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POSITIVE ENTIRE SOLUTIONS OF QUASILINEAR ELLIPTIC PROBLEMS VIA NONSMOOTH CRITICAL POINT THEORY

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We prove that a variational quasilinear elliptic equation admits a positive weak solution on $\mathbb{R}^n$. Our results extend to a wider class of equations some known results about semilinear and quasilinear problems: all the coefficients involved (also the ones in the principal part) depend both on the variable x and on the unknown function u; moreover, they are not homogeneous with respect to u.

1. Introduction

We investigate the existence of a positive function $u \in H^1(\mathbb{R}^n)$ $(n \geq 3)$ solving in distributional sense the quasilinear elliptic equation

$$(1) \qquad -\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u = -b(x)u + g(x,u) \quad \text{in } \mathbb{R}^n;$$

here $H^1 := H^1(\mathbb{R}^n)$ denotes the completion of $C^\infty_{\rm c} := C^\infty_{\rm c}(\mathbb{R}^n)$ (the space of smooth functions with compact support in $\mathbb{R}^n$) with respect to the norm

$$\forall u \in C^\infty_{\rm c} \quad \|u\| = \left(\int_{\mathbb{R}^n} (|\nabla u|^2 + u^2)\right)^{1/2};$$

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it is well known that there exist continuous imbeddings $H^1 \subset L^p(\mathbb{R}^n)$ for all $p \in [2, 2^*]$, where $2^* = 2n/(n-2)$ is the critical Sobolev exponent. The assumptions on the coefficients a_{ij}, b, g and the exact statements of our results are quoted in Section 2.

To determine weak solutions of (1) we look for critical points of the functional $J : H^1 \to \mathbb{R}$ defined by

$$\forall u \in H^1 \quad J(u) = \frac{1}{2}\int_{\mathbb{R}^n} \sum_{i,j=1}^{n} a_{ij}(x,u) D_i u D_j u \;+\; \frac{1}{2}\int_{\mathbb{R}^n} b(x)u^2 - \int_{\mathbb{R}^n} G(x,u)$$

where $G(x,\xi) = \int_0^\xi g(x,t)\,dt$. The first difficulty we have to face is that we cannot work in the classical framework of critical point theory; indeed, under reasonable assumptions on a_{ij}, b, g, the functional J is continuous but not even locally Lipschitz unless either the functions $a_{ij}(x,s)$ are independent of s or $n = 1$ (see [10]). Nevertheless, the derivative of J exists in the smooth directions, i.e. for all $u \in H^1$ and $\varphi \in C_c^\infty$ we can define

$$J'(u)[\varphi] \\ = \int_{\mathbb{R}^n} \bigg(\sum_{i,j=1}^{n} \bigg[a_{ij}(x,u) D_i u D_j \varphi + \frac{1}{2}\frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u \varphi \bigg] + b(x) u \varphi - g(x,u)\varphi \bigg).$$

According to the nonsmooth critical point theory developed in (for the reader's convenience we quote the basic tools in Section 5), we know that critical points u (in a suitable sense) of J satisfy $J'(u)[\varphi] = 0$ for $\varphi \in C_c^\infty$ and hence solve (1) in distributional sense. Therefore we follow this theory as it seems to be the natural framework to study by variational methods quasilinear equations of the kind of (1) (see [3, 10, 11, 12]).

In the last few years there has been a growing interest in the existence of positive solutions to variational semilinear and quasilinear equations on unbounded domains; these problems are suggested by various branches of mathematical physics (see [8, 21] and references therein). It seems difficult to give complete references of the results existing in the literature; however, let us make an attempt to indicate the ones which are more closely related to our problem.

Semilinear and quasilinear problems in bounded domains may be studied and solved by standard variational techniques as in [1, 23]; it is well known that in unbounded domains these arguments do not apply due to the lack of compactness of the problem (the PS condition does not hold); the a priori estimate techniques fail as well, as such estimates are not, in general, sufficient to guarantee a "good" behaviour at infinity of the solution or to prevent the solution from being the trivial one (see [22]). However, for some problems, also in unbounded domains a form of compactness can be recovered by using the techniques of [21]; a typical situation is when the coefficients involved in the problem tend to some limits at infinity: in this case the related problem at infinity allows us to find a range of levels at which the PS sequences are in fact relatively compact (see [7, 21, 22] for semilinear problems and [5] for a quasilinear case). Quasilinear equations on unbounded domains have been studied, among others, in [5, 14, 18, 20, 24]; in all these papers, the principal part of the differential equation is of the kind $\mathrm{div}[\varphi(\nabla u)]$ for suitable functions φ.

The structure of (1) is different, the coefficients involved depend both on x and u and this yields some further difficulties. First, we cannot obtain the critical point of J as a constrained critical point on a suitable unit ball as in [21] because the terms involved in (1) are not homogeneous with respect to u; moreover, we cannot follow the approximation procedure of [14, 17] because it requires a certain monotonicity of the principal part of the functional (see Section 3). Second, in the autonomous case, the equation does not necessarily admit a radially symmetric solution on $\mathbb{R}^n$: a form of compactness induced by this symmetry can be exploited to prove existence results (see [8] and references therein); in this paper we obtain a positive solution of (1) (when the coefficients are independent of x) by applying the concentration-compactness principle [21] directly on PS sequences. Third, we cannot give a representation result for PS sequences as in [5, 6, 7] because the gradient of the functional J is not defined; however, if the quasilinear equation (1) "converges" to a semilinear problem at infinity we can still prove a weak form of the representation result and obtain a positive solution of (1). We point out that to prove our results we do not wonder about the relative compactness of PS sequences of the functional J but we determine a solution of (1) only by means of the weak convergence of PS sequences.

2. Main existence results

Throughout this paper we require the coefficients a_{ij} $(i, j = 1, \ldots, n)$ to satisfy

$$\begin{cases} a_{ij} \equiv a_{ji}, \\ a_{ij}(x, \cdot) \in C^1(\mathbb{R}) \quad \text{for a.e. } x \in \mathbb{R}^n, \\ a_{ij}(x, s), \dfrac{\partial a_{ij}}{\partial s}(x, s) \in L^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}); \end{cases} \tag{2}$$

moreover, on the matrices $[a_{ij}(x, s)]$ and $[s(\partial a_{ij}/\partial s)(x, s)]$ we make the following assumptions:

$$\exists \nu > 0, \quad \sum_{i,j=1}^{n} a_{ij}(x, s)\xi_i\xi_j \geq \nu|\xi|^2 \quad \text{for a.e. } x \in \mathbb{R}^n, \ \forall s \in \mathbb{R}, \ \forall \xi \in \mathbb{R}^n, \tag{3}$$

$$(4)\qquad \begin{cases} \exists p \in (2, 2^*), \ \gamma \in (0, p-2), \\ 0 \le s \displaystyle\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial s}(x,s)\xi_i\xi_j \le \gamma \sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j \\ \qquad\qquad\qquad\qquad\qquad \text{for a.e. } x \in \mathbb{R}^n, \ \forall s \in \mathbb{R}, \ \forall \xi \in \mathbb{R}^n. \end{cases}$$

We will first prove an existence result for the following autonomous equation:

$$(5)\qquad -\sum_{i,j=1}^{n} D_j(a_{ij}(u)D_iu) + \frac{1}{2}\sum_{i,j=1}^{n} a'_{ij}(u)D_iuD_ju = -\lambda u + |u|^{p-2}u \quad \text{in } \mathbb{R}^n.$$

Theorem 1. *Assume that the functions a_{ij} do not depend on x, i.e. $a_{ij}(x,s) = a_{ij}(s)$ and that* (2)–(4) *hold; then, for all $\lambda > 0$, problem* (5) *admits a positive nontrivial solution $\overline{u} \in H^1(\mathbb{R}^n)$.*

To prove an existence result for a nonautonomous case some other assumptions are needed. We first require that $b \in L^\infty(\mathbb{R}^n)$ is strictly positive:

$$(6)\qquad \exists \overline{b}, \underline{b} > 0 \quad \overline{b} \ge b(x) \ge \underline{b} \quad \text{for a.e. } x \in \mathbb{R}^n;$$

let p be as in (4) and assume that there exist $\beta > 0$, $\alpha \in L^r(\mathbb{R}^n)$ for some $r \in [2n/(n+2), 2)$ and $q \in (2, 2^*)$ such that

$$(7)\qquad \begin{cases} g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function,} \\ g(x,0) = 0 \quad \text{for a.e. } x \in \mathbb{R}^n, \\ g(x,s) \le \alpha(x) + \beta s^{q-1} \quad \forall s > 0 \text{ and for a.e. } x \in \mathbb{R}^n, \\ 0 \le pG(x,s) \le sg(x,s) \quad \forall s > 0 \text{ and for a.e. } x \in \mathbb{R}^n. \end{cases}$$

If we assume that

$$(8)\qquad \begin{cases} \displaystyle\lim_{|x|\to\infty} a_{ij}(x,s) = \delta_{ij} & \text{uniformly in } s \in \mathbb{R} \ \forall i,j = 1,\dots,n, \\ \displaystyle\lim_{|x|\to\infty} s \cdot \frac{\partial a_{ij}}{\partial s}(x,s) = 0 & \text{uniformly in } s \in \mathbb{R} \ \forall i,j = 1,\dots,n, \\ \displaystyle\lim_{|x|\to\infty} b(x) = \lambda & \text{for some } \lambda > 0, \\ \displaystyle\lim_{|x|\to\infty} \frac{g(x,s)}{s^{p-1}} = 1 & \text{uniformly in } s > 0, \end{cases}$$

then, as $|x| \to \infty$, the quasilinear equation (1) becomes a semilinear equation: for positive solutions the following problem at infinity is obtained:

$$(9)\qquad -\Delta u + \lambda u = u^{p-1} \quad \text{in } \mathbb{R}^n.$$

Equation (9) has been exhaustively studied in the literature (see e.g. [7, 8]): it admits a strictly positive solution for all $\lambda > 0$ and $p \in (2, 2^*)$. Assumptions (8) state that the quasilinear equation (1) and the related functional J tend to regularize as $|x| \to \infty$: this nicer behaviour will allow us to prove

Theorem 2. *Assume* (2)–(4) *and* (6)–(8)*; moreover, assume that*

$$
(10) \qquad \begin{cases} \displaystyle\sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j \le |\xi|^2 \quad \forall s \in \mathbb{R}\ \forall \xi \in \mathbb{R}^n \text{ for a.e. } x \in \mathbb{R}^n, \\ b(x) \le \lambda \quad \text{for a.e. } x \in \mathbb{R}^n, \\ g(x,s) \ge s^{p-1} \quad \forall s > 0 \text{ and for a.e. } x \in \mathbb{R}^n. \end{cases}
$$

Then (1) *admits a nontrivial positive solution in* $H^1(\mathbb{R}^n)$.

In the particular case where (1) is a semilinear problem of the kind $-\Delta u + b(x)u = g(x,u)$, the existence of positive entire solutions has been determined under various assumptions on the nonlinearity $g(x,\cdot)$ (see [6, 13, 17] and the rich references therein). Theorem 2 generalizes in some sense such existence results to the quasilinear case.

3. Some remarks on the assumptions

• The assumption (4) is typical of quasilinear problems: it appears, for instance, in [2, 10] where different techniques are employed.

• By assumptions (2) and (4) we have

$$
(11) \qquad u \in H^1 \Rightarrow \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u u \in L^1(\mathbb{R}^n)
$$

and therefore $J'(u)[u]$ can be written in integral form.

• A particular attention must be paid when in (7) we have the limit case $\alpha \in L^{2n/(n+2)}(\mathbb{R}^n)$: take $r \in (2n/(n+2), 2)$; then for all $\varepsilon > 0$ there exist $\alpha_1 \in L^r$ and $\alpha_2 \in L^{2n/(n+2)}$ such that

$$
(12) \qquad \alpha = \alpha_1 + \alpha_2 \quad \text{and} \quad \|\alpha_2\|_{2n/(n+2)} \le \varepsilon;
$$

this will be used in Section 4.4.

• Let us explain why we cannot use the procedure of [14, 17]: one should minimize the functional J constrained on the set $M = \{u \in H^1 \setminus \{0\} : J'(u)[u] = 0\}$ and one should prove the crucial implication

$$
u \in M \Rightarrow J(u) = \max_{t \ge 0} J(tu).
$$

In our context, by (11) it still makes sense to define the set M; consider the simple case where $g(x,s) = |s|^{p-2}s$, take $u \in M$ and define the function

$$
f(t) = J(tu) = \frac{t^2}{2} \int_{\mathbb{R}^n} \sum_{i,j=1}^{n} a_{ij}(x,tu) D_i u D_j u + \frac{t^2}{2} \int_{\mathbb{R}^n} b(x) u^2 - \frac{t^p}{p} \int_{\mathbb{R}^n} |u|^p.
$$

We have

$$f'(t) = t\int_{\mathbb{R}^n}\sum_{i,j=1}^n a_{ij}(x,tu)D_iuD_ju + \frac{t^2}{2}\int_{\mathbb{R}^n}\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,tu)D_iuD_juu$$
$$+ t\int_{\mathbb{R}^n} b(x)u^2 - t^{p-1}\int_{\mathbb{R}^n}|u|^p$$

and, since $J'(u)[u] = 0$,

$$f'(t) = \frac{t}{2}\int_{\mathbb{R}^n}\langle[\Psi(x,tu)-\Psi(x,u)]\nabla u,\nabla u\rangle + t(1-t^{p-2})\int_{\mathbb{R}^n}|u|^p$$

where $\Psi(x,s) = \big[2a_{ij}(x,s) + \frac{\partial a_{ij}}{\partial s}(x,s)\cdot s\big]$. Observe that $f'(1) = 0$ and that to ensure that $t = 1$ is at least a local maximum we would need the following "monotonicity" assumption:

$$\forall s_2 > s_1 \geq 0 \quad \text{the matrix } \Psi(x,s_1) - \Psi(x,s_2) \text{ is positive semidefinite}$$

for a.e. $x \in \mathbb{R}^n$, which, together with (4), implies that the coefficients a_{ij} do not depend on s.

• Some information about the (local) behaviour of solutions of (1) follows by applying Theorem 2.2.5 of [12]: if in (7) we also require that $\alpha \in L^s$ with $s > n/2$ then any solution is locally bounded, and further results can be obtained by well-known techniques of regularity theory.

• Our last remark states that we can obtain positive solutions of (1) by determining critical points of the modified functional J_+ defined by

$$J_+(u) := \frac{1}{2}\int_{\mathbb{R}^n}\sum_{i,j=1}^n a_{ij}(x,u)D_iuD_ju + \frac{1}{2}\int_{\mathbb{R}^n} b(x)u^2 - \int_{\mathbb{R}^n} G(x,u^+) \quad \forall u \in H^1,$$

where u^+ denotes the positive part of u, i.e. $u^+(x) = \max(u(x),0)$.

Lemma 1. *Assume* (2)–(4), (6), (7) *and let* $u \in H^1$ *satisfy* $J'_+(u)[\varphi] = 0$ *for all* $\varphi \in C^\infty_{\mathrm{c}}$*; then* u *is a weak positive solution of* (1).

Proof. We first prove that $u \geq 0$: consider the function Tu defined by

$$Tu = \begin{cases} 0 & \text{if } u \geq 0,\\ u & \text{if } 0 \geq u \geq -1,\\ -1 & \text{if } u \leq -1;\end{cases}$$

as $Tu \in H^1 \cap L^\infty$, we can evaluate $J'_+(u)$ on Tu and we obtain

$$0 = J'_+(u)[Tu] = \int_{\{0\geq u\geq -1\}}\sum_{i,j=1}^n a_{ij}(x,u)D_iuD_ju$$
$$+ \frac{1}{2}\int_{\mathbb{R}^n}\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_ju\cdot Tu + \int_{\mathbb{R}^n} b(x)u\cdot Tu.$$

By (3), (4), (6) we know that the r.h.s. of the above expression is positive and can vanish only if $Tu \equiv 0$; therefore, $u \geq 0$. Since $J'_+(v)[\varphi] = J'(v)[\varphi]$ for all $\varphi \in C_c^\infty$ and all v belonging to the cone of positive functions of H^1, u solves equation (1). □

Therefore, without loss of generality we can suppose that

$$g(x,s) = 0 \quad \forall s \leq 0 \text{ for a.e. } x \in \mathbb{R}^n, \tag{13}$$

and, from now on, we make this assumption.

4. Proofs of the results

4.1. The behaviour of PS sequences. We first prove the following boundedness criterion which applies, in particular, to PS sequences:

Lemma 2. *Assume* (2)–(4), (6), (7)*; then every sequence* $\{u_m\} \subset H^1$ *satisfying*

$$|J(u_m)| \leq C_1 \quad \text{and} \quad |J'(u_m)[u_m]| \leq C_2 \|u_m\|$$

is bounded in H^1.

Proof. Consider $\{u_m\} \subset H^1$ such that $|J(u_m)| \leq C_1$. Then by (7) (and (13)) we get

$$I_m := \frac{1}{2}\int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij}(x,u_m) D_i u_m D_j u_m - \frac{1}{p}\int_{\mathbb{R}^n} g(x,u_m)u_m + \frac{1}{2}\int_{\mathbb{R}^n} b(x) u_m^2 \leq C_1;$$

by (11) we can compute $J'(u_m)[u_m]$ and by the assumptions we have

$$\bigg| \int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij}(x,u_m) D_i u_m D_j u_m + \frac{1}{2}\int_{\mathbb{R}^n} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_m) D_i u_m D_j u_m u_m \\ - \int_{\mathbb{R}^n} g(x,u_m)u_m + \int_{\mathbb{R}^n} b(x)u_m^2 \bigg| \leq C_2 \|u_m\|.$$

Therefore, by (4) and computing $I_m - \frac{1}{p} J'(u_m)[u_m]$ we get

$$\frac{p-2-\gamma}{2p}\int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij}(x,u_m) D_i u_m D_j u_m + \frac{p-2}{2p}\int_{\mathbb{R}^n} b(x)u_m^2 \leq C_3\|u_m\| + C_1;$$

finally, by (3) and (6) there exists $C_4 > 0$ such that $C_4\|u_m\|^2 \leq C_3 \|u_m\| + C_1$ and the result follows. □

From now on by $\omega \Subset \mathbb{R}^n$ we mean that ω is an open bounded subset of $\mathbb{R}^n$; we prove a local compactness property which is a slightly more general version of a result of [10]:

Lemma 3. *Assume* (2)–(4) *and let* $\{u_m\}$ *be a bounded sequence in* H^1 *satisfying*

$$\int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij}(x,u_m) D_i u_m D_j \varphi + \frac{1}{2} \int_{\mathbb{R}^n} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_m) D_i u_m D_j u_m \varphi = \langle \beta_m, \varphi \rangle \quad \forall \varphi \in C_c^\infty$$

with $\{\beta_m\}$ *strongly convergent in* $H^{-1}(\omega)$ *for all* $\omega \Subset \mathbb{R}^n$ *to some* $\beta \in H^{-1}(\mathbb{R}^n)$. *Then, up to a subsequence,* $\{u_m\}$ *converges strongly in* $H^1(\omega)$ *for all* $\omega \Subset \mathbb{R}^n$.

Proof. Up to a subsequence we have $u_m \rightharpoonup u$ in H^1 for some $u \in H^1$, $u_m \to u$ in $L^2(\omega)$ for every $\omega \Subset \mathbb{R}^n$ and $u_m(x) \to u(x)$ for a.e. $x \in \mathbb{R}^n$. Moreover, by a trivial extension to unbounded domains of Theorem 2.1 of [9], $\nabla u_m(x) \to \nabla u(x)$ for a.e. $x \in \mathbb{R}^n$. By arguing just as in Lemma 2.3 of [10] we obtain

$$\int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j \varphi + \frac{1}{2} \int_{\mathbb{R}^n} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u \varphi = \langle \beta, \varphi \rangle \quad \forall \varphi \in C_c^\infty. \tag{14}$$

Now choose $\omega \Subset \mathbb{R}^n$ and a positive smooth cut-off function $\chi : \mathbb{R}^n \to \mathbb{R}$ such that $\chi = 1$ on ω and $\Omega := \operatorname{supp} \chi \Subset \mathbb{R}^n$. From (11), (14) and by a density argument, we have

$$\int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j(\chi u) + \frac{1}{2} \int_{\mathbb{R}^n} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u (\chi u) = \langle \beta, \chi u \rangle$$

and by Fatou's Lemma, we get

$$\liminf_{m\to\infty} \int_{\mathbb{R}^n} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_m) D_i u_m D_j u_m (\chi u_m) \geq \int_{\mathbb{R}^n} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u (\chi u);$$

therefore,

$$\limsup_{m\to\infty} \int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij}(x,u_m) D_i u_m D_j(\chi u_m) \leq \int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j(\chi u). \tag{15}$$

This allows us to show that $\nabla u_m \to \nabla u$ in $L^2(\omega)$; indeed, by (3),

$$\begin{aligned}\int_\omega |\nabla u_m - \nabla u|^2 &\le \frac{1}{\nu}\int_\omega \sum_{i,j=1}^n a_{ij}(x,u_m) D_i(u_m-u) D_j(u_m-u)\\ &\le \frac{1}{\nu}\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_m) D_i(u_m-u) D_j u_m \cdot \chi\\ &\quad - \frac{1}{\nu}\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_m) D_i(u_m-u) D_j u \cdot \chi =: I_m;\end{aligned}$$

we claim that $I_m \to 0$ as $m \to \infty$. The second term in I_m vanishes because $a_{ij}(x,u_m)D_i(u_m-u) \rightharpoonup 0$ in $L^2(\Omega)$. So, let us treat the first term in I_m: we split it as

$$\frac{1}{\nu}\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_m) D_i(\chi(u_m-u)) D_j u_m \\ - \frac{1}{\nu}\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_m) D_i\chi D_j u_m \cdot (u_m-u);$$

since $u_m \to u$ in $L^2(\Omega)$, the last term vanishes, hence

$$\begin{aligned}\int_\omega |\nabla u_m - \nabla u|^2 \le I_m &\le \frac{1}{\nu}\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_m) D_i(\chi(u_m-u)) D_j u_m + o(1)\\ &\le \frac{1}{\nu}\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i(\chi u) D_j u\\ &\quad - \frac{1}{\nu}\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_m) D_i(\chi u) D_j u_m + o(1)\end{aligned}$$

where the last inequality comes from (15). Now the assertion follows from the fact that

$$a_{ij}(x,u_m)D_j u_m \rightharpoonup a_{ij}(x,u)D_j u \quad \text{in } L^2(\Omega)$$

for all $i = 1,\dots,n$. □

The previous results allow us to prove

PROPOSITION 1. *Assume* (2)–(4), (6), (7) *and that* $\{u_m\} \subset H^1$ *is a PS sequence for* J*; then there exists* $\overline{u} \in H^1$ *such that* (*up to a subsequence*)

(i) $u_m \rightharpoonup \overline{u}$ *in* H^1,
(ii) $u_m \to \overline{u}$ *in* $H^1(\omega)$ *for every* $\omega \Subset \mathbb{R}^n$,
(iii) $\overline{u} \ge 0$ *solves* (1) *in distributional sense.*

PROOF. Note first that $\{u_m\}$ is bounded by Lemma 2 and (i) follows. To obtain (ii) it suffices to apply Lemma 3 with $\beta_m = \alpha_m + g(x,u_m) - b(x)u_m \in H^{-1}$

where $\alpha_m \to 0$ in H^{-1} (see also Proposition 4 in the appendix): indeed, if $u_m \rightharpoonup u$ in H^1, then $\beta_m \to \beta$ in $H^{-1}(\omega)$ for all $\omega \Subset \mathbb{R}^n$ with $\beta = g(x,u) - b(x)u$ (see Theorem 2.2.7 of [12]). Finally, (iii) follows from (14) and Lemma 1. $\square$

To conclude this section we prove a technical result that will be used in the proof of Theorem 2:

Lemma 4. *Assume* (2)–(4), (6), (7) *and let* $\{u_m\} \subset H^1$ *be a PS sequence for* J. *Then for all* $\varepsilon > 0$ *there exists* $R > 0$ *such that*

$$\int_{\{|u_m| \le R\}} |\nabla u_m|^2 \le \varepsilon$$

for m *large enough.*

Proof. We use the same device as for Theorem 2.2.9 of [12]. Fix $\varepsilon > 0$, take $\delta \in (0,1)$ and for all $R > 0$ define the function

$$\varphi_\delta(s) = \begin{cases} s & \text{if } |s| \le R, \\ R + \delta R - \delta s & \text{if } s \in (R, R + R/\delta), \\ \delta s - R - \delta R & \text{if } s \in (-R - R/\delta, -R), \\ 0 & \text{if } |s| > R + R/\delta. \end{cases}$$

Let

$$\begin{aligned} w_m = & -\sum_{i,j=1}^n D_j(a_{ij}(x,u_m) D_i u_m) \\ & + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_m) D_i u_m D_j u_m + b(x) u_m - g(x,u_m). \end{aligned}$$

Then computing $J'(u_m)$ on $\varphi_\delta(u_m) \in H^1 \cap L^\infty$, by (4), (6) we have

$$\begin{aligned} \int_{\mathbb{R}^n} \sum_{i,j=1}^n & a_{ij}(x,u_m) D_i u_m D_j(\varphi_\delta(u_m)) \\ & \le \int_{\mathbb{R}^n} g(x,u_m)\varphi_\delta(u_m) + \frac{1}{4\delta} \|w_m\|^2_{H^{-1}} + \delta \|u_m\|^2. \end{aligned}$$

Choose $\delta > 0$ such that $\delta \|u_m\|^2 \le \varepsilon\nu/6$ and $\delta \int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij}(x,u_m) D_i u_m D_j u_m \le \varepsilon\nu/2$, with m large enough so that $\frac{1}{4\delta}\|w_m\|^2_{H^{-1}} \le \varepsilon\nu/6$ and R so that $\int_{\mathbb{R}^n} g(x,u_m)\varphi_\delta(u_m) \le \varepsilon\nu/6$: this is possible because as $R \to 0$, by (7) and by interpolation we have ($r' = r/(r-1)$)

$$\int_{\mathbb{R}^n} g(x,u_m)\varphi_\delta(u_m) \le \int_{\{|u_m|\le R+R/\delta\}} g(x,u_m)u_m$$
$$\le \|\alpha\|_r \left(\int_{\{|u_m|\le R+R/\delta\}} |u_m|^{r'} \right)^{1/r'} + \beta \int_{\{|u_m|\le R+R/\delta\}} |u_m|^q \to 0.$$

Therefore, we obtain

$$\int_{\{|u_m|\le R+R/\delta\}} \sum_{i,j=1}^n a_{ij}(x,u_m) D_i u_m D_j(\varphi_\delta(u_m)) \le \varepsilon\nu/2,$$

that is,

$$\int_{\{|u_m|\le R\}} \sum_{i,j=1}^n a_{ij}(x,u_m) D_i u_m D_j u_m \le \varepsilon\nu;$$

the result follows by (3). □

4.2. The variational characterization. In this section we build a PS sequence for the functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j u + \frac{1}{2}\int_{\mathbb{R}^n} b(x)u^2 - \int_{\mathbb{R}^n} G(x,u)$$

under the assumptions (2)–(4), (6), (7) and (13).

As the function G is superquadratic at $+\infty$, for every positive function $v \in H^1$ we have $\lim_{t\to\infty} J(tv) = -\infty$; we choose in particular a nontrivial function e such that

$$e \in C_c^\infty, \quad e \ge 0 \quad \text{and} \quad J(te) < 0 \quad \forall t > 1, \tag{16}$$

to define the class

$$\Gamma := \{\gamma \in C([0,1]; H^1) : \gamma(0) = 0,\ \gamma(1) = e\} \tag{17}$$

and the minimax value

$$\alpha := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)). \tag{18}$$

We obtain a PS sequence for J at level α by applying the mountain pass lemma [1] in the nonsmooth version [16]. Let us briefly verify that the functional J has such geometrical structure:

- $J(0) = 0$.
- Choosing e as in (16) we have $J(e) \le 0$.

• There are $\varrho, \delta > 0$ such that $\varrho < \|e\|$ and $J(u) \geq \delta$ if $\|u\| = \varrho$; indeed, by (7) and (13) we infer

$$\forall \varepsilon > 0\ \exists C_\varepsilon > 0 \quad 0 \leq G(x,s) \leq \varepsilon s^2 + C_\varepsilon s^{2^*} \quad \forall s \in \mathbb{R} \text{ and for a.e. } x \in \mathbb{R}^n;$$

hence, by (3) and (6) we have $J(u) \geq C_1\|u\|^2 - C_2\|u\|^{2^*}$.

We have thus proved

Proposition 2. *Let Γ and α be as in* (17), (18)*; then J admits a PS sequence $\{u_m\}$ at level α.*

As the imbedding $H^1(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ is not compact, we cannot infer that the above PS sequence converges strongly; however, using Proposition 1, we will prove the existence of a nontrivial solution of (1) by means of its weak limit.

4.3. Proof of Theorem 1. We apply the concentration-compactness principle [21] to PS sequences as in [4].

Let $J : H^1 \to \mathbb{R}$ be the "positive" functional associated with problem (5), that is,

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij}(u) D_i u D_j u + \frac{\lambda}{2}\int_{\mathbb{R}^n} u^2 - \frac{1}{p}\int_{\mathbb{R}^n} |u^+|^p;$$

let $\{u_m\}$ be the PS sequence found in Proposition 2; then $\{u_m\}$ is bounded in H^1 by Proposition 1. Since $J'(u_m)[u_m] = o(1)$ and $J(u_m) = \alpha + o(1)$, by assumption (4) we have

$$\begin{aligned}2\alpha &= 2J(u_m) - J'(u_m)[u_m] + o(1) \\ &= \int_{\mathbb{R}^n} |u_m^+|^p - \frac{2}{p}\int_{\mathbb{R}^n} |u_m^+|^p - \frac{1}{2}\int_{\mathbb{R}^n} \sum_{i,j=1}^n a'_{ij}(u_m) D_i u_m D_j u_m u_m + o(1) \\ &\leq \frac{p-2}{p}\|u_m^+\|_p^p + o(1),\end{aligned}$$

hence $\|u_m^+\|_p \geq c > 0$ and $\{u_m\}$ does not converge strongly to 0 in L^p. Taking into account that $\|u_m^+\|_2$ and $\|\nabla u_m^+\|_2$ are bounded, by Lemma I.1, p. 231 of [21], we infer that the sequence $\{u_m^+\}$ "does not vanish" in L^2, i.e. there exists a sequence $\{y_m\} \subset \mathbb{R}^n$ and $C > 0$ such that

$$\int_{y_m + B_R} |u_m^+|^2 \geq C$$

for some R. Defining the sequence of functions $v_m(x) = u_m(x - y_m)$, we have

$$\int_{B_R} |v_m^+|^2 \geq C; \tag{19}$$

moreover, by the translation invariance of J and $|dJ|$ (see the appendix), $\{v_m\}$ is a PS sequence for J at the same level α. Hence $\{v_m\}$ converges strongly in

$H^1(B_R)$ to its weak limit $\overline{v}$ by Proposition 1 and $\overline{v} \not\equiv 0$ by (19): $\overline{v}$ is a nontrivial solution of equation (5) and it is positive by Lemma 1. $\square$

4.4. The weak splitting. In this section we assume that the hypotheses of Theorem 2 hold and we prove a weak form of the representation result for PS sequences given in [5, 6, 7]. Consider the problem at infinity (9) and the corresponding functional

$$J_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 + \frac{\lambda}{2}\int_{\mathbb{R}^n} u^2 - \frac{1}{p}\int_{\mathbb{R}^n} |u^+|^p \quad \forall u \in H^1,$$

which is of class C^1. We can prove

LEMMA 5. *Let $\{u_m\}$ be a PS sequence for J and let u be its weak limit; then*

$$J(u_m) = J(u) + J_\infty(u_m - u) + o(1) \quad \textit{as } m \to \infty.$$

PROOF. The splittings

$$\int_{\mathbb{R}^n} G(x, u_m) - \int_{\mathbb{R}^n} G(x, u) - \frac{1}{p}\int_{\mathbb{R}^n} |(u_m - u)^+|^p = o(1),$$
$$\int_{\mathbb{R}^n} b(x)u_m^2 - \int_{\mathbb{R}^n} b(x)u^2 - \lambda\int_{\mathbb{R}^n} (u_m - u)^2 = o(1)$$

are standard (see e.g. Lemma 2.2 of [13]); therefore we must only treat the principal part.

For all $\varepsilon > 0$ there exists R_ε such that

$$\Big| \int_{|x|>R_\varepsilon} \sum_{i,j=1}^n a_{ij}(x, u_m) D_i u_m D_j u_m - \int_{|x|>R_\varepsilon} \sum_{i,j=1}^n a_{ij}(x, u) D_i u D_j u \\ - \int_{|x|>R_\varepsilon} |\nabla(u_m - u)|^2 \Big| < c\varepsilon$$

for some $c > 0$: indeed, since u is given (i.e. $\|D_i u\|_{L^2(\{|x|>R_\varepsilon\})} \le c\varepsilon$ for all i), by applying Hölder's inequality it suffices to prove that $|\int_{|x|>R_\varepsilon} \sum_{i,j=1}^n [a_{ij}(x, u_m) - \delta_{ij}] D_i u_m D_j u_m| < c\varepsilon$ and this follows by (8). On the other hand, by Proposition 1 we infer $\nabla u_m \to \nabla u$ in $[L^2(B_{R_\varepsilon})]^n$ and hence

$$\int_{|x|\le R_\varepsilon} \sum_{i,j=1}^n [a_{ij}(x, u_m) D_i u_m D_j u_m - a_{ij}(x, u) D_i u D_j u] \\ = \int_{|x|\le R_\varepsilon} \sum_{i,j=1}^n [a_{ij}(x, u_m) D_i(u_m - u) D_j u_m - a_{ij}(x, u) D_i u D_j u \\ + a_{ij}(x, u_m) D_i u D_j u_m] = o(1).$$

We have thus proved

$$\int_{|x|\le R_\varepsilon}\sum_{i,j=1}^n a_{ij}(x,u_m)D_iu_mD_ju_m-\int_{|x|\le R_\varepsilon}\sum_{i,j=1}^n a_{ij}(x,u)D_iuD_ju -\int_{|x|\le R_\varepsilon}|\nabla(u_m-u)|^2=o(1)$$

and the result follows by the arbitrariness of ε. □

Next, one should prove that $J'_\infty(u_m-u)=J'(u_m-u)+o(1)$ as in [5], but we cannot obtain such a result because $J\notin C^1(H^1,\mathbb{R})$; however, we can prove

Lemma 6. *Let $\{u_m\}$ be a PS sequence for J and let u be its weak limit; then (up to a subsequence)*

$$J'(u_m)[u_m]=J'(u)[u]+J'_\infty(u_m-u)[u_m-u]+o(1).$$

Proof. By the proof of Lemma 5 and (8) it suffices to prove that, up to a subsequence, we have

$$\int_{\mathbb{R}^n}\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial s}(x,u_m)D_iu_mD_ju_mu_m-\int_{\mathbb{R}^n}\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_juu=o(1).$$

By (8), for all $\varepsilon>0$ there exists R_ε such that $\left|\frac{\partial a_{ij}}{\partial s}(x,s)\cdot s\right|\le\varepsilon$ if $|x|>R_\varepsilon$ and $s\in\mathbb{R}$; therefore, by Hölder's inequality,

$$\int_{|x|>R_\varepsilon}\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial s}(x,u_m)D_iu_mD_ju_mu_m\le c\varepsilon,$$
$$\int_{|x|>R_\varepsilon}\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_juu\le c'\varepsilon.$$

On the other hand, Proposition 1 and (4) yield

$$\frac{\partial a_{ij}}{\partial s}(x,u_m)D_iu_mD_ju_mu_m\to\frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_juu\quad\text{for a.e. } x\in\mathbb{R}^n$$

and there exists $\psi\in L^1(B_{R_\varepsilon})$ such that

$$\frac{\partial a_{ij}}{\partial s}(x,u_m)D_iu_mD_ju_mu_m\le\psi(x)\quad\text{for a.e. } x\in B_{R_\varepsilon}$$

(up to a subsequence); hence, by the Lebesgue Theorem,

$$\int_{B_{R_\varepsilon}}\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial s}(x,u_m)D_iu_mD_ju_mu_m-\int_{B_{R_\varepsilon}}\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_juu=o(1).$$

The result follows by the arbitrariness of ε. □

Let $\{u_m\}$ denote a PS sequence for J; we now prove two results in the case where $u_m \rightharpoonup 0$. If $r \in (2n/(n+2), 2)$ in (7), then reasoning as for Theorem 1 and by (7) we have

$$\begin{aligned} 2\alpha = 2J(u_m) - J'(u_m)[u_m] + o(1) &\le \int_{\mathbb{R}^n} g(x, u_m)u_m + o(1) \\ &\le \|\alpha\|_r \|u_m\|_{r'} + \beta \|u_m\|_p^p + o(1), \end{aligned}$$

where $r' = r/(r-1) \in (2, 2^*)$; hence, either $\|u_m\|_{r'}$ or $\|u_m\|_p$ does not converge to 0 and the sequence $\{u_m\}$ does not vanish. If $r = 2n/(n+2)$, then the same result can be obtained by (12). Therefore, there exist $\overline{u} \not\equiv 0$ and a sequence $\{y_m\} \subset \mathbb{R}^n$ such that $|y_m| \to \infty$ and

$$\tau_m u_m \rightharpoonup \overline{u} \quad \text{in } H^1 \tag{20}$$

where $\tau_m u_m(x) := u_m(x - y_m)$. We prove that $\overline{u}$ is a solution of (9):

LEMMA 7. *Let $\{u_m\}$ be a PS sequence for J and assume that $u_m \rightharpoonup 0$, and let $\overline{u}$ be as in* (20)*; then $J'_\infty(\overline{u}) = 0$ and $\overline{u} > 0$.*

PROOF. For all $\varphi \in C_c^\infty$ define $\tau^m \varphi(x) := \varphi(x + y_m)$; since $\{u_m\}$ is a PS sequence we have

$$\forall \varphi \in C_c^\infty \quad J'(u_m)[\tau^m \varphi] = o(1) \quad \text{as } m \to \infty,$$

that is,

$$\begin{aligned} \int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij}(x, u_m) D_i u_m D_j(\tau^m \varphi) + \frac{1}{2} \int_{\mathbb{R}^n} \frac{\partial a_{ij}}{\partial s}(x, u_m) D_i u_m D_j u_m (\tau^m \varphi) \\ + \int_{\mathbb{R}^n} b(x) u_m (\tau^m \varphi) - \int_{\mathbb{R}^n} g(x, u_m)(\tau^m \varphi) = o(1). \end{aligned}$$

Obviously, as $m \to \infty$,

$$\begin{aligned} \int_{\mathbb{R}^n} b(x) u_m (\tau^m \varphi) &= \int_{\operatorname{supp} \varphi} b(x - y_m)(\tau_m u_m)\varphi = \lambda \int_{\mathbb{R}^n} \overline{u}\varphi + o(1), \\ \int_{\mathbb{R}^n} g(x, u_m)(\tau^m \varphi) &= \int_{\operatorname{supp} \varphi} g(x - y_m, \tau_m u_m)\varphi = \int_{\mathbb{R}^n} |\overline{u}^+|^{p-1}\varphi + o(1); \end{aligned}$$

here we have used (13). Next, note that by (8),

$$\begin{aligned} &\int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij}(x, u_m) D_i u_m D_j(\tau^m \varphi) \\ &\quad = \int_{\operatorname{supp} \varphi} \sum_{i,j=1}^n a_{ij}(x - y_m, \tau_m u_m) D_i(\tau_m u_m) D_j \varphi = \int_{\mathbb{R}^n} \nabla \overline{u} \nabla \varphi + o(1). \end{aligned}$$

Finally, take $\varepsilon > 0$; then by Lemma 4 we have

$$\int_{\mathbb{R}^n} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_m) D_i u_m D_j u_m (\tau^m \varphi) \\ \leq c\varepsilon + \int_{\{|u_m|>R\}} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_m) D_i u_m D_j u_m (\tau^m \varphi)$$

and again by (8),

$$\int_{\{|u_m|>R\}} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_m) D_i u_m D_j u_m (\tau^m \varphi) \\ = \int_{\operatorname{supp}\varphi \cap \{|\tau_m u_m|>R\}} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x-y_m, \tau_m u_m) D_i(\tau_m u_m) D_j(\tau_m u_m)\varphi = o(1);$$

by arbitrariness of ε, this, together with Lemma 1 and a density argument, gives $J'_\infty(\overline{u}) = 0$ and $\overline{u} \geq 0$; $\overline{u} > 0$ follows by the maximum principle. □

We can now prove a lower semicontinuity property of J_∞ on the "translated" PS sequence $\{\tau_m u_m\}$:

Lemma 8. *Let $\{u_m\}$ be a PS sequence for J and assume that $u_m \rightharpoonup 0$, and let $\overline{u}$ be as in* (20)*; then $J_\infty(\overline{u}) \leq \liminf_{m\to\infty} J_\infty(\tau_m u_m)$.*

Proof. Since $u_m \rightharpoonup 0$, by Lemma 6 we have $J'_\infty(u_m)[u_m] = o(1)$ as $m \to \infty$ and by the translation invariance of J'_∞ we get $J'_\infty(\tau_m u_m)[\tau_m u_m] = o(1)$, which yields

$$\int_{\mathbb{R}^n} |\nabla(\tau_m u_m)|^2 + \lambda \int_{\mathbb{R}^n} |\tau_m u_m|^2 = \int_{\mathbb{R}^n} |(\tau_m u_m)^+|^p + o(1);$$

therefore,

$$J_\infty(\tau_m u_m) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^n} |(\tau_m u_m)^+|^p + o(1).$$

Similarly, by Lemma 7 we infer

$$J_\infty(\overline{u}) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^n} |\overline{u}|^p$$

and the result follows by Fatou's Lemma. □

If $u_m \rightharpoonup 0$, by Lemma 5 and by the translation invariance of J_∞ we obtain $J(u_m) = J_\infty(u_m) + o(1) = J_\infty(\tau_m u_m) + o(1)$; therefore, we can summarize the above results in the following

Proposition 3. *Assume that the hypotheses of Theorem 2 hold; let $\{u_m\}$ be the PS sequence for J found in Proposition* 2 *and assume that $u_m \rightharpoonup 0$; then $\alpha \geq J_\infty(\overline{u})$ where $\overline{u}$ is given by* (20).

4.5. Proof of Theorem 2. Let $\{u_m\}$ be the PS sequence at level α given by Proposition 2; by Proposition 1, it converges weakly (up to a subsequence) to a positive limit $u \in H^1$ that solves (1). Therefore, if $u \not\equiv 0$ Theorem 2 is proved.

If $u \equiv 0$, consider $\overline{u} > 0$ as in (20); we claim that $\overline{u}$ is in fact a critical point for J at level α. To see this, we build a path $\gamma \in \Gamma$ (Γ as in (17)) for which $\max_{[0,1]} J(\gamma(t)) = \alpha$. Let e be as in (16), and define the set $V := \{a\overline{u}+be : a \geq 0, b \geq 0\}$. For all $v \in V$ we have $\lim_{t\to\infty} J_\infty(tv) = -\infty$ and since V is a two-dimensional manifold, by a compactness argument we can choose R large enough to ensure that

$$J_\infty(a\overline{u} + be) \leq 0 \quad \forall a, b \geq 0,\ a + b = R.$$

Define the path $\gamma : [0,1] \to H^1$ by

$$\gamma(t) := \begin{cases} 3Rt\overline{u} & \text{if } t \in [0,1/3], \\ (3t-1)Re + (2-3t)R\overline{u} & \text{if } t \in (1/3,2/3), \\ (3R+3t-3Rt-2)e & \text{if } t \in [2/3,1]; \end{cases}$$

we obviously have $\gamma \in \Gamma$. Moreover, $J_\infty(\gamma(t)) < 0$ if $t \in (1/3,1]$ and $\max_{[0,1/3]} J_\infty(\gamma(t)) = J_\infty(\overline{u})$ by the results of [17]. Hence, (18), (10) and Proposition 3 imply

$$\alpha \leq \max_{[0,1]} J(\gamma(t)) \leq \max_{[0,1]} J_\infty(\gamma(t)) = J_\infty(\overline{u}) \leq \alpha;$$

therefore, the path γ is "optimal" in Γ and the deformation lemma in its nonsmooth version [15] implies that there exists $\overline{t} \in (0,1)$ such that $\gamma(\overline{t})$ is a critical point of J at level α. Moreover, $\gamma(\overline{t}) = \overline{u}$; if not, by (10) and the results of [17] we obtain

$$J(\gamma(\overline{t})) \leq J_\infty(\gamma(\overline{t})) < J_\infty(\overline{u}) = \alpha,$$

contradicting $J(\gamma(\overline{t})) = \alpha$. Therefore, $\overline{u}$ is a (strictly) positive solution of (1) and Theorem 2 is proved. $\square$

Remark. If $\overline{u}$ solves (1), then either (1) reduces to the semilinear autonomous problem (9) or there exists $\omega \subset \mathbb{R}^n$ of positive measure such that the inequalities in (10) become strict for all $x \in \omega$ and for some $\xi \in \mathbb{R}^n$ and $s > 0$ outside the range of values attained by $\nabla\overline{u}$ and $\overline{u}$ respectively. Moreover, we obviously have $b(x) \equiv \lambda$.

5. Appendix: basic tools in nonsmooth critical point theory

In this section we quote some tools of the nonsmooth critical point theory introduced in [15, 16] (see also [19]).

Definition 1. Let (X,d) be a metric space, $I \in C(X,\mathbb{R})$ and let $x \in X$. We denote by $|dI|(x)$ the supremum of the $\sigma \in [0,\infty)$ such that there exist $\delta > 0$

and a continuous map

$$\mathcal{H} : B(x,\delta) \times [0,\delta] \to B(x, 2\delta)$$

such that for all $y \in B(x,\delta)$ and $t \in [0,\delta]$ we have

$$d(\mathcal{H}(y,t), y) \le t \quad \text{and} \quad I(\mathcal{H}(y,t)) \le I(y) - \sigma t$$

where $B(x,r) := \{y \in X : d(x,y) < r\}$; $|dI|(x)$ is called the *weak slope* of I at x.

We observe that if $\Psi : X \to X$ is any surjective isometry in X, then $|dI|(\Psi(x)) = |dI|(x)$ for all $x \in X$; in particular, if $X = H^1$ and I is invariant under translations, so is $|dI|$.

Definition 2. Let $I \in C(X,\mathbb{R})$; a point $x \in X$ is said to be *critical* for I if $|dI|(x) = 0$. A real number c is said to be a *critical value* for I if there exists $x \in X$ such that $I(x) = c$ and $|dI|(x) = 0$.

Let us now turn to PS sequences:

Definition 3. Let $I \in C(X,\mathbb{R})$; we say that a sequence $\{x_m\} \subset X$ is a *Palais–Smale sequence* (*PS sequence*) for I if $\{I(x_m)\}$ is bounded and $|dI|(x_m) \to 0$. We say that the functional I satisfies the *PS condition* if every PS sequence is relatively compact.

Following [3] we have

Definition 4. Let X be a Banach space, let $I \in C(X,\mathbb{R})$ and let Y be a dense subspace of X. If the directional derivative of I exists for all $x \in X$ in all the directions $y \in Y$ (i.e. $I'(x)[y]$ exists for all $x \in X$ and $y \in Y$) we say that I is *weakly Y-differentiable* and we call the extended real number

$$\|I'_Y(x)\| := \sup\{I'(x)[y] : y \in Y,\ \|y\|_X = 1\}$$

the *weak Y-slope* at x.

We can obtain a crucial lower estimate of the weak slope by means of the weak C^∞_{c}-slope; indeed, Theorem 1.5 of [10] states the following:

Proposition 4. *Assume* (2)–(4), (6), (7)*; then* $J \in C(H^1,\mathbb{R})$ *and* J *is weakly C^∞_{c}-differentiable. Furthermore, for all $u \in H^1$ we have*

$$|dJ|(u) \ge \sup\{J'(u)[\varphi] : \varphi \in C^\infty_{\mathrm{c}},\ \|\varphi\|_{H^1} = 1\} =: \|J'_{C^\infty_{\mathrm{c}}}(u)\|;$$

in particular, if $u \in H^1$ is a critical point of J (in the sense of Definition 2) then $J'(u)[\varphi] = 0$ for all $\varphi \in C^\infty_{\mathrm{c}}$ and u is a weak solution of (1).

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