Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 5, 1995, 345–356

# PERIODIC SOLUTIONS AND OPTIMIZATION PROBLEMS FOR A CLASS OF SEMILINEAR PARABOLIC CONTROL SYSTEMS

Walter Allegretto – Paolo Nistri

*Dedicated to Ky Fan on the occasion of his 80th birthday*

## 0. Introduction

This paper is devoted to the study of the existence of periodic solutions for a class of control problems described by a semilinear parabolic equation. Related optimization problems are also considered. Periodic control problems and optimal periodic control problems for evolution equations described both by ordinary differential equations and by parabolic equations arise in many different situations. Examples are found in the theory of chemical processes [7], biomedical models [13], competing species ([6], and the extensive references therein), thermostat problems [3], [5], [18] and [20] (and the references therein). Several authors have also treated the existence of periodic solutions of differential inclusions in Banach spaces. These differential inclusions can model periodic control problems. In fact, under suitable assumptions, a large class of control problems can be reduced to the problem of finding solutions of a differential inclusion. We mention here the following papers: [8], [9], [10], [15], [16] and the references therein. The variety of techniques used in the quoted papers is quite large. Referring only to those papers dealing with parabolic equations, we recall that in [6] the properties of the Poincaré map and the theory of ordered spaces

1991 *Mathematics Subject Classification.* 35B10, 35K55, 49J20.
Research supported in part by a bilateral project, by NSERC (Canada) and CNR (Italy) and by the research project MURST (40%) "Teoria del controllo dei sistemi dinamici".

are employed, in [3], [5], [18] the theory of linear parabolic equations together with suitable (different) models of the hysteresis process, while in [20] topological transversality methods are used to show the existence of periodic solutions.

The approach in Banach spaces employed in [8], [9] and [10] is based mainly on the theory of multivalued condensing operators with convex values and the related fixed point theory. In these papers the linear part of the equation is invertible. Finally, in [15] and [16] the selection properties of nonempty, closed, nonconvex valued operators and the viability theory in infinite-dimensional spaces are used respectively.

The control system we consider in this paper represents a possible model for a class of semilinear parabolic equations with distributed parameters. The paper is organized as follows. In Section 1, Theorem 1 provides sufficient conditions to guarantee the existence of periodic solutions corresponding to any control function from a certain given class $\mathcal{V}$ of admissible controls. In order to prove Theorem 1 we use a completely different approach from those considered in the cited papers. Namely, since the linear part of the considered parabolic equation is at resonance, we use the Lyapunov–Schmidt method to split our equation into two equations: one in an infinite-dimensional space and the other in a finite-dimensional space. Then, under suitable assumptions on the nonlinear part, we apply to this system the Leray–Schauder topological degree theory to obtain the existence of a periodic solution for any control function $v \in \mathcal{V}$. Furthermore, a nice topological property of the resulting solution set is obtained.

In Section 2, we associate three different cost functionals to the considered control problem. Under different assumptions which depend on the specific cost functional we consider, we prove the existence of a solution of the corresponding optimization problem. That is, among all the pairs $(u, v)$ where $u$ is a periodic solution corresponding to the control $v \in \mathcal{V}$, we prove the existence of a minimum of the cost functional. In [14] an approach similar to that outlined in this paper has been used to treat periodic optimization problems for a class of control systems described by a nonlinear system of ordinary differential equations in finite-dimensional spaces.

## 1. Existence of periodic solutions

We consider the nonlinear boundary value control problem described by the following semilinear parabolic equation:

$$(1)\quad \begin{cases} \frac{\partial}{\partial t}u(x,t) + \mathcal{M}u(x,t) = f(x,t,u(x,t),v(x,t)), & (x,t) \in \Omega \times (0,1), \\ D^\alpha u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,1), & \text{for all } \alpha \text{ with } |\alpha| \le l-1, \\ u(x,0) = u(x,1), & x \in \Omega, \end{cases}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ (e.g. of class $C^{2l}$), and $\mathcal{M}$ is a formally selfadjoint strongly elliptic operator of order $2l$, $l \geq 1$, with smooth coefficients in $\Omega$. That is,

$$\mathcal{M}u(x,t) = \sum_{\substack{|\alpha|\leq l\\ |\beta|\leq l}} (-1)^{|\alpha|} D^\alpha [A_{\alpha,\beta} D^\beta u(x,t)],$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an $n$-tuple of nonnegative integers (a multi-index), $|\alpha| = \alpha_1 + \ldots + \alpha_n$ (the length of the multi-index) and $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$. Furthermore, for any multi-indices $\alpha, \beta$ the function $A_{\alpha,\beta} : \Omega \to \mathbb{R}$ is smooth, $A_{\alpha,\beta} = A_{\beta,\alpha}$, and there exists $c > 0$ such that

$$\sum_{|\beta|=|\alpha|=l} A_{\alpha,\beta}(x)\xi^\alpha \xi^\beta \geq c|\xi|^{2l},$$

for any $x \in \overline{\Omega}$ and any $\xi \in \mathbb{R}^n$.

The control function $v = v(x,t)$ belongs to the control space $\mathcal{V} = L^\infty(\Omega \times (0,1), V)$, where $V \subset \mathbb{R}^m$ is a given connected, compact set.

The nonlinear term $f : \Omega \times (0,1) \times \mathbb{R} \times V \to \mathbb{R}$ satisfies the following conditions:

- ($\mathrm{f}_1$) the map $(x,t) \to f(x,t,p,q)$ is measurable for any $(p,q) \in \mathbb{R} \times V$;
- ($\mathrm{f}_2$) the map $(p,q) \to f(x,t,p,q)$ is continuous for almost all (a.a.) $(x,t) \in \Omega \times (0,1)$;
- ($\mathrm{f}_3$) $|f(x,t,p,q)| \leq a(x,t) + b|p|$ for a.a. $(x,t) \in \Omega \times (0,1)$, for any $(p,q) \in \mathbb{R} \times V$, where $a \in L^2(\Omega \times (0,1), \mathbb{R}_+)$, $b > 0$.

The function $t \to f(x,t,p,q)$ is extended from $(0,1)$ to $\mathbb{R}$ by 1-periodicity.

Consider now the multivalued function $F : \Omega \times (0,1) \times \mathbb{R} \multimap \mathbb{R}$ defined by

$$F(x,t,p) = \{f(x,t,p,q) : q \in V\} = f(x,t,p,V)$$

for a.a. $(x,t) \in \Omega \times (0,1)$ and for any $p \in \mathbb{R}$. We have the following result (see [1]).

Proposition 1. *The multivalued map $(x,t,p) \multimap F(x,t,p)$ has compact convex values; it is measurable with respect to $(x,t)$ and upper semicontinuous with respect to $p$.*

Let $X = L^2(\Omega \times (0,1), \mathbb{R})$, and consider the multivalued Nemytskiĭ operator $\mathcal{F} : X \multimap X$ generated by $F$, that is,

$$\mathcal{F}(u) = \{z \in X : \ z(x,t) \in F(x,t,u(x,t)) \text{ for a.a. } (x,t) \in \Omega \times (0,1)\}.$$

We have the following result (see [11]).

Proposition 2. *The multivalued map $\mathcal{F} : X \multimap X$ has closed convex values; it maps bounded sets into bounded sets and its composition with a linear compact map $K : X \to X$ is upper semicontinuous. Moreover, $\mathcal{F}(u) = \{\mathcal{F}_v(u) : v \in \mathcal{V}\}$*

*for any* $u \in X$, *where* $\mathcal{F}_v(u)(x,t) = f(t,x,u(x,t),v(x,t))$ *for a.a.* $(x,t) \in \Omega \times (0,1)$.

The operator $\mathcal{M}$ together with the homogeneous Dirichlet boundary conditions induces a selfadjoint differential operator $M : L^2(\Omega,\mathbb{R}) \supset \mathcal{D}(M) \to L^2(\Omega,\mathbb{R})$, where

$$\mathcal{D}(M) = W_2^{2l}(\Omega,\mathbb{R}) \cap W_{2,0}^{l}(\Omega,\mathbb{R}) \quad \text{and} \quad Mu = \mathcal{M}u \quad \text{for } u \in \mathcal{D}(M).$$

It is well known (see e.g. [17], [21]) that if $\mathcal{M}$ is a strongly elliptic operator then there exist two constants $c_0 > 0$ and $\lambda_0 \geq 0$ such that for each $u \in \mathcal{D}(M)$ the following Gårding inequality holds:

$$\langle Mu, u\rangle \geq c_0\|u\|^2_{W_2^l} - \lambda_0\|u\|^2_{L^2}.$$

Here $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\Omega,\mathbb{R})$. Furthermore, the smallest eigenvalue $\mu_1$ of $M$ is given by

$$\mu_1 = \inf_{\|u\|_{L^2}=1}\langle Mu, u\rangle.$$

Let $L$ be the differential operator formally defined by

$$Lu = \frac{\partial u}{\partial t} + Mu, \quad \text{where} \quad u(x,0) = u(x,1) \text{ for any } x \in \Omega.$$

Observe that the results in (e.g.) [21] coupled with an eigenfunction expansion show that $L$ admits a compact right inverse $H : X \to X$ on $N(M)^\perp$, the space perpendicular to the kernel of $M$, $N(M)$, in $X$. We recall that we assume that $N(M) \neq \{0\}$ and so $\mu_1 \leq 0$.

Therefore, for any $v \in \mathcal{V}$, the nonlinear boundary value control problem (1) can be rewritten as follows:

$$\begin{cases} u_1 = HQ\mathcal{F}_v(u_0 + u_1), \\ 0 = (I - Q)\mathcal{F}_v(u_0 + u_1), \end{cases} \tag{2}$$

where $u = u_0 + u_1 \in N(M) \oplus N(M)^\perp = X$ and $Q$ is the projection of $X$ onto $N(M)^\perp$ parallel to $N(M)$.

By [11, Corollaire 5.4], for any $v \in \mathcal{V}$ the operator $HQ\mathcal{F}_v : X \to X$ is continuous and compact. The same holds for the operator $(I-Q)\mathcal{F}_v : X \to X$ with finite-dimensional range. Denote by $S \subset X$ the set of all possible solutions of (2) corresponding to the controls $v \in \mathcal{V}$.

Proposition 3. *If* $S$ *is bounded in* $X$ *then it is compact in* $X$.

Proof. Let $\{u_n\} \subset S$ with corresponding $\{v_n\} \subset \mathcal{V}$. Then for any $n \in \mathbb{N}$ we have

$$u_{1,n} = HQ\mathcal{F}_{v_n}(u_{0,n} + u_{1,n}), \qquad 0 = (I-Q)\mathcal{F}_{v_n}(u_{0,n} + u_{1,n}),$$

where $u_n = u_{0,n} + u_{1,n}$. Let $z_n = \mathcal{F}_{v_n}(u_n)$, and observe that $\{z_n\} \subset \bigcup_n \mathcal{F}(u_n)$. Consequently, by the boundedness of $\{u_n\}$ and Proposition 2 the sequence $\{z_n\}$ is bounded in $X$. Thus, by passing to a subsequence if necessary, we see that $z_n \rightharpoonup z$ weakly in $X$, and $u_n \to u$ a.e. in $\Omega\times(0,1)$. Therefore, from the inclusion $z_n(x,t) \in f(x,t,u_n(x,t),V)$ and the convexity of $f(x,t,p,V)$, which is an interval, we get by taking the limit as $n \to \infty$,

$$z(x,t) \in f(x,t,u(x,t),V)$$

for a.a. $(x,t) \in \Omega \times (0,1)$. By [19] there exists $v \in \mathcal{V}$ such that

$$z(x,t) = f(x,t,u(x,t),v(x,t))$$

for a.a. $(x,t) \in \Omega\times(0,1)$. On the other hand, the linear operators $HQ$ and $I-Q$ are compact and so weakly continuous, hence

$$\lim_{n\to\infty} HQz_n = HQz \quad \text{and} \quad \lim_{n\to\infty}(I-Q)z_n = (I-Q)z$$

with $z = \mathcal{F}_v(u)$ and $u_1 = HQz$, i.e. $u \in S$. □

Assume the following condition.

($f_4$) There exist constants $C > 1$ and $\mu_1^* < \mu_1$ such that

$$f(x,t,p,q)p \le \mu_1^* p^2$$

for a.a. $(x,t) \in \Omega \times (0,1)$, any $p \in \mathbb{R}$ with $|p| > C$ and any $q \in V$.

We are now in a position to prove the main result of this section.

Theorem 1. *Assume* ($f_1$)–($f_4$). *For any* $v \in \mathcal{V}$ *the nonlinear boundary value control problem* (1) *has a* 1*-periodic solution.*

Proof. The proof is divided in two steps. In the first step we prove that the set $S$ is bounded in $X$, say by a constant $r > 0$. In the second one we show that

$$\deg(I - HQ\mathcal{F}_v, (I-Q)\mathcal{F}_v, B_r, 0) \ne 0$$

for any $v \in \mathcal{V}$. Here deg denotes the Leray–Schauder topological degree and $B_r$ denotes the ball in $X$ centered at zero with radius $r$.

First Step. It is convenient to reduce the nonlinear boundary value control problem (1) to the periodic control problem represented by the ordinary differential equation

$$\tag{3} \begin{cases} \dot u(t) + Mu(t) = \widehat{f}(t,u(t),v(t)) & \text{for a.a. } t \in (0,1), \\ u(0) = u(1), \end{cases}$$

in the Banach space $U = \{u \in L^2((0,1),W^l_{2,0}(\Omega,\mathbb{R})) : \dot u \in L^2((0,1),(W^l_{2,0}(\Omega,\mathbb{R}))^*)\}$, where $v \in L^\infty((0,1),L^\infty(\Omega,V))$ and $\widehat{f} : (0,1) \times W^l_{2,0}(\Omega,\mathbb{R}) \times$

$L^\infty(\Omega, V) \to L^2(\Omega, \mathbb{R})$ is defined by

$$\widehat{f}(t, u, v)(\cdot) = f(\cdot, t, u(\cdot), v(\cdot)).$$

Assume that $u \in U$ is a solution to (3) for some control function $v$. Then

$$\langle \dot{u}(t), u(t) \rangle_{W^l_{2,0}} + \langle Mu(t), u(t) \rangle = \langle \widehat{f}(t, u(t), v(t)), u(t) \rangle. \tag{4}$$

Here $\langle \cdot, \cdot \rangle_{W^l_{2,0}}$, denotes the scalar product in $W^l_{2,0}(\Omega, \mathbb{R})$, and $\langle \cdot, \cdot \rangle$ denotes the one in $L^2(\Omega, \mathbb{R})$ and in the sequel $\| \cdot \|$ will denote the norm in the latter space. Equality (4) can be rewritten as follows:

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + \langle Mu(t), u(t) \rangle = \langle \widehat{f}(t, u(t), v(t)), u(t) \rangle. \tag{5}$$

Using ($\mathrm{f}_3$) we get

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + \langle Mu(t), u(t) \rangle \le \|a(t)\| \cdot \|u(t)\| + b\|u(t)\|^2.$$

Dividing by $\|u(t)\|^2 + 1$ and integrating on the interval $[\tau, t]$, where $\tau \in \mathbb{R}$ and $t \in [\tau, \tau + 1]$, we obtain

$$\frac{1}{2} [\log(\|u(t)\|^2 + 1) - \log(\|u(\tau)\|^2 + 1)] \le C_1(u),$$

where

$$C_1(u) = -\mu_1^* + \int_\tau^t \left[ \frac{\|a(s)\| \cdot \|u(s)\|}{1 + \|u(s)\|^2} + b \right] ds.$$

By the 1-periodicity of $u$ we obtain

$$\log \sup_{t \in [0,1]} (\|u(t)\|^2 + 1) \le \log \inf_{t \in [0,1]} (\|u(t)\|^2 + 1) + 2C_1(u). \tag{6}$$

Suppose that there exists a sequence $\{u_n\}$ of solutions to (3) with corresponding $\{v_n\}$ such that $\sup_{t \in [0,1]} \|u_n(t)\| \to \infty$ as $n \to \infty$. Then (6) implies that

$$\inf_{t \in [0,1]} \|u_n(t)\| \to \infty$$

as $n \to \infty$, since $\{C_1(u_n)\}$ is a bounded sequence in $\mathbb{R}$. Furthermore, for $n$ sufficiently large, from (5) dividing by $\|u_n(t)\|^2$ and integrating on $(0, 1)$ we obtain

$$0 = \int_0^1 \left[ \frac{\langle Mu_n(t), u_n(t) \rangle}{\|u_n(t)\|^2} - \frac{\langle \widehat{f}(t, u_n(t), v_n(t)), u_n(t) \rangle}{\|u_n(t)\|^2} \right] dt. \tag{7}$$

We want to show that under our assumptions we have

$$\liminf_{n \to \infty} \int_0^1 \left[ \frac{\langle Mu_n(t), u_n(t) \rangle}{\|u_n(t)\|^2} - \frac{\langle \widehat{f}(t, u_n(t), v_n(t)), u_n(t) \rangle}{\|u_n(t)\|^2} \right] dt > 0,$$

contradicting (7). For this, consider

$$\begin{aligned}
\liminf_{n\to\infty} \int_0^1 & \left[\frac{\langle Mu_n(t), u_n(t)\rangle}{\|u_n(t)\|^2} - \frac{\langle \widehat{f}(t, u_n(t), v_n(t)), u_n(t)\rangle}{\|u_n(t)\|^2}\right] dt \\
&\geq \int_0^1 \left[\liminf_{n\to\infty} \frac{\langle Mu_n(t), u_n(t)\rangle}{\|u_n(t)\|^2}\right. \\
&\qquad\qquad \left. - \limsup_{n\to\infty} \frac{\langle \widehat{f}(t, u_n(t), v_n(t)), u_n(t)\rangle}{\|u_n(t)\|^2}\right] dt \\
&\geq \int_0^1 \left[\mu_1 - \limsup_{n\to\infty} \frac{\langle \widehat{f}(t, u_n(t), v_n(t)), u_n(t)\rangle}{\|u_n(t)\|^2}\right] dt.
\end{aligned}$$

It is now sufficient to prove that

$$\limsup_{n\to\infty} \frac{\langle \widehat{f}(t, u_n(t), v_n(t)), u_n(t)\rangle}{\|u_n(t)\|^2} < \mu_1. \tag{8}$$

For $n$ sufficiently large and for fixed $t \in (0,1)$, consider

$$\langle \widehat{f}(t, u_n(t), v_n(t)), u_n(t)\rangle = \int_\Omega f(x, t, u_n(x,t), v_n(x,t)) u_n(x,t)\, dx.$$

Let $C$ represent the constant of $(\mathrm{f}_4)$ and define $\Omega_n = \{x \in \Omega : |u_n(x,t)| \leq C\}$ and $\Omega_n' = \{x \in \Omega : |u_n(x,t)| > C\}$. By $(\mathrm{f}_4)$ we have

$$\begin{aligned}
\int_\Omega f(x, t, u_n(x,t), v_n(x,t)) u_n(x,t)\, dx &= \int_{\Omega_n} f(x, t, u_n(x,t), v_n(x,t)) u_n(x,t)\, dx \\
&\quad + \int_{\Omega_n'} f(x, t, u_n(x,t), v_n(x,t)) u_n(x,t)\, dx \\
&\leq C \int_\Omega a(x,t) dx + C^2 b \operatorname{meas}(\Omega) \\
&\quad + \mu_1^* \int_{\Omega_n'} u_n^2(x,t)\, dx \\
&\leq K(t, a, b, C, \Omega) + \mu_1^* \int_{\Omega_n'} u_n^2(x,t)\, dx
\end{aligned}$$

and we obtain

$$\frac{\langle \widehat{f}(t, u_n(t), v_n(t)), u_n(t)\rangle}{\|u_n(t)\|^2} < \frac{K(t, a, b, C, \Omega)}{\|u_n(t)\|^2} + \mu_1^* \tag{9}$$

since

$$\frac{\int_{\Omega_n'} u_n^2(x,t)\, dx}{\|u_n(t)\|^2} \leq 1.$$

Taking the $\limsup$ in (9) we obtain (8). Therefore there exists a constant $r > 0$ such that $\sup_{t\in[0,1]} \|u(t)\| < r$ for any solution $u = u(t)$, $t \in (0,1)$, of (3) corresponding to some control $v \in \mathcal{V}$. Hence the set $S$ is bounded in $X$ by the constant $r$.

Second Step. First of all observe that the above proof also shows that any solution of the periodic control problem

$$\begin{cases} \dot{u}(t) + Mu(t) = \lambda \widehat{f}(t, u(t), v(t)) + (1-\lambda)\mu u(t), & t \in (0,1), \\ u(0) = u(1), \end{cases} \tag{10}$$

where $\mu < \mu_1 \leq 0$ and $\lambda \in [0,1]$, is bounded by a constant, which we denote again by $r > 0$, independent of $\lambda$ and $v$.

In other words, the homotopy (10) is admissible. Now, for $\lambda = 0$ we have

$$\begin{cases} \dot{u}(t) + Mu(t) - \mu u(t) = 0 & \text{for a.a. } t \in (0,1), \\ u(0) = u(1). \end{cases}$$

Or equivalently,

$$\begin{cases} u_1 - \mu HQ(u_0 + u_1) = 0, \\ (I - Q)(u_0 + u_1) = 0. \end{cases}$$

Since $\mu < \mu_1 \leq 0$, the linear operator

$$\Phi = (I - \mu HQ, (I - Q)) : X \supset B_r \to X$$

is a one-to-one map of $\overline{B}_r$ onto $\Phi(\overline{B}_r)$ with $0 \in \Phi(B_r)$, and thus

$$|\deg(\Phi, B_r, 0)| = 1$$

(see [12, Theorem 4.3.14]; cf. also [12, Theorem 4.3.11]), and in conclusion for $\lambda = 1$ we have

$$|\deg(I - HQ\mathcal{F}_v, (I - Q)\mathcal{F}_v, B_r, 0)| = 1$$

for any $v \in \mathcal{V}$. This concludes the proof. $\square$

## 2. Optimization problems

In this section we consider the solvability of different optimization problems for the nonlinear boundary value control problem (1). Specifically, together with (1) we consider the following possible cost functionals:

1. $J_1 : X \to \overline{\mathbb{R}}$ is defined by

$$J_1(u) = \int_0^1 \int_\Omega f_1(x, t, u(x,t))\, dx\, dt + \int_\Omega u^2(x, 0)\, dx,$$

where $f_1 : \Omega \times (0,1) \times \mathbb{R} \to \overline{\mathbb{R}}$ satisfies the following conditions:

(1a) the function $(x, t, p) \to f_1(x, t, p)$ is measurable;

(1b) the function $p \to f_1(x, t, p)$ is lower semicontinuous for a.a. $(x, t) \in \Omega \times (0,1)$;

(1c) there exist $\phi_1 \in L^1(\Omega \times (0,1), \mathbb{R})$ and $\psi_1 \in L^1(\Omega, \mathbb{R}_+)$ such that $\phi_1(x,t) - \psi_1(x)|p| \leq f_1(x, t, p)$ for a.a. $(x, t, p) \in \Omega \times (0,1) \times \mathbb{R}$;

2. $J_2 : \mathcal{V} \to \overline{\mathbb{R}}$ is defined by

$$J_2(v) = \int_0^1 \int_\Omega f_2(x,t,v(x,t))\,dx\,dt,$$

where $f_2 : \Omega \times (0,1) \times \mathbb{R} \to \overline{\mathbb{R}}$ satisfies the following conditions:

(2a) the function $(x,t,q) \to f_2(x,t,q)$ is measurable;

(2b) the function $q \to f_2(x,t,q)$ is lower semicontinuous and convex for a.a. $(x,t) \in \Omega \times (0,1)$;

(2c) there exist $\phi_2 \in L^1(\Omega\times(0,1),\mathbb{R})$ and $\psi_2 \in L^1(\Omega,\mathbb{R}_+)$ such that $\phi_2(x,t) - \psi_2(x)|q| \le f_2(x,t,q)$ for a.a. $(x,t,q) \in \Omega \times (0,1) \times \mathbb{R}^m$,

(2d) the set $V \subset \mathbb{R}^m$ where the controls take their values is convex and compact.

3. $J_3 : X \times \mathcal{V} \to \overline{\mathbb{R}}$ is defined by

$$J_3(u,v) = \int_0^1 \int_\Omega f_3(x,t,u(x,t),v(x,t))\,dx\,dt + \int_\Omega u^2(x,0)\,dx,$$

where $f_3 : \Omega \times (0,1) \times \mathbb{R} \times \mathbb{R}^m \to \overline{\mathbb{R}}$ satisfies the following conditions:

(3a) the function $(x,t,p,q) \to f_3(x,t,p,q)$ is measurable;

(3b) the function $(p,q) \to f_3(x,t,p,q)$ is continuous;

(3c) $|f_3(x,t,p,q)| \le a_3(x,t) + b_3|p|$ for a.a. $(x,t) \in \Omega \times (0,1)$ and for any $(p,q) \in \mathbb{R} \times V$, where $a_3 \in L^2(\Omega \times (0,1),\mathbb{R}_+), b_3 > 0$.

We proceed now to solve the proposed optimization problems.

1. Consider the optimization problem

$$\begin{cases} J_1(u) \to \inf = m_1 \\ \text{with } u \text{ solution of (1).} \end{cases} \tag{11}$$

The following result holds.

Theorem 2. *Under the assumptions of Theorem* 1 *and* (1a)–(1c) *the optimization problem* (11) *has a solution.*

Proof. Define $\widehat{f}_1 :\ (0,1) \times L^2(\Omega,\mathbb{R}) \to \mathbb{R}$ as follows:

$$\widehat{f}_1(t,u) = \int_\Omega f_1(x,t,u(x))\,dx.$$

The function $(t,u) \to \widehat{f}_1(t,u)$ is measurable and by [2] the function $u \to \widehat{f}_1(t,u)$ is lower semicontinuous. Therefore, the cost functional

$$J_1(u) = \int_0^1 \widehat{f}_1(t,u(t))\,dt + \int_\Omega u^2(x,0)\,dx$$

is lower semicontinuous in $X$. Hence it attains its minimum on the solution set $S \subset X$ which is bounded and thus compact by Proposition 3. $\square$

2. Consider the problem

$$\begin{cases} J_2(v) \to \inf = m_2 \\ \text{with } v \in \mathcal{V}. \end{cases} \tag{12}$$

We can prove the following.

Theorem 3. *Under the assumptions of Theorem* 1 *and* (2a)–(2d) *the optimization problem* (12) *has a solution.*

Proof. Define $\widehat{f}_2 : (0,1) \times L^\infty(\Omega, V) \to \mathbb{R}$ as follows:

$$\widehat{f}_2(t,v) = \int_\Omega f_2(x,t,v(x))\,dx.$$

The function $(t,v) \to \widehat{f}_2(t,v)$ is measurable and by [2] the function $v \to \widehat{f}_2(t,v)$ is lower semicontinuous. Clearly $v \to \widehat{f}_2(t,v)$ is convex since $f_2$ is. Consider now a minimizing sequence $\{v_n\} \subset \mathcal{V}$. By passing to a subsequence if necessary, we see that $v_n \rightharpoonup v_0$ weakly in $X$. On the other hand, the control set $\mathcal{V}$ is convex and closed in $X$, since $V$ is (see [4], p. 117), thus it is weakly closed in $X$ and so $v_0 \in \mathcal{V}$. The cost functional

$$J_2(v) = \int_0^1 \widehat{f}_2(t,v(t))\,dt$$

is sequentially weakly lower semicontinuous in $\mathcal{V}$, since it is lower semicontinuous and convex in the variable $v$. Therefore

$$m_2 = \liminf_{n\to\infty} J_2(v_n) \geq J_2(v_0). \qquad \square$$

3. Finally, consider the following problem:

$$\begin{cases} J_3(u,v) \to \inf = m_3 \\ \text{where } u \text{ is a solution of (1)} \\ \text{with corresponding controls } v. \end{cases} \tag{13}$$

We have the following.

Theorem 4. *Under the assumptions of Theorem* 1 *and* (3a)–(3c) *the optimization problem* (13) *has a solution.*

Proof. For any $v \in \mathcal{V}$, let $\mathcal{G}_v : X \to X$ be the Nemytskiĭ operator generated by the function $f_3$, that is,

$$\mathcal{G}_v(u)(x,t) = f_3(x,t,u(x,t),v(x,t))$$

for a.a. $(x,t) \in \Omega \times (0,1)$. By our assumptions on $f_3$, this operator is well defined and continuous. Consider the linear operator $K : X \to L^2((0,1),\mathbb{R})$ defined by

$$w(t) = (Ky)(t) = \int_0^t \left( \int_\Omega y(x,s)\,dx \right) ds, \qquad y \in X.$$

Clearly the operator $K$ is continuous and compact and so is the composition $K\mathcal{G}_v : X \to X_0$ for any $v \in \mathcal{V}$.

Let $\{(u_n, v_n)\}$ be a minimizing sequence, where $\{u_n\} \subset S$ and $\{v_n\} \subset \mathcal{V}$ are corresponding controls. Define a sequence $\{w_n\} \subset L^2((0,1),\mathbb{R})$ as follows:

$$w_n = K\mathcal{G}_{v_n}(u_n) + c_n,$$

where $c_n = \int_\Omega u_n^2(x,0)\,dx$ is considered as a constant function with respect to time. Therefore the pair $(u_n, w_n) \in S \times L^2((0,1),\mathbb{R})$ satisfies the system

$$\begin{aligned} u_{1,n} &= HQ\mathcal{F}_{v_n}(u_{0,n} + u_{1,n}), \\ 0 &= (I - Q)\mathcal{F}_{v_n}(u_{0,n} + u_{1,n}), \\ w_n &= K\mathcal{G}_{v_n}(u_{0,n} + u_{1,n}) + c_n, \end{aligned} \tag{14}$$

for any $n \in \mathbb{N}$, or equivalently, $z_n = T_{v_n}(u_n)$, where $z_n = (u_n, w_n)$ and $T_{v_n} : X \to X \times L^2((0,1),\mathbb{R})$ is defined by

$$T_{v_n}(u_n) = (HQ\mathcal{F}_{v_n}(u_n), u_{0,n} + (I - Q)\mathcal{F}_{v_n}(u_n), K\mathcal{G}_{v_n}(u_n) + c_n).$$

Obviously, $T_{v_n}$ is continuous and compact for any $v_n \in \mathcal{V}$. Taking into account the convexity of the vector field $(x,t,p) \to (f(x,t,p,V), f_3(x,t,p,V))$, the boundedness of $\{u_n\} \subset S$ and $\{v_n\} \subset \mathcal{V}$, we can use the arguments of the proof of Proposition 3 to show the existence of a solution $z^* = (u^*, w^*)$ of (14) corresponding to a control $v^* \in \mathcal{V}$. Therefore

$$\begin{aligned} u_1^* &= HQ\mathcal{F}_{v^*}(u^*), \\ 0 &= (I - Q)\mathcal{F}_{v^*}(u^*), \\ w^* &= K\mathcal{G}_{v^*}(u^*) + \left(\int_\Omega u^{*2}(x,0)\,dx\right), \end{aligned}$$

and so $u^* \in S$ with corresponding $v^* \in \mathcal{V}$ and $w^*(1) = J_3(u^*, v^*) = m_3$. □

## References


[1] J. P. Aubin and A. Cellina, *Differential Inclusions*, Grundlehren Math. Wiss., vol. 264, Springer-Verlag, Berlin, 1984.

[2] E. J. Balder, *Necessary and sufficient conditions for $L_1$-strong-weak lower semicontinuity of integral functionals*, Nonlinear Anal. **11** (1987), 1399–1404.

[3] M. Brokate and A. Friedman, *Optimal design for heat conduction problems with hysteresis*, SIAM J. Control Optim. **27** (1989), 697–717.

[4] R. Conti, *Problemi di Controllo e di Controllo Ottimale*, UTET, 1974.

[5] A. Friedman and L.-S. Jiang, *Periodic solutions for a thermostat control problem*, Comm. Partial Differential Equations **13** (1988), 515–550.

[6] P. Hess, *Periodic-Parabolic Boundary Value Problems and Positivity*, Longman Scientific and Technical, 1990.

[7] F. J. Horn and J. E. Bailey, *An application of the theorem of relaxed control to the problem of increasing catalyst selectivity*, J. Optim. Theory Appl. **2** (1968), 441–449.

[8] M. I. Kamenskiĭ, P. Nistri, V. V. Obukhovskiĭ and P. Zecca, *Optimal feedback control for a semilinear evolution equation*, J. Optim. Theory Appl. **82** (1994), 503–517.

[9] M. I. Kamenskiĭ and V. V. Obukhovskiĭ, *Condensing multioperators and periodic solutions of parabolic functional-differential inclusions in Banach spaces*, Nonlinear Anal. **20** (1993), 781–792.

[10] ———, *On periodic solutions of differential inclusions with unbounded operators in Banach spaces*, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **21** (1991), 173–191.

[11] J. M. Lasry et R. Robert, *Analyse nonlinéaire multivoque*, Cahiers Math. de la Décision 7611, Université de Paris, Dauphine.

[12] N. G. Lloyd, *Degree Theory*, Cambridge University Press, Cambridge, 1978.

[13] L. Markus, *Optimal control of limit cycles or what control theory can do to care a heart attack or to cause one*, Symposium on Ordinary Differential Equations, Minneapolis, 1972 (W. A. Harris and Y. Sibuya, eds.), Lecture Notes in Math., vol. 132, Springer-Verlag, Berlin, 1973, pp. 108–134.

[14] P. Nistri, *Periodic control problems for a class of nonlinear periodic differential systems*, Nonlinear Anal. **7** (1983), 79–90.

[15] N. S. Papageorgiou, *Viable and periodic solutions for differential inclusions in Banach spaces*, Kobe J. Math. **5** (1988), 29–42.

[16] ———, *Periodic solutions of nonlinear evolution inclusions*, Technical Report.

[17] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci., vol. 44, Springer-Verlag, Berlin, 1983.

[18] J. Prüss, *Periodic solutions of the thermostat problem*, Differential Equations in Banach Spaces, Lecture Notes in Math., vol. 1223, Springer-Verlag, Berlin, 1986, pp. 216–226.

[19] E. Roxin, *The existence of optimal controls*, Michigan Math. J. **9** (1962), 109-119.

[20] T. I. Seidman, *5 papers on switching systems*, Mathematics Research Reports, Mathematics Programs at U.M.B.C., Catonsville, Maryland.

[21] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, vol. II/A: Linear Monotone Operators, Springer-Verlag, Berlin, 1990.

*Manuscript received July 15, 1994*

Walter Allegretto
Department of Mathematics
University of Alberta
Edmonton, Alberta, CANADA T6G 2G1

Paolo Nistri
Dipartimento di Sistemi e Informatica
Università di Firenze
Via di Santa Marta 3, Firenze, ITALY