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BORSUK–ULAM TYPE THEOREMS ON PRODUCT SPACES II

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Dedicated to the memory of Juliusz P. Schauder

Abstract. A generalization of the theorem of Zhong on the product of spheres to multivalued maps is given. We prove also a stronger result of Bourgin–Yang type.

1. Introduction

Let S^n denote the unit sphere in the Euclidean space R^{n+1}. The famous Borsuk–Ulam theorem states that for every continuous map $f : S^n \to R^n$ there exists a point $x \in S^n$ such that $f(x) = f(-x)$ (see [1], [14]). It can be formulated also in the equivalent form:

Theorem 1.1. *Let $f : S^n \to R^n$ be an odd map, i.e. $f(-x) = -f(x)$ for every $x \in S^n$. Then the set $f^{-1}(0)$ is nonempty.*

One of the most important generalizations of it is the Bourgin–Yang theorem (see [2], [15]):

Theorem 1.2. *Let $f : S^n \to R^k$ be an odd map. Then the covering dimension $\dim f^{-1}(0) \ge n - k$.*

In 1992 Zhong [16] extended the Borsuk–Ulam theorem for maps on the product of two spheres.

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Theorem 1.3. *Suppose that $f=(f_1,f_2): S^n\times S^m\to R^n\times R^m$ is a continuous map satisfying:*

(1) *$f_1(-x,y)=-f_1(x,y)$, $f_1(x,-y)=f_1(x,y)$ for every $(x,y)\in S^n\times S^m$,*
(2) *$f_2(-x,y)=f_2(x,y)$, $f_2(x,-y)=-f_1(x,y)$ for every $(x,y)\in S^n\times S^m$.*

Then there exists a point $(x,y)\in S^n\times S^m$ such that $f(x,y)=0$.

It is easily seen that f is equivariant under a suitable action of the group $Z_2\times Z_2$ on $R^{n+1}\times R^{m+1}$.

The aim of the first part of our paper (see [5]) was to give a natural generalization of the theorem of Zhong to the product of q spheres with the natural free action of the group $(Z_2)^r$, $r\in N$. In fact, we have also generalized Theorem 1.2 to that case. As the main tool we used the ideal-valued G-index defined by Fadell and Husseini in [6]. Our proof was different from that of Zhong and gave a more general result. In this paper we present further generalizations of the above results to multivalued maps. Multivalued versions of the Borsuk–Ulam type theorems were considered also by Gęba and Górniewicz [7], and Izydorek [11], [12]. Here we extend their results to the product of spheres.

2. Preliminaries

Throughout the paper we will use the Čech cohomology with coefficients in Z_2, the group of integers mod 2. This particular cohomology is chosen because it is defined for paracompact spaces and has the continuity property, i.e.

$$H^*(X,Z_2)=\varprojlim H^*(X_n,Z_2),$$

where $X=\bigcap_{n\in N}X_n$.

Let G be the direct sum of r copies of the group Z_2, $G=(Z_2)^r$, for some $r\in N$. Assume that G acts freely on a paracompact Hausdorff space $\widetilde{X}$, i.e. for $g\in G$ and $\widetilde{x}\in\widetilde{X}$ $g\widetilde{x}=\widetilde{x}$ implies $g=0$ in G. We call $\widetilde{X}$ a free *G-space*.

It is well known that any free G-space $\widetilde{X}$ admits an equivariant map $\widetilde{h}:\widetilde{X}\to EG$ into a contractible free G-space EG (see [4]); any two such maps are equivariantly homotopic (see [4, Theorems 8.12 and 6.14]). The map $\widetilde{h}$ induces a map $h:X\to BG$ on the orbit spaces $X:=\widetilde{X}/G$ and $BG:=EG/G$ which is unique up to homotopy. Consequently we are given the unique ring homomorphism

$$h^*:H^*(BG,Z_2)\to H^*(X,Z_2).$$

For $G=(Z_2)^r$ the space EG can be identified with the r-fold Cartesian product of spheres S^∞ of infinite dimension $EG=S^\infty\times\ldots\times S^\infty$ with a free action of G defined by

$$g_k(x_1,\ldots,x_k,\ldots,x_r)=(x_1,\ldots,-x_k,\ldots,x_r),$$

where g_k are fixed generators of G, $k = 1, \dots, r$. We easily find the orbit space BG which is the Cartesian product of r copies of infinite dimensional real projective spaces $BG = P^\infty \times \dots \times P^\infty$.

It is well known that $H^*(P^\infty, Z_2)$ is the polynomial ring $Z_2[x]$, where x corresponds to the generator of $H^1(P^\infty, Z_2) = Z_2$. By the Künneth formula we obtain

$$H^*(BG, Z_2) = Z_2[x_1, \dots, x_r],$$

the ring of polynomials of r variables. Elements $x_1, \dots, x_r$ correspond to generators $g_1, \dots, g_r$ of $H^1(BG, Z_2) = (Z_2)^r$.

Let us recall the Fadell and Husseini definition of the G-index, $I^G(\widetilde{X})$, for a G-space $\widetilde{X}$ (see [6]) formulated for the particular case when $\widetilde{X}$ is a free $(Z_2)^r$-space.

Definition 2.1. The *G-index* of a free G-space $\widetilde{X}$ is the ideal $I^G(\widetilde{X}) = \ker h^*$ in the ring $H^*(BG, Z_2) = Z_2[x_1, \dots, x_r]$.

Most of the properties of the G-index are immediate consequences of the definition. In particular, we have:

(a) (Monotonicity) If G acts freely on $\widetilde{X}$ and $\widetilde{Y}$, and $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is an equivariant map, then $I^G(\widetilde{Y}) \subset I^G(\widetilde{X})$.

(b) (Dimension) If $\dim \widetilde{X} < m$, then $x_1^{t_1} \dots x_r^{t_r} \in I^G(\widetilde{X})$ whenever $t_1 + \dots + t_r \geq m$ where dim denotes the covering dimension.

An important special case of the above is:

(c) (Nontriviality) If $I^G(\widetilde{X}) \neq Z_2[x_1, \dots, x_r]$, then $\widetilde{X} \neq \emptyset$.

Let G act freely on $\widetilde{X}$ and let $\widetilde{A} \subset \widetilde{X}$ be a compact G-space. Since the Čech cohomology has the continuity property and ring $Z_2[x_1, \dots, x_r]$ is Noetherian we obtain:

(d) (Continuity) There is an open neighbourhood $\widetilde{U}$ of $\widetilde{A}$ in $\widetilde{X}$ which is a G-space such that $I^G(\widetilde{U}) = I^G(\widetilde{A})$.

The concept of the G-index was introduced by Yang [15] for $G = Z_2$ and next extended to other more general settings by several authors, notably to actions of compact Lie groups by Fadell and Husseini [6].

3. Multivalued maps

Let X, Y be two Hausdorff topological spaces. We say that $\varphi : X \to Y$ is *a multivalued map* if for every point $x \in X$ a nonempty subset $\varphi(x)$ of Y is given.

A *graph* of a multivalued map φ is the set

$$\Gamma_\varphi := \{(x, y) \in X \times Y \mid y \in \varphi(x)\}.$$

An *image* of a subset $A \subset X$ is the set $\varphi(A) := \bigcup_{x \in A} \varphi(x)$.

For a subset $B \subset Y$ we can define two types of *a counterimage*:

$$\varphi^{-1}(B) := \{x \in X \mid \varphi(x) \subset B\}, \quad \varphi_+^{-1}(B) := \{x \in X \mid \varphi(x) \cap B \neq \emptyset\}.$$

They both coincide if φ is a singlevalued map.

One defines a composition of $\varphi : X \to Y$ and $\psi : Y \to Z$ as a map $\gamma : X \to Z$ given by $\gamma(x) = \psi(\varphi(x))$.

A multivalued map $\varphi : X \to Y$ is *upper semicontinuous* (*u.s.c.*) provided

(i) for each $x \in X$ $\varphi(x) \subset Y$ is compact,
(ii) for every open subset $V \subset Y$ the set $\varphi^{-1}(V)$ is open in X.

Let us recall some basic properties of u.s.c. maps:

(1) The image of a compact set is a compact set.
(2) The graph Γ_φ is a closed subset of $X \times Y$.
(3) The composition of two u.s.c. maps is an u.s.c. map, too.

Now we recall an important class of admissible multivalued maps considered by Górniewicz [8], [9].

We say that a space X is *acyclic* if $H^*(X) = H^*(point)$.

Definition 3.1. An u.s.c. map $\varphi : X \to Y$ is *acyclic* if all the values $\varphi(x)$ are acyclic sets.

A continuous map $p : X \to Y$ is *a Vietoris map* if:

(i) $p(X) = Y$,
(ii) p is proper (i.e. $p^{-1}(A)$ is compact whenever $A \subset Y$ is compact),
(iii) for every $y \in Y$ the set $p^{-1}(y)$ is acyclic.

An important feature of Vietoris maps is the famous Vietoris–Begle mapping theorem (see [13]) which says that if X, Y are paracompact spaces and $p : X \to Y$ is a Vietoris map, then it induces an isomorphism on the Čech cohomology.

Definition 3.2. An u.s.c. map $\varphi : X \to Y$ is *admissible* provided there exists a space Γ, and two continuous maps $p : \Gamma \to X$, $q : \Gamma \to Y$ such that

(i) p is a Vietoris map,
(ii) for every $x \in X$ $q(p^{-1}(x)) \subset \varphi(x)$.

We call every such a pair (p, q) of maps *a selected pair* for φ.

The class of admissible maps includes all u.s.c maps with acyclic values, and in particular with convex values, when Y is a normed space. Moreover, a composition of two admissible maps is also admissible (see [8], [9]). Many results from the topological fixed point theory of singlevalued maps carry onto this class of maps.

A multivalued map $\varphi : X \to Y$ is a *G-map* if X, Y are G-spaces and $\varphi(gx) = g(\varphi(x))$ for all $x \in X$ and $g \in G$.

It is easily seen that each acyclic G-map admits a selected pair of G-maps (see Remark 4.1). However, there are admissible maps, even convex-valued maps, which are not G-maps still admitting a selected pair of G-maps.

4. Generalization of Zhong's theorem

Let us fix a sequence of natural numbers $n_1, \ldots, n_r$. For $k = 1, \ldots, r$ consider a subspace of EG

$$\widetilde{M}_k = S^\infty \times \ldots \times S^\infty \times S^{n_k-1} \times S^\infty \times \ldots \times S^\infty$$

which is a G-space itself. Clearly,

$$M_k = P^\infty \times \ldots \times P^\infty \times P^{n_k-1} \times P^\infty \times \ldots \times P^\infty.$$

It is well known that the cohomology ring $H^*(P^m, Z_2)$ is equal to the truncated polynomial ring $Z_2[x]/(x^{m+1})$, $m \geq 0$ (see [3], [4], [10], [13]). By the Künneth formula we obtain

$$H^*(M_k, Z_2) = Z_2[x_1, \ldots, x_r]/(x_k^{n_k}).$$

Lemma 4.1. *If $\widetilde{M} = \bigcup_{k=1}^r \widetilde{M}_k$, then $x_1^{n_1} \cdot \ldots \cdot x_r^{n_r} \in I^G(\widetilde{M})$.*

Proof. Since a G-map $\widetilde{h}$ is unique up to G-homotopy we can choose $\widetilde{h} = \widetilde{\iota}$ – the natural inclusion. Using the diagram

$$\begin{array}{ccc} \widetilde{M}_k & \xrightarrow{\widetilde{\iota}} & EG \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ M_k & \xrightarrow{\iota} & BG \end{array}$$

we find that $\iota^* : Z_2[x_1, \ldots, x_r] \to Z_2[x_1, \ldots, x_r]/(x_k^{n_k})$ maps x_k onto x_k, $k = 1, \ldots, r$. Since ι^* is a ring homomorphism

$$\iota^*(x_k^{n_k}) = [\iota^*(x_k)]^{n_k} = 0.$$

Therefore $x_k^{n_k}$ is an element of $I^G(\widetilde{M}_k)$.

Put $M = \bigcup_{k=1}^r M_k$ and consider the long exact sequence of the pair (M, M_k), for $k = 1, \ldots, r$,

$$\cdots \longrightarrow H^{n_k}(M, M_k) \xrightarrow{j^*} H^{n_k}(M) \xrightarrow{i^*} H^{n_k}(M_k) \longrightarrow \cdots.$$

Let $\widetilde{h} : \widetilde{M} \to EG$ be the natural inclusion map and let $h^* : Z_2[x_1, \ldots, x_r] \to H^*(M)$ be the corresponding ring homomorphism. Now, $i^*(h^*(x_k^{n_k})) = 0$ because $x_k^{n_k} \in I^G(\widetilde{M}_k)$. Thus, there is an element $\alpha_k \in H^{n_k}(M, M_k)$ such that $j^*(\alpha_k) = h^*(x_k^{n_k})$.

From the following commutative diagram

$$\begin{array}{ccc}
H^{n_1}(M) \otimes \ldots \otimes H^{n_r}(M) & \xrightarrow{\cup} & H^{n_r+\ldots+n_r}(M, M) = 0 \\
\Big\downarrow{\scriptstyle j_1^*\otimes\ldots\otimes j_r^*} & & \Big\downarrow{\scriptstyle j^*} \\
H^{n_1}(M, M_1) \otimes \ldots \otimes H^{n_r}(M, M_r) & \xrightarrow{\cup} & H^{n_1+\ldots+n_r}(M)
\end{array}$$

where $\cup$ denotes the cup-product (see [3]), we conclude

$$\alpha_1 \cup \ldots \cup \alpha_r = 0, \quad j_1^* \otimes \ldots \otimes j_r^*(\alpha_1 \otimes \ldots \otimes \alpha_r) = h^*(x_1^{n_1}) \otimes \ldots \otimes h^*(x_r^{n_r}),$$

and finally

$$0 = j^*(0) = h^*(x_1^{n_1}) \cup \ldots \cup h^*(x_r^{n_r}) = h^*(x_1^{n_1} \cdot \ldots \cdot x_r^{n_r}).$$

This proves Lemma 4.1. □

Let $\widetilde{X} = S^{n_1} \times \ldots \times S^{n_r}$ be a standard G-subspace of EG and let $\widetilde{\Gamma}$ be a free G-space. Consider the following diagram of G-maps

$$\widetilde{X} = S^{n_1} \times \ldots \times S^{n_r} \xleftarrow{p} \widetilde{\Gamma} \xrightarrow{q} R^{n_1} \times \ldots \times R^{n_r}.$$

Proposition 4.1. *If p is a Vietoris map, then there is no G-equivariant map $\widetilde{f} : \widetilde{\Gamma} \to \widetilde{M}$.*

Proof. In [5] we have observed that

$$I^G(\widetilde{X}) = (x_1^{n_1+1}, \ldots , x_r^{n_r+1}) \subset Z_2[x_1, \ldots , x_r].$$

Since p is a Vietoris G-map on free G-spaces, and the group G is finite, it is easy to check that the induced map on orbit spaces is also a Vietoris map. Therefore it induces an isomorphism of cohomology algebras (with coefficients in Z_2). Hence

$$I^G(\widetilde{\Gamma}) = I^G(\widetilde{X}).$$

Suppose that there exists $\widetilde{f} : \widetilde{\Gamma} \xrightarrow{G} \widetilde{M}$. By the monotonicity property of the G-index it follows $I^G(\widetilde{M}) \subset I^G(\widetilde{\Gamma})$. But $x_1^{n_1} \ldots x_r^{n_r} \notin I^G(\widetilde{\Gamma})$. This contradicts Lemma 4.1. □

Let $R^{n_1} \times \ldots \times R^{n_r}$ be a representation of $G = (Z_2)^r$ with the action given by $g_k(x_1, \ldots , x_k, \ldots , x_r) = (x_1, \ldots , -x_k, \ldots , x_r)$, where g_k are generators of G as before ($k = 1, \ldots , r$). In [5] we have proved the following

Theorem 4.1. *If $\widetilde{f} : S^{n_1} \times \ldots \times S^{n_r} \to R^{n_1} \times \ldots \times R^{n_r}$ is a G-map, then $\widetilde{f}^{-1}(0) \neq \emptyset$.*

A multivalued version of it is the following

Theorem 4.2. *If an admissible map* $\widetilde{\varphi} : S^{n_1} \times \ldots \times S^{n_r} \to R^{n_1} \times \ldots \times R^{n_r}$ *has a selected pair* (p, q) *of the form*

$$S^{n_1} \times \cdots \times S^{n_r} \xleftarrow{p} \widetilde{\Gamma} \xrightarrow{q} R^{n_1} \times \ldots \times R^{n_r}$$

where p *and* q *are* G*-maps, then* $\widetilde{\varphi}^{-1}(0) = \{x \mid 0 \in \varphi(x)\} \neq \emptyset$.

Proof. It is enough to proof that $q^{-1}(0) \neq \emptyset$. Therefore we can proceed the same lines as in the proof of Theorem 3.1 in [5]. The difference is that we use our Proposition 4.1 instead of Proposition 3.1 in [5]. $\square$

Let $d_1, \ldots, d_r$ be natural numbers and let $S^{n_1+d_1} \times \ldots \times S^{n_r+d_r}$ be the standard G-subspace of EG.

Theorem 4.3. *If* $\widetilde{\varphi} : S^{n_1+d_1} \times \ldots \times S^{n_r+d_r} \to R^{n_1} \times \ldots \times R^{n_r}$ *is an admissible map with a selected pair* (p, q) *of* G*-maps, then*

$$x_1^{d_1} \cdot \ldots \cdot x_r^{d_r} \notin I^G(\{x \mid 0 \in \widetilde{\varphi}(x)\}).$$

Proof. Observe that p induces a Vietoris map on orbit spaces, therefore the cohomology algebras are isomorphic. Thus, by repeating the algebraic arguments in the proof of Theorem 3.2 in [5], we obtain $x^{d_1} \cdot \ldots \cdot x^{d_r} \notin I^G(q^{-1}(0))$.

But on the other hand $A = \{x \mid 0 \in \varphi(x)\} = p(q^{-1}(0))$ and therefore $I^G(A) \subset I^G(q^{-1}(0))$. This ends the proof. $\square$

Corollary 4.1. *Let* $\widetilde{\varphi} : S^{n_1+d_1} \times \ldots \times S^{n_r+d_r} \to R^{n_1} \times \ldots \times R^{n_r}$ *be an admissible map with a selected pair* (p, q) *of* G*-maps. Then the covering dimension*

$$\dim\{x \mid 0 \in \widetilde{\varphi}(x)\} \geq d_1 + \ldots + d_r.$$

Proof. It is an immediate consequence of the dimension property of the G-index and Theorem 4.3. $\square$

Remark 4.1. If X, Y are G-spaces and $\varphi : X \to Y$ is an acyclic G-map, then the projections $p_\varphi : \Gamma_\varphi \to X$, $q_\varphi : \Gamma_\varphi \to Y$ define a natural selected pair of G-maps. Therefore Theorems 4.1 and 4.2 hold true for acyclic G-maps and their compositions.

References

[1] K. Borsuk, *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math. **20** (1933), 177–190.

[2] D. G. Bourgin, *On some separation and mapping theorems*, Comment. Math. Helv. **29** (1955), 199–214.

[3] A. Dold, *Lectures on Algebraic Topology*, Springer-Verlag, 1972.

[4] T. Dieck, *Transformation Groups*, Walter de Gruyter Studies in Math., vol. 8, Berlin New York, 1987.

[5] Z. Dzedzej, A. Idzik and M. Izydorek, *Borsuk–Ulam type theorems on product spaces* (to appear).

[6] E. Fadell and S. Husseini, *An ideal-valued cohomological index theory with applications to Borsuk–Ulam and Bourgin–Yang theorems*, Ergodic Theory Dynam. Systems **8*** (1988), 73–85.

[7] K. Gba and L. Grniewicz, *On the Bourgin–Yang theorem for multivalued maps I*, Bull. Polish Sci. Math. **34** (1986), 315–322.

[8] L. Górniewicz, *Homological methods in fixed point theory of multivalued maps*, Dissertationes Math. **129** (1976), 1–71.

[9] ______, *Topological Fixed Point Theory for Multivalued Mappings*, Kluwer, Amsterdam, 1999.

[10] Wu Yi Hsiang, *Cohomology Theory of Topological Transformation Groups*, Ergebnisse der Mathematik und ihrer Grenzgebierte, vol. 85, Springer-Verlag, Berlin Heidelberg New York, 1975.

[11] M. Izydorek, *Remarks on Borsuk-Ulam theorem for multivalued maps*, Bull. Polish Acad. Sci. Math. **15** (1987), 501–504.

[12] M. Izydorek, *Nonsymmetric version of Bourgin–Yang theorem for multivalued maps and free Z_p-actions*, J. Math. Anal. Appl. **137** (1989), 349–353.

[13] E. H. Spanier, *Algebraic Topology*, McGraw Hill, New York, 1966.

[14] H. Steinlein, *Borsuk's antipodal theorem and its generalizations and applications: A survey*, Meth. Topol. en Anal. Non Lineaire, Sém. Math. Sup. **95** (1985), Presses Univ. Montréal, Montreal, Quebec, 166–235.

[15] C. T. Yang, *On theorems of Borsuk–Ulam, Kakutani–Yamabe–Yujobô and Dyson I*, Ann. of Math. **60** (1954), 262–282.

[16] C. Zhong, *The Borsuk–Ulam theorem on product space*, Fixed point theory and its applications (K. K. Tan, ed.), World Scientific, Singapore, New Yersey, London, Hong Kong, 1992.

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