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VARIATIONAL INEQUALITIES AND SURJECTIVITY FOR SET-VALUED MONOTONE MAPPINGS

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Throughout this paper, Φ and 2^X denote the real field (or the complex field) and the family of all nonempty subsets of a vector space over Φ, respectively. Let E and F be vector spaces over Φ and $\langle \cdot\,, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional. For each $x_0 \in E$ and $\varepsilon > 0$, let

$$\omega(x_0, \varepsilon) = \{y \in F : |\langle y, x_0 \rangle| < \varepsilon\}.$$

We denote by $\sigma(F, E)$ the topology on F generated by the family $\{\omega(x, \varepsilon) : x \in E,\ \varepsilon > 0\}$ as a subbase for the neighbourhood system at 0.

It is easy to show that, if F possesses the $\sigma(F, E)$-topology, F becomes a locally convex topological vector space. The $\sigma(E, F)$-topology on E is defined analogously. A subset X of E is said to be $\sigma(E, F)$-compact if X is compact related to the $\sigma(E, F)$-topology.

Let X be a nonempty subset of E. A set-valued mapping $T : X \to 2^F$ is said to be monotone relative to the bilinear functional $\langle \cdot\,, \cdot \rangle : F \times E \to \Phi$ (monotone for short) if, for all $x, y \in X$, $u \in T(x)$ and $w \in T(y)$,

$$\operatorname{Re} \langle u - w, x - y \rangle \geq 0.$$

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The mapping T is said to be maximal monotone relative to the bilinear functional $\langle\,\cdot\,,\,\cdot\,\rangle : F\times E\to\Phi$ (maximal monotone) if, for any $y\in X$ and $g\in T(y)$, $\operatorname{Re}\langle f-g, x-y\rangle\ge 0$ implies that $x\in X$ and $f\in T(x)$. A bilinear functional $\langle\,\cdot\,,\,\cdot\,\rangle : F\times E\to\Phi$ is said to be variable related if, for any $f\in F$, $\langle f,x\rangle=0$ for all $x\in E$ implies $f=0$.

In this paper, we study a class of variational inequalities and surjectivity for set-valued monotone mappings in topological vector spaces. Our results generalize the results of Shih and Tan ([9]) and others ([1], [6], [7]).

Throughout this paper, let E be a locally convex Hausdorff topological vector space, F be a locally convex Hausdorff topological vector space equipped with the $\sigma(F,E)$-topology and the bilinear functional $\langle\,\cdot\,,\,\cdot\,\rangle : F\times E\to\Phi$ variable related.

For our main theorem, we need the following lemmas:

Lemma 1. *For any $f\in F$, the mapping $x\mapsto\langle f,x\rangle$ is continuous with respect to the $\sigma(E,F)$-topology in E, and for any $x\in E$, the mapping $f\mapsto\langle f,x\rangle$ is also continuous on the $\sigma(F,E)$-topology in F.*

Lemma 2. *Let X be a nonempty convex subset of E and $T : X\to 2^F$ be upper semi-continuous on each line segment of X. If, for each $\overline{y}\in X$,*

$$\sup_{u\in T(x)}\operatorname{Re}\langle u,\overline{y}-x\rangle\le 0\quad\text{for all } x\in X, \tag{1}$$

then

$$\inf_{w\in T(\overline{y})}\operatorname{Re}\langle w,\overline{y}-x\rangle\le 0\quad\text{for all } x\in X. \tag{2}$$

Proof. For any $x\in X$ and $t\in[0,1]$, let

$$x_t = tx+(1-t)\overline{y}=\overline{y}-t(\overline{y}-x).$$

Since X is convex, we have $x_t\in X$ and so, by (1),

$$\sup_{u\in T(x_t)}\operatorname{Re}\langle u,\overline{y}-x_t\rangle\le 0,$$

from which follows that

$$\sup_{u\in T(x_t)}\operatorname{Re}\langle u,\overline{y}-x\rangle\le 0. \tag{3}$$

For any $f\in T(\overline{y})$ and $\varepsilon>0$, let

$$u(f)=\{w\in F : |\langle w-f,\overline{y}-x\rangle|<\varepsilon\}.$$

Then $u(f)$ is an open neighbourhood at f and so $G=\bigcup_{f\in T(\overline{y})}u(f)$ is an open neighbourhood at $T(\overline{y})$. Since T is upper semi-continuous on each line segment

$L = \{x_t : t \in [0,1]\} \subset X$, for any open neighbourhood G at $T(\overline{y})$, there exists an open neighbourhood N of $\overline{y}$ in L such that $T(y) \subset G$ for all $y \in N$.

Letting $t \to 0^+$, then $x_t \to \overline{y}$, and so there exists $\delta \in (0,1)$ such that $x_t \in N$ for all $t \in (0,\delta)$. Thus we have $T(x_t) \subset G$. Let $t_0 \in (0,\delta)$ and $u_0 \in T(x_{t_0}) \subset G$, then there exists $f_0 \in T(\overline{y})$ such that $u_0 \in u(f_0)$. Thus we have

$$|\langle u_0 - f_0, \overline{y} - x\rangle| < \varepsilon,$$

and so

$$|\mathrm{Re}\,\langle f_0 - u_0, \overline{y} - x\rangle| \le |\langle u_0 - f_0, \overline{y} - x\rangle| < \varepsilon. \tag{4}$$

Combining (3) and (4), we have

$$\mathrm{Re}\,\langle f_0, \overline{y} - x\rangle \; < \mathrm{Re}\,\langle u_0, \overline{y} - x\rangle + \varepsilon \le \varepsilon,$$

which implies that

$$\inf_{w \in T(\overline{y})} \mathrm{Re}\,\langle w, \overline{y} - x\rangle \; \le \varepsilon.$$

Since ε is arbitrary, we have

$$\inf_{w \in T(\overline{y})} \mathrm{Re}\,\langle w, \overline{y} - x\rangle \le 0$$

for all $x \in X$. This completes the proof. □

Remark 1. If E is a Banach space, $F = E^*$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ is the pairing between E and E^*, then the topology in F coincides with the weak-star topology in E^*. From Lemma 2, we can obtain Lemma 2 in [9] and the condition "for all $x \in X$, $T(x)$ is a weak-star compact subset in E^*" may be dropped. Further, Lemma 2 generalizes the corresponding results in [6].

Lemma 3. *Let $T : E \to 2^F$ be a set-valued monotone mapping. Then T is a maximal monotone mapping if and only if any monotone mapping $T^* : E \to 2^F$ which satisfies $T(y) \subset T^*(y)$ for all $y \in E$ must be equal to T.*

Proof. We suppose that T is maximal monotone and T^* is a monotone mapping such that $T(y) \subset T^*(y)$ for all $y \in E$ and assume that $T \ne T^*$. Then there exists $y_0 \in E$ such that $T(y_0) \ne T^*(y_0)$ and so there exists $f_0 \in T^*(y_0)$ such that $f_0 \notin T(y_0)$. Since T is maximal monotone, for any $y \in E$ and $g \in T(y)$,

$$\mathrm{Re}\,\langle f_0 - g, y_0 - y\rangle \ge 0,$$

and so $y_0 \in E$ and $f_0 \in T(y_0)$, which is a contradiction. Therefore, we have $T = T^*$. Conversely, we suppose that T is monotone and, for all $x, y \in E$, $f \in F$ and $g \in T(y)$,

$$\mathrm{Re}\,\langle f - g, x - y\rangle \ge 0.$$

We define $T^* : E \to 2^F$ by

$$T^* = \begin{cases} T(z) & \text{for } z \neq x, \\ T(x) \cup \{f\} & \text{for } z = x, \end{cases}$$

for all $z \in E$. Then T^* is monotone and $T(z) \subset T^*(z)$ for all $z \in E$. Thus, by assumption, $T = T^*$ and so $T^*(x) = T(x)$ and $f \in T(x)$. Therefore, T is maximal monotone. This completes the proof. □

Lemma 4. *Let $T : E \to 2^F$ be a set-valued monotone mapping with compact convex values and T be upper semi-continuous on each line segment of E. Then T is maximal monotone.*

Proof. Let $T^* : E \to 2^F$ be monotone and $T(y) \subset T^*(y)$ for all $y \in E$. Since T^* is monotone, for all $x, y_0 \in E$, $w_0 \in T^*(y_0)$ and $u \in T^*(x)$,

$$\operatorname{Re}\langle u - w_0, y_0 - x\rangle \leq 0,$$

and so, for all $x \in E$,

$$\sup_{u \in T(x)} \operatorname{Re}\langle u - w_0, y_0 - x\rangle \leq 0.$$

By Lemma 2, we have

$$\sup_{x \in E} \inf_{w \in T(y_0)} \operatorname{Re}\langle w - w_0, y_0 - x\rangle \leq 0.$$

From Lemma 1, it follows that, for all $x \in E$, $w \mapsto \operatorname{Re}\langle w - w_0, y_0 - x\rangle$ is a continuous affine functional on $T(y_0)$ and, for all $w \in T(y_0)$, $x \mapsto \operatorname{Re}\langle w - w_0, y_0 - x\rangle$ is clearly a concave functional. Noting that $T(y_0)$ is a compact convex set and so, by the max–min theorem of Kneser ([5]), we have

$$\inf_{w \in T(y_0)} \sup_{x \in E} \operatorname{Re}\langle w - w_0, y_0 - x\rangle \leq 0.$$

Since $T(y_0)$ is compact, there exists $\overline{w} \in T(y_0)$ such that

$$\sup_{x \in E} \operatorname{Re}\langle \overline{w} - w_0, y_0 - x\rangle \leq 0.$$

For all $y \in E$, letting $x = y_0 + y$, we have $\operatorname{Re}\langle \overline{w} - w_0, y\rangle \geq 0$. On the other hand, letting $x = y_0 - y$, we have $\operatorname{Re}\langle \overline{w} - w_0, y\rangle \leq 0$. Thus, for all $y \in E$, $\operatorname{Re}\langle \overline{w} - w_0, y\rangle = 0$. Since the bilinear functional $\langle \cdot , \cdot \rangle$ is variable related, we have $\overline{w} = w_0$ and so $w_0 \in T(y_0)$, which means that $T = T^*$. Therefore, by Lemma 3, T is maximal monotone. This completes the proof. □

LEMMA 5 (Fan–Knuster–Kuratowski–Mazurkiewicz Theorem, [10]). *Let Y be a nonempty convex subset of a topological vector space E and $\emptyset \neq X \subset Y$. For each $x \in X$, let $F(x)$ be a relatively closed subset of Y such that the convex hull of each finite subset $\{x_1, \dots, x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_0)$. Then, for each nonempty subset X_0 of X such that X_0 is contained in a compact convex subset of Y, $\cap_{x \in X_0} F(x) \neq \emptyset$. Furthermore, if, for such a X_0 (i.e., X_0 is contained in a compact convex subset of Y), the nonempty set $\bigcap_{x \in X_0} F(x)$ is compact, then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

Now, using Lemmas 1–5, we have our main theorems.

THEOREM 1. *Let X be a nonempty convex subset of E, $T : X \to 2^F$ be a set-valued monotone mapping and T be upper semi-continuous on each line segment of X. If there exists a $\sigma(E, F)$-compact set K in E and $x_0 \in X$ such that, for all $y \in X - K$,*

$$\inf_{w \in T(y)} \operatorname{Re} \langle w, y - x_0 \rangle > 0,$$

then there exists $\overline{y} \in X$ such that

$$\inf_{w \in T(y)} \operatorname{Re} \langle w, \overline{y} - x \rangle \leq 0$$

for all $x \in X$. Further, if $T(\overline{y})$ is a compact convex set, then there exists $\overline{w} \in T(\overline{y})$ such that

$$\operatorname{Re} \langle \overline{w}, \overline{y} - x \rangle \leq 0 \quad \textit{for all } x \in X.$$

PROOF. For all $x \in X$, let

$$F(x) = \left\{ y \in X : \inf_{w \in T(y)} \operatorname{Re} \langle w, y - x \rangle \leq 0 \right\},$$
$$G(x) = \left\{ y \in X : \sup_{u \in T(x)} \operatorname{Re} \langle u, y - x \rangle \leq 0 \right\}.$$

(i) First, we show that $\bigcap_{x \in X} F(x) = \bigcap_{x \in X} G(x)$. Since T is monotone, for all $x, y \in X$, $u \in T(x)$ and $w \in T(y)$,

$$\operatorname{Re} \langle w, y - x \rangle \geq \operatorname{Re} \langle u, y - x \rangle,$$

and so

$$\inf_{w \in T(y)} \operatorname{Re} \langle w, y - x \rangle \geq \sup_{u \in T(x)} \operatorname{Re} \langle u, y - x \rangle. \tag{5}$$

Thus $F(x) \subset G(x)$ for all $x \in X$, which implies that $\bigcap_{x \in X} F(x) \subset \bigcap_{x \in X} G(x)$. By Lemma 2, if $\sup_{u \in T(x)} \operatorname{Re} \langle u, y - x \rangle \leq 0$ for all $x, y \in X$, then

$$\inf_{w \in T(y)} \operatorname{Re} \langle w, y - x \rangle \leq 0$$

for all $x, y \in X$. Thus $G(x) \subset F(x)$ for all $x \in X$, which implies that $\bigcap_{x\in X} G(x) \subset \bigcap_{x\in X} F(x)$. Combining the above results, we have

$$\bigcap_{x\in X} F(x) = \bigcap_{x\in X} G(x).$$

(ii) Next, we show that, for each finite set $\{x_1, \dots, x_n\} \subset X$,

$$\operatorname{co}\{x_1, \dots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i),$$

where $\operatorname{co}\{x_1, \dots, x_n\}$ denotes the convex hull of $\{x_1, \dots, x_n\}$. Let's assume that our conclusion is not true. Then there exists $\overline{y} \in \operatorname{co}\{x_1, \dots, x_n\}$ and $\overline{y} = \sum_{i=1}^{n} \lambda_i x_i$, $\lambda_i \geq 0$, $i = 1, \dots, n$, $\sum_{i=1}^{n} \lambda_i = 1$, such that

$$\overline{y} \notin \bigcup_{i=1}^{n} G(x_i).$$

By (5), we have $\overline{y} \notin \bigcup_{i=1}^{n} F(x_i)$ and so

$$\inf_{w\in T(y)} \operatorname{Re}\langle w, \overline{y} - x_i\rangle > 0$$

for $i = 1, \dots, n$. Therefore, we have

$$\begin{aligned} 0 &= \inf_{w\in T(y)} \operatorname{Re}\langle w, \overline{y} - \overline{y}\rangle = \inf_{w\in T(y)} \operatorname{Re}\left\langle w, \overline{y} - \sum_{i=1}^{n} \lambda_i x_i \right\rangle \\ &\geq \sum_{i=1}^{n} \lambda_i \inf_{w\in T(y)} \operatorname{Re}\langle w, \overline{y} - x_i\rangle > 0, \end{aligned}$$

which is a contradiction and we have the conclusion.

(iii) Finally, we show that

$$\bigcap_{x\in X} F(x) = \bigcap_{x\in X} G(x) \neq \emptyset,$$

and the conclusion of the theorem is true. We suppose that there exists a $\sigma(E, F)$-compact set K in E and $x_0 \in X$ such that, for all $y \in X - K$,

$$\inf_{w\in T(y)} \operatorname{Re}\langle w, y - x_0\rangle > 0.$$

Then $y \notin F(x_0)$ and so $F(x_0) \subset K$. From the proof of (i), it follows that $G(x_0) \subset K$. By Lemma 1, for all $u \in F$ and $x \in X$, $y \mapsto \operatorname{Re}\langle u, y - x\rangle$ is continuous on $\sigma(E, F)$-topology in X and, by Proposition 1.4.6 in [2], $y \mapsto \sup_{u\in T(x)} \operatorname{Re}\langle u, y - x\rangle$ is lower semi-continuous on $\sigma(E, F)$-topology in X. Thus $G(x_0)$ is a $\sigma(E, F)$-compact set. By Lemma 5, we have $\bigcap_{x\in X} G(x) \neq \emptyset$ and so $\bigcap_{x\in X} F(x) \neq \emptyset$. Taking $\overline{y} \in \bigcap_{x\in X} F(x)$, then we have

$$\inf_{w\in T(y)} \operatorname{Re}\langle w, \overline{y} - x\rangle \leq 0 \tag{6}$$

for all $x \in X$. To show the conclusion of the theorem, suppose that $T(\overline{y})$ is a compact convex set. By Lemma 1, for all $x \in X$, $w \mapsto \operatorname{Re}\langle w, \overline{y} - x\rangle$ is a continuous affine functional on $T(\overline{y})$ and, for all $w \in T(\overline{y})$, $x \mapsto \operatorname{Re}\langle w, \overline{y} - x\rangle$ is a concave functional on X. By the Kneser max–min theorem ([5]), we have

$$\inf_{w\in T(y)} \sup_{x\in X} \operatorname{Re}\langle w, \overline{y} - x\rangle = \sup_{x\in X} \inf_{w\in T(y)} \operatorname{Re}\langle w, \overline{y} - x\rangle.$$

By (6), it follows that

$$\inf_{w\in T(t)} \sup_{x\in X} \operatorname{Re}\langle w, \overline{y} - x\rangle \leq 0.$$

Since $T(\overline{y})$ is compact, there exists $\overline{w} \in T(\overline{y})$ such that

$$\operatorname{Re}\langle \overline{w}, \overline{y} - x\rangle \leq 0$$

for all $x \in X$. This completes the proof. □

As an immediate consequence of Theorem 1, we have the following:

Corollary 2. *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a nonempty convex subset of E and $T : X \mapsto 2^{E^*}$ be a set-valued monotone mapping which is upper semi-continuous in the topology of E and the weak topology of E^* on each line segment of X. If there exists $x_0 \in X$ such that*

$$\lim_{\substack{\|y\|\to\infty \\ y\in X}} \inf_{w\in T(y)} \operatorname{Re}\langle w, y - x_0\rangle > 0, \tag{7}$$

then there exists $\overline{y} \in X$ such that

$$\sup_{x\in X} \inf_{w\in T(y)} \operatorname{Re}\langle w, \overline{y} - x\rangle \leq 0. \tag{8}$$

Further, if $T(\overline{y})$ is a weakly compact convex set in E^*, then there exists $\overline{w} \in T(\overline{y})$ such that $\operatorname{Re}\langle \overline{w}, \overline{y} - x\rangle \leq 0$ *for all $x \in X$.*

Proof. Let $F = E^*$ in Theorem 1 and $\langle\cdot,\cdot\rangle$ be the pairing between E and E^*. Then the $\sigma(E, F)$-topology on F coincides with the weak-star topology on E^*. Since E is reflexive, the weak-star topology on E^* is consistent with the weak topology on E^*. By (7), there exists $R > 0$ such that, for all $y \in X$ with $\|y\| > R$,

$$\inf_{w\in T(y)} \operatorname{Re}\langle w, y - x_0\rangle > 0. \tag{9}$$

Putting $K = \{y \in X : \|y\| \leq R\}$, we find that K is a weakly compact subset of X and for all $y \in X - K$, (9) holds. Therefore, all the conditions of Theorem 1 are satisfied and so the conclusions of Corollary 2 follow. This completes the proof. □

Remark 2. Corollary 2 improves Theorem 1 in [9], i.e., Corollary 2 says that Theorem 1 in [9] is true even though the conditions "X is a closed subset of E" and "for all $x \in X$, $T(x)$ is a weakly compact subset of E^*" are dropped in Theorem 1.

Using Theorem 1, we obtain results on the surjectivity for multi-valued monotone mappings as follows

THEOREM 3. *Let $T : E \to 2^F$ be a set-valued monotone mapping with compact convex values and T be upper semi-continuous on each line segment of E. If, for any $w_0 \in F$, there exists a $\sigma(E,F)$-compact set K in E and $x_0 \in E$ such that, for all $y \in E - K$,*

$$\inf_{w+w_0 \in T(y)} \operatorname{Re} \langle w, y - x_0 \rangle > 0, \tag{10}$$

then T is surjective, and the solution set $S(w_0) = \{y \in E : w_0 \in T(y)\}$ is a nonempty $\sigma(E,F)$-closed convex set.

PROOF. For any $w_0 \in F$, we define the mapping $T^* : E \to 2^F$ by

$$T^*(y) = T(y) - w_0$$

for all $y \in E$. Then T^* is a monotone mapping with compact convex values and T^* is upper semi-continuous on each line segment of E. Since there exists a $\sigma(E,F)$-compact set K and $x_0 \in E$ such that, for all $y \in E - K$,

$$\inf_{w \in T^*(y)} \operatorname{Re} \langle w, y - x_0 \rangle > 0,$$

it follows from Theorem 1 that there exist $\overline{y} \in E$ and $v \in T^*(\overline{y})$ such that, for all $x \in X$,

$$\operatorname{Re} \langle v, \overline{y} - x \rangle \le 0,$$

and so, for $\overline{w} = v + w_0 \in T(\overline{y})$,

$$\operatorname{Re} \langle \overline{w} - w_0,\ \overline{y} - x \rangle \le 0$$

for all $x \in X$. Therefore, by the proof of Lemma 4, we have $w_0 = \overline{w} \in T(\overline{y})$, which means that T is surjective.

Next, we show that the solution set $S(w_0) = \{y \in E : w_0 \in T(y)\}$ is a nonempty $\sigma(E,F)$-closed convex set. Since T is surjective, the set $S(w_0) = \{y \in E : w_0 \in T(y)\}$ is nonempty. To show that $S(w_0)$ is a $\sigma(E,F)$-closed convex set, let

$$H = \bigcap_{y \in E} \bigcap_{v \in T(y)} \{z \in E : \operatorname{Re} \langle w_0 - v, z - y \rangle \ge 0\}.$$

By Lemma 1, the mapping $z \mapsto \operatorname{Re} \langle w_0 - v, z - y \rangle$ is continuous with respect to the $\sigma(E,F)$-topology in E and so, for all $y \in E$ and $v \in T(y)$,

$$\{z \in E : \operatorname{Re} \langle w_0 - v, z - y \rangle \ge 0\},$$

is a $\sigma(E,F)$-closed set and it is also clearly convex. Thus H is a $\sigma(E,F)$-closed convex set.

Now, we show that $S(w_0) = H$. Let $z \in S(w_0)$. Then $z \in E$ and $w_0 \in T(z)$. Since T is monotone, for all $y \in E$ and $v \in T(y)$,

$$\operatorname{Re}\langle w_0 - v, z - y\rangle \geq 0,$$

and so $z \in H$, i.e., $S(w_0) \subset H$.

Conversely, let $z \in H$. Then $z \in E$ and $\operatorname{Re}\langle w_0 - v, z - y\rangle \geq 0$ for all $y \in E$ and $v \in T(y)$. We define a mapping $T^* : E \to 2^F$ by

$$T^*(y) = \begin{cases} T(y) & \text{for} \quad y \neq z, \\ T(z) \cup \{w_0\} & \text{for} \quad y = z. \end{cases}$$

Then T^* is monotone and, for all $y \in E$, $T(y) \subset T^*(y)$. Thus, by Lemma 4, T is maximal monotone and, from Lemma 3, it follows that $T = T^*$. Hence $w_0 \in T(z)$ and so $z \in S(w_0)$, i.e., $H \subset S(w_0)$. Therefore we have $H = S(w_0)$. This completes the proof. □

REMARK 3. (1) From the proof of Theorem 3, we can show easily that, in the case that the bilinear functional $\langle\,\cdot\,,\,\cdot\,\rangle : F \times E \to \Phi$ is continuous with respect to the locally convex topology in the second variable, $S(w_0)$ is a nonempty closed convex set in E.

(2) Theorem 3 generalizes the corresponding results in [1], [7] and [9]. If E is a reflexive Banach space, $F = E^*$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ is the pairing between E and E^*, then, from Theorem 3, we obtain the result in [9]. If T is injective, from Theorem 3, we also obtain the corresponding results in [1] and [7].

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