > 0$ *such that, for all* $0 < \tau < \bar{\tau}_\ell$, *there exists* $u_{\ell,\tau} \in V_\ell(2,\varepsilon)$ *which solves* (3.1).

If (3.2) occurs, then the conclusion of Theorems 0.4 and 0.5 can be deduced without much difficulty. Otherwise, for $\varepsilon > 0$ small and $\ell \geq \bar{\ell}_\varepsilon$, it follows from the proof of Theorem 0.2 that, along a subsequence, $u_\ell = \lim_{\tau\to 0} u_{\ell,\tau} \in V_\ell(2,\varepsilon)$ is a solution of (0.3). It is not difficult to see that all these solutions are different.

4. Outline of the proof of Theorem 0.3

We only outline the proof of Theorem 0.3 for $n = 3$, $k = 1$, $m = 2$ and $K \in C^1(\mathbf{S}^3)$. The proof of the more general result as stated in Theorem 0.3 is similar in nature. The details are in [19] and [21] (Theorem 3.1 in [19] and Theorem 0.7 in [21]). Let $x_0 \in \mathbf{S}^3$ be the north pole and make a stereographic projection to the equatorial plane of $\mathbf{S}^3$. The equation (0.2) is transformed to

$$\begin{cases} -\Delta u = (1/8)K(x)u^5 & \text{in } \mathbb{R}^3, \\ u > 0 & \text{in } \mathbb{R}^3, \end{cases}$$

where $K \in C^1(\mathbf{R}^3)$ and $\lim_{|x|\to\infty} K(x) = K_\infty > 0$. Let $\psi \in C^\infty(\mathbb{R}^3)$ satisfy

$$\|\psi\|_{C^2(\mathbb{R}^3)} < \infty, \tag{4.1}$$

$$\lim_{|x|\to\infty} \psi(x) = 1, \tag{4.2}$$

and

$$\sum_{i=1}^{3} \psi_{x_i}(x)x_i < 0 \qquad \forall x \neq 0. \tag{4.3}$$

REMARK 4.1. It is well known that under (4.1), (4.2) and (4.3), ψ violates the Kazdan-Warner condition and therefore

$$-\Delta u = \psi u^5 \qquad \text{in } \mathbb{R}^3 \tag{4.4}$$

has no nontrivial solution in E. Recall that the Kazdan-Warner condition is

$$\int \sum_{i=1}^{3} \psi_{x_i}(x)x_i u^6 = 0$$

for any solution $u \in E$ of (4.4).

For $\varepsilon > 0$ small, as in the statement of Theorem 0.3 and for large positive integer ℓ, we set

$$K_\ell(x_1, x_2, x_3) = K(x_1, , x_2, x_3) + \varepsilon\{\psi(x_1 + \ell, x_2, x_3) + \psi(x_1 - \ell, x_2, x_3) - 2\},$$

$$O_\ell^{(1)} = \{x = (x_1, x_2, x_3) \,|\, |(x_1 + \ell, x_2, x_3)| < 1\},$$

and

$$O_\ell^{(2)} = \{x = (x_1, x_2, x_3) \,|\, |(x_1 - \ell, x_2, x_3)| < 1\}.$$

Consider the equation corresponding to K_ℓ:

$$\begin{cases} -\Delta u = (1/8)K_\ell(x)u^5 & \text{in } \mathbb{R}^3, \\ u > 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{4.5}$$

Using an argument similar to the proof of Theorem 0.4, we can prove that, for any $\varepsilon > 0$ small, there exists $\overline{\ell}_\varepsilon$ such that for $\ell \geq \overline{\ell}_\varepsilon$ equation (4.5) has at least one solution $u \in V_\ell(2, \varepsilon) \subset E$. This completes the proof of Theorem 0.3 in the case $n = 3$, $k = 1$, $m = 2$ and $K \in C^1(\mathbf{S}^3)$.

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