Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 23, 2004, 323–337

OPTIMAL FEEDBACK CONTROL IN THE PROBLEM OF THE MOTION OF A VISCOELASTIC FLUID

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Abstract. We study an optimization problem for the feedback control system emerging as a regularized model for the motion of a viscoelastic fluid subject to the Jeffris–Oldroyd rheological relation. The approach includes systems governed by the classical Navier–Stokes equation as a particular case. Using the topological degree theory for condensing multimaps we prove the solvability of the approximating problem and demonstrate the convergence of approximate solutions to a solution of a regularized one. At last we show the existence of a solution minimizing a given convex, lower semicontinuous functional.

1. Introduction

In recent decades a cospicuous number of works were devoted to various aspects of mathematical control theory for systems governed by partial differential equations. In this connection, an essential assumption in many approaches (see e.g. the well known monograph [9] of J. L. Lions), is the uniqueness of the solution corresponding to a given boundary value problem. However, this suggestion

2000 *Mathematics Subject Classification.* 76D55, 76D05, 47H04, 47H09, 47H11.

Key words and phrases. Optimal control, feedback control, viscoelastic fluid, Navier–Stokes equation, Jeffris–Oldroyd rheological relation, topological degree, condensing multivalued map.

The work of the first and third author is supported by RFBR Grants 04-01-00081, 02-01-00189 and U.S. CRDF-R.F. Ministry of Education Adward VZ-010-0.

The work of the second author is supported by a grant of MIUR.

looks fairly restrictive for a large variety of nonlinear equations of mathematical physics. In particular it is the case for equations arising in problems of hydrodynamics (a natural example is the well known Navier–Stokes equation).

For Navier–Stokes and Euler systems and some of their generalizations, solutions for control problems, when uniqueness theorems are unknown, were obtained by A. V. Fursikov (see [4]–[6]). Various problems of optimal control for systems governed by Navier–Stokes equations were also considered in a number of works (see, e.g. [1], [7]).

In the present paper we study an optimal control problem for a system emerging as a regularized model for the motion of a viscoelastic fluid subject to the Jeffris–Oldroyd rheological relation. Let us note that in the last few years many authors studied the solvability of the initial-boundary problem for equations with the Jeffris–Oldroyd rheological relation. In this paper we deal with the feedback control system governed by equations whose solvability was proved in the works [11] and [12].

The paper is organized as follows. In the first section, we give the statement of the optimal control problem and define the main notions. Then, we present the regularized problem describing the motion of a viscoelastic fluid. The approach includes the classical Navier–Stokes equation as a particular case. In the second section, we prove the solvability of the approximating inclusion, using some a priori estimates and topological degree theory for condensing multimaps as a main tool.

In the next section we prove the convergence (in some generalized sense) of solutions of approximating problems to a solution of the regularized one. In the last section we prove the existence of a solution minimizing a given convex, lower semicontinuous functional.

2. The setting of the problem

We consider an optimal control problem for the motion of a viscoelastic fluid filling a domain $\Omega \subset \mathbb{R}^n$, $2 \le n \le 4$. It will be assumed that Ω is a bounded domain with locally Lipschitz boundary Γ.

Let the function $v\colon [0,T] \times \Omega \to \mathbb{R}^n$ describes the velocity vector of a particle at the moment $t \in [0,T]$ in the point $x \in \Omega$. The density ρ of the fluid is assumed to be constant; the pressure of the fluid at the moment t in the point $x \in \Omega$ is characterized by the value $p(t,x)$. By the symbol $\mathcal{E}(v)$ we will denote the tensor of velocities of deformations

$$\begin{aligned} \mathcal{E}_{ij}(v) &= \frac{1}{2}\bigg(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\bigg), \quad 1 \le i,j \le n, \\ \mathcal{E}(v) &= (\mathcal{E}_{ij}(v)). \end{aligned}$$

Trajectories of the motion of fluid particles are determined by the field of velocities v as solutions of the integral equation

$$z(\tau; t, x) = x + \int_t^\tau v(s, z(s; t, x))\, ds, \quad \tau, t \in [0, T],\ x \in \Omega. \tag{2.1}$$

We will use $L_2(\Omega)^n$, $L_p(\Omega)^n$, $W_2^1(\Omega)^n$, $C(\Omega)^n$ and $C^1(\Omega)^n$ as standard notations for the corresponding spaces of functions $\varphi: \Omega \to \mathbb{R}^n$. The scalar product in $L_2(\Omega)^n$ will be denoted by $(v, w)_{L_2(\Omega)^n}$.

Let us denote

$$V = \{v \in W_2^1(\Omega)^n : v_{|\Gamma} = 0,\ \operatorname{div} v = 0\}.$$

We will consider V as a Hilbert space with the scalar product

$$(v, w)_V = \int_\Omega \sum_{i,j=1}^n \mathcal{E}_{ij}(v) \cdot \mathcal{E}_{ij}(w)\, dx$$

generating the corresponding norm $\|v\|$. In the space V this norm is equivalent to the norm induced from the space $W_2^1(\Omega)^n$.

Let H be the closure of V with respect to the norm of the space $L_2(\Omega)^n$ and V^* the dual space of V.

By the symbol $C^1 D(\overline{\Omega})$ we denote the set of all continuously differentiable bijective maps $\zeta: \overline{\Omega} \to \overline{\Omega}$ such that $\zeta_{|\Gamma}$ is the identity, and

$$\det \left(\frac{\partial \zeta}{\partial x} \right) = 1, \quad x \in \Omega.$$

Let us suppose that this set is endowed with the norm from the space $C(\overline{\Omega})^n$.

We consider also the set

$$CG = C([0, T] \times [0, T]; C^1 D(\overline{\Omega})).$$

Notice that

$$CG \subset C([0, T] \times [0, T]; C^1(\overline{\Omega})^n).$$

Then, CG can be considered as a metric space with the metric induced by the norm of the space $C([0, T] \times [0, T]; C(\overline{\Omega})^n)$.

Let us mention that the solvability of the equation (2.1) for a given v is known only for the case $v \in L_1([0, T]; C(\overline{\Omega}))$. moreover, the uniqueness can be guaranteed for $v \in L_1([0, T]; C^1(\overline{\Omega}))$ with $v_{|[0,T]\times\Gamma} = 0$. In our case $v \in L_2([0, T]; V)$, so one of the possible ways to avoid this difficulty is the smoothing of the field of velocities. To do so, let us introduce the regularization map

$$\gamma_\delta: H \to C^1(\overline{\Omega})^n \cap V, \quad \delta > 0$$

with the properties that $\gamma_\delta(v) \to v$ in H while $\delta \to 0$ and the operator

$$S_\delta: L_2([0, T]; H) \to L_2([0, T]; C^1(\overline{\Omega})^n \cap V)$$

generated by γ_δ is continuous. The construction of S_δ is given in [12].

Let us substitute the equation (2.1) by the equation

$$z(\tau;t,x) = x + \int_t^\tau S_\delta v(s, z(s;t,x))\,ds, \quad \tau, t \in [0,T],\ x \in \Omega. \tag{2.2}$$

For each $v \in L_2([0,T];V)$ this equation has a unique solution $Z_\delta(v)$ in the class CG, i.e.

$$z(\tau;t,x) = Z_\delta(v)(\tau;t,x).$$

In the sequel we will use the following notations for functional spaces:

$$E = L_2([0,T];V), \quad E^* = L_2([0,T];V^*), \quad E_1^* = L_1([0,T];V^*).$$

We introduce a feedback control in the system, realized by the choice of the external force from the set $U(v) \subset E^*$. As a domain of the feedback multimap U we consider the space $W_1 \subset C([0,T];V^*)$

$$W_1 = \{v : v \in E,\ v' \in E_1^*\}$$

endowed with the norm $\|v\|_{W_1} = \|v\|_E + \|v'\|_{E_1^*}$.

It will be assumed that the feedback multimap U satisfies the following conditions:

(U1) U is upper semicontinuous and takes values in the collection $Kv(E^*)$ of all convex, compact subsets of E^*,

(U2) U is globally bounded, i.e. there exists a constant $M > 0$ such that

$$\|U(v)\|_{E^*} := \sup\{\|u\|_{E^*} : u \in U(v)\} \le M$$

for all $v \in W_1$,

(U3) the value $U(D)$ of every bounded set $D \subset W_1$ is a relatively compact set in E^*,

(U4) U is weakly closed in the following sense: if $\{v_l\}_{l=1}^\infty \subset W_1$, $v_l \rightharpoonup v_0$, $u_l \in U(v_l)$ and $u_l \xrightarrow{E^*} u_0$, then $u_0 \in U(v_0)$.

Remark 2.1. Condition (U_4) is fulfilled when the feedback multimap U satisfies the following "convexity" condition:

$$U(\lambda v_0 + (1-\lambda)v_1) \supseteq \lambda U(v_0) + (1-\lambda)U(v_1)$$

for every $v_0, v_1 \in W_1$, $0 \le \lambda \le 1$.

In fact, from Mazur's lemma (see e.g. [3]) it follows that there exists a double sequence of nonnegative numbers $\{\alpha_{ik}\}_{i,k=1}^\infty$ such that:

(a) $\sum_{k=1}^\infty \alpha_{ik} = 1$ for all $i = 1, 2, \ldots$,

(b) for every $i = 1, 2, \ldots$ there exists a number $k_0(i)$ such that $\alpha_{ik} = 0$ for all $k > k_0(i)$,

(c) the sequence $\widetilde{v}_i = \sum_{k=1}^\infty \alpha_{ik} v_k$ strongly converges to v_0.

Then,

$$\widetilde{u}_i = \sum_{k=1}^{\infty} \alpha_{ik} u_k \in \sum_{k=1}^{\infty} \alpha_{ik} U(v_k) \subseteq U(\widetilde{v}_i).$$

Since obviously $\widetilde{u}_i \to u_0$ and the multimap U is closed ([8], Theorem 1.1.4) we obtain the desired $u_0 \in U(v_0)$.

Under above conditions the controlled motion of the fluid can be described in the form of the following regularized problem (cf. [11])

$$\rho\biggl(\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i}\biggr) - \mu_1 \mathrm{Div} \int_0^t e^{-(t-s)/\lambda} \mathcal{E}(v)(s, Z_\delta(v)(s;t,x))\, ds \\ - \mu_0 \mathrm{Div}\, \mathcal{E}(v) + \mathrm{grad}\, p \in \rho U(v), \quad (t,x) \in [0,T] \times \Omega, \tag{2.3}$$

$$\mathrm{div}\, v = 0, \quad (t,x) \in [0,T] \times \Omega, \tag{2.4}$$

$$v_{|[0,T]\times\Omega} = 0, \tag{2.5}$$

$$v(0,x) = v^0(x), \quad x \in \Omega, \tag{2.6}$$

$$\int_\Omega p\, dx = 0, \tag{2.7}$$

where μ_0, μ_1 are constants ($\mu_0 > 0$).

Remark 2.2. Taking $\mu_1 = 0$ we obtain the control system governed by the classical Navier–Stokes equation.

Denote by (f, v) the action of the functional f from V^* on the function $v \in V$.

Let us introduce the following operators:

(1) $A\colon V \to V^*$,

$$(A(v), h) = \mu_0 (\mathcal{E}(v), \mathcal{E}(h))_{L_2(\Omega)}, \quad v, h \in V,$$

(2) $K\colon V \to V^*$,

$$(K(v), h) = \sum_{i,j=1}^{n} \rho\biggl(v_i v_j, \frac{\partial h_i}{\partial x_j}\biggr)_{L_2(\Omega)}, \quad v, h \in V,$$

(3) for $v \in E$ and $z \in CG$ the functional $C(v,z)$ on V for every fixed $t \in [0,T]$ can be given by the formula

$$(C(v,z)(t), h) = \mu_1 \biggl(\int_0^t e^{-(t-s)/\lambda} \mathcal{E}(v)(s, z(s;t,x))\, ds, \mathcal{E}(h)\biggr)_{L_2(\Omega)^n}.$$

In the sequel we will identify $H \equiv H^*$. Then taking into account the embeddings $V \subseteq H \equiv H^* \subseteq V^*$, the action of an element $v \in E$ on $h \in V$ for almost all $t \in [0,T]$ will be determined by the equality

$$(v(t), h) = (v(t), h)_{L_2(\Omega)^n}.$$

So we have the representation

$$\frac{d}{dt}(v(t),h)_{L_2(\Omega)^n} = \frac{d}{dt}(v(t),h) = (v'(t),h)$$

where $v'(t)$ is considered as a locally integrable function with values in V^*.

Definition 2.3. Given $v^0 \in H$, by *weak solution* of the regularized problem (2.3)–(2.7) we mean a function $v \in E$ with derivative $v' \in E_1^*$ satisfying the relations

$$\rho v' + A(v) - K(v) + C(v, Z_\delta(v)) \in U(v), \tag{2.8}$$

$$v(0) = v^0. \tag{2.9}$$

It is clear that each weak solution belongs to the space W_1.

Denote $Q_T = [0,T] \times \Omega$.

Definition 2.4. If v is a weak solution of the problem (2.3)–(2.7) then the control $u \in U(v)$ satisfying the relation

$$\rho v' + A(v) - K(v) + C(v, Z_\delta(v)) = u \tag{2.10}$$

is said to be the *control corresponding to* v. The pair (v,u) satisfying (2.10) will be called an *admissible solution* of the regularized control problem (2.8)–(2.9) and hence of the problem (2.3)–(2.7).

We will consider the following optimization problem: to find an admissible solution $(\overline{v},\overline{u})$ minimizing a given convex, lower semicontinuous cost functional $J: E \times E^* \to \mathbb{R}$ (see Section 5 below).

3. The approximating problem

To find an admissible solution of problem (2.3)–(2.7) we construct first approximating inclusions substituting the operator K by operators $K_\varepsilon, \varepsilon > 0$ in such a way that all the members of the inclusion (2.8) will belong to the same space E^*.

For a given $\varepsilon > 0$, define the operator $K_\varepsilon: V \to V^*$ by the formula

$$(K_\varepsilon(v),h) = \rho \bigg(\sum_{i,j=1}^n \frac{v_i v_j}{1+\varepsilon\|v\|^2}, \frac{\partial h_i}{\partial x_j} \bigg)_{L_2(\Omega)}$$

and consider the approximating problem

$$\rho v' + A(v) - K_\varepsilon(v) + C(v, Z_\delta(v)) \in U(v), \tag{3.1}$$

$$v(0) = v^0. \tag{3.2}$$

in the space $W = \{v : v \in E,\ v' \in E^*\}$ with norm $\|v\|_W = \|v\|_E + \|v'\|_{E^*}$.

It is known (see [9], [10]) that W is a Banach space and $W \subset C([0,T];H)$.

Introduce the maps $L, G, N_\varepsilon: W \to E^* \times H$ and the multimap $\widetilde{U}: W \to Kv(E^* \times H)$ in the following way:

$$\begin{aligned} L(v) &= (\rho v' + A(v), v(0, \cdot)), \\ G(v) &= (C(v, Z_\delta(v)), 0), \\ N_\varepsilon(v) &= (K_\varepsilon(v), 0), \\ \widetilde{U}(v) &= (U(v), v^0). \end{aligned}$$

Then, problem (3.1)–(3.2) can be written in the form of the following operator inclusion:

$$L(v) - N_\varepsilon(v) + G(v) \in \widetilde{U}(v) \tag{3.3}$$

Our first goal is to prove the following existence result.

Theorem 3.1. *For every $\varepsilon > 0$ and $v^0 \in H$ inclusion* (3.3) *and hence problem* (3.1)–(3.2) *have a solution $(v_\varepsilon, u_\varepsilon) \in W \times E^*$.*

To prove Theorem 3.1 let us describe some properties of operators involved in the inclusion (3.3).

As it was mentioned above, $W \subset E \cap C([0,T]; H)$. For functions $v \in E \cap C([0,T]; H)$ let us introduce the norm

$$\|v\|_{EC} = \max_{0 \le t \le T} \|v(t)\|_H + \|v\|_E$$

and the equivalent norms

$$\|v\|_{k,EC} = \|\overline{v}\|_{EC}$$

where $\overline{v}(t) = e^{-kt} \cdot v(t)$, $k \ge 0$. Equivalent norms $\|\cdot\|_{k,E}$, $\|\cdot\|_{k,E^* \times H}$, $\|\cdot\|_{k,L_2(Q_T)}$ are defined similarly.

Proposition 3.2 ([2], [12]).

(a) *The linear operator $L: W \to E^* \times H$ is bounded and invertible, and for any couple of functions $v, w \in W$ the following estimate holds:*

$$\|v - w\|_{k,EC} \le C_1 \|L(v) - L(w)\|_{k,E^* \times H}$$

for every $k \ge 0$, where the constant C_1 does not depend on v, w and k.

(b) *The map $N_\varepsilon: W \to E^* \times H$ is completely continuous for $\varepsilon > 0$ and for any $v \in W$ the following estimate holds*

$$\|N_\varepsilon(v)\|_{E^* \times H} \le \frac{C_2}{\varepsilon}.$$

(c) *$C(v, z) \in E^*$ for every $v \in E$, $z \in CG$ and the map $C: E \times CG \to E^*$ is continuous and bounded. Moreover, for every $k > 0$ the following* *estimate holds:*

$$\|C(v,z) - C(w,z)\|_{k,E^*} \le \mu_1 \sqrt{\frac{T}{2k}} \|v-w\|_{k,E}. \tag{3.4}$$

(d) *The map* $Z_\delta : W_1 \to CG$ *is completely continuous.*

From (a) of the above statement it follows that solving inclusion (3.3) is equivalent to to the studying of of the fixed point problem in the space W

$$v \in F_\varepsilon(v) \tag{3.5}$$

where the multimap $F_\varepsilon : W \to Kv(W)$ is defined as

$$F_\varepsilon(v) = L^{-1}N_\varepsilon(v) - L^{-1}G(v) + L^{-1}\widetilde{U}(v). \tag{3.6}$$

It follows from basic properties of multivalued maps (see, e.g. [8]) that F_ε is an upper semicontinuous multimap with convex, compact values. We will show that this multimap is condensing with respect to the Hausdorff measure of noncompactness (MNC) in W (see [8]).

Let χ_k be the Hausdorff MNC in W generated by the norm $\|\cdot\|_{k,E}$.

Proposition 3.3. *The multimap* F_ε $(\varepsilon > 0)$ *is* χ_k*-condensing provided* k *is sufficienty large.*

Proof. Let $D \subset W$ be an arbitrary bounded set. Consider the map $\overline{G} : W \times W \to W$ defined as

$$\overline{G}(v,w) = L^{-1}(C(v, Z_\delta(w)), 0).$$

From Proposition 3.2(a), (c), (d) it follows that for every fixed $v \in W$ the set $\overline{G}(v, D)$ is relatively compact. Further, for every fixed $z \in Z_\delta(D)$ the map $C(\,\cdot\,, z)$ is $\mu_1\sqrt{T/2k}$-Lipschitz with respect to the norms $\|\cdot\|_{k,E}$ and $\|\cdot\|_{k,E^*}$, hence $(C(\,\cdot\,,z), 0)$ is Lipschitz with the same constant with respect to $\|\cdot\|_{k,E}$ and $\|\cdot\|_{k,E^*\times H}$. Then from Proposition 3.2(a) it follows that $\overline{G}(\cdot, w)$ is $C_1\mu_1\sqrt{T/2k}$-Lipschitz with respect to norms $\|\cdot\|_{k,E}$ and $\|\cdot\|_{k,EC}$. Since, for any $v \in W$ it is

$$\|v\|_{k,E} \le \|v\|_{k,EC}$$

we obtain that

$$\overline{G}(\,\cdot\,, w) : W \to W$$

is $C_1\mu_1\sqrt{T/2k}$-Lipschitz with respect to the $\|\cdot\|_{k,E}$ norm.

Now, choosing $k > 0$ so that

$$C_1\mu_1\sqrt{\frac{T}{2k}} < 1$$

and applying Proposition 2.2.2 in [8] we obtain that the map $G(v) = \overline{G}(v,v)$ is χ_k-condensing.

From Proposition 3.2(b) and (a) it follows that the set $L^{-1}N_\varepsilon(D)$ is relatively compact and property (U3) implies the same for the set $L^{-1}\widetilde{U}(D)$. Applying the well known properties of the Hausdorff MNC (see e.g. [8]) we conclude the proof. □

Consider now the one-parameter family

$$\rho v' + A(v) - \lambda K_\varepsilon(v) + \lambda C(v, Z_\delta(v)) \in \lambda U(v), \quad \lambda \in [0,1] \tag{3.7}$$

including the approximation problem (3.1) for $\lambda = 1$.

We are going to obtain an apriori estimate for the solutions of this family.

Proposition 3.4. *For any solution $v \in W$ of the initial problem* (3.7), (3.2) *the following estimates hold:*

$$\|v\|_{EC} \le C_3(1 + M + \|v^0\|_H), \tag{3.8}$$

$$\|v'\|_{E^*} \le C_4(1 + M + \|v^0\|_H), \tag{3.9}$$

where M is the constant in condition (U2) *and the constants C_3 and C_4 depend on ε.*

Proof. Let $v \in W$ be any solution of problem (3.7), (3.2). Then

$$L(v) \in \lambda(N_\varepsilon(v) - G(v) + \widetilde{U}(v)), \quad \lambda \in [0,1]. \tag{3.10}$$

Since $L(0) = 0$, from Proposition 3.2(a) it follows that

$$\|v\|_{k,EC} \le C_1 \|L(v)\|_{k,E^* \times H}. \tag{3.11}$$

Analogously, $C(0, Z_\delta(v)) = 0$ and from (3.4) we have that

$$\|C(v, Z_\delta(v))\|_{k,E^*} \le \mu_1 \sqrt{\frac{T}{2k}} \|v\|_{k,E}. \tag{3.12}$$

From (3.10), (3.12), Proposition 3.2(b) and property (U3), applying the estimate (3.11) we obtain

$$\begin{aligned} \|v\|_{k,EC} &\le C_1 \bigg(\frac{C_2}{\varepsilon} + \mu_1 \sqrt{\frac{T}{2k}} \|v\|_{k,E} + M + \|v^0\|_H \bigg) \\ &\le C_1 \bigg(\frac{C_2}{\varepsilon} + \mu_1 \sqrt{\frac{T}{2k}} \|v\|_{k,EC} + M + \|v^0\|_H \bigg) \end{aligned}$$

i.e.

$$\bigg(1 - C_1 \mu_1 \sqrt{\frac{T}{2k}}\bigg) \|v\|_{k,EC} \le C_1 \bigg(\frac{C_2}{\varepsilon} + M + \|v^0\|_H \bigg).$$

Choosing k sufficiently large and taking into account the equivalence of norms $\|\cdot\|_{k,EC}$ and $\|\cdot\|_{EC}$ we get the estimate (3.8). To obtain the estimate (3.9) it is sufficient to express explicitely v' from inclusion (3.7):

$$v' \in -\frac{1}{\rho}(A(v) - \lambda K_\varepsilon(v) + \lambda C(v, Z_\delta(v))) + \frac{\lambda}{\rho} U(v), \tag{3.13}$$

to note that all operators in the right hand side of (3.13) are bounded in E and apply estimate (3.8). □

Proof of Theorem 3.1. From Proposition 3.3 it follows that for every $\varepsilon > 0$ there exists a ball $B_R \subset W$ centered at the origin and of a sufficiently large radious R such that

$$v \notin \lambda F_\varepsilon(v) \quad \text{for all } v \in \partial B_R,\ \lambda \in [0,1].$$

Applying the topological degree theory to the condensing family of multifields $i - \lambda F_\varepsilon$ (see [8, Chapter 3]) we obtain that

$$\deg(i - F_\varepsilon, \partial B_R) = \deg(i, \partial B_R) = 1.$$

This result implies the existence of a fixed point of the multimap F_ε in the ball B_R and hence the solvability of the approximating problem (3.1)–(3.2). □

4. The regularizated problem

In this section we show that for $\varepsilon \to 0$ the solutions of the approximating problems are converging, in the sense of distributions, to a solution of the regularized problem (2.8)–(2.9). It is known from [10] that $W_1 \subset E \cap L_\infty([0,T]; H)$. For functions $v \in E \cap L_\infty([0,T]; H)$ introduce the norm

$$\|v\|_{EL} = \|v\|_E + \|v\|_{L_\infty([0,T];H)}$$

and the equivalent norms

$$\|v\|_{k,EL} = \|\overline{v}\|_{EL}, \quad \text{where } \overline{v}(t) = e^{-kt} v(t),\ k \geq 0.$$

Proposition 4.1. *For every solution $v_\varepsilon \in W_1$ of the approximating problem* ((3.1)–(3.2) *with $\varepsilon > 0$ the following estimates hold*

$$\|v_\varepsilon\|_{EL} \leq C_5(M + \|v^0\|_H), \tag{4.1}$$

$$\|v_\varepsilon'\|_{E_1^*} \leq C_6(M + \|v^0\|_H)^2, \tag{4.2}$$

where constants C_5 and C_6 are independent from ε.

Proof. To prove (4.1) let us make the substitution

$$v_\varepsilon(t) = e^{kt}\overline{v}_\varepsilon(t)$$

and multiply the approximating equation

$$\rho v_\varepsilon' + A(v_\varepsilon) - K_\varepsilon(v_\varepsilon) + C(v_\varepsilon, Z_\delta(v_\varepsilon)) = u_\varepsilon \tag{4.3}$$

by e^{-kt}. Then we obtain

$$\rho \overline{v}_\varepsilon' + \rho \overline{v}_\varepsilon + A(\overline{v}_\varepsilon) - \overline{K}_\varepsilon(\overline{v}_\varepsilon) + \overline{C}(\overline{v}, Z_\delta(e^{kt}\overline{v}_\varepsilon)) = \overline{u}_\varepsilon \tag{4.4}$$

where

$$\begin{aligned}\overline{K}_\varepsilon(\overline{v}_\varepsilon) &= e^{-kt}K_\varepsilon(e^{kt}\overline{v}_\varepsilon),\\ \overline{C}(\overline{v}_\varepsilon, Z_\delta(e^{kt}\overline{v}_\varepsilon)) &= e^{-kt}C(e^{kt}\overline{v}_\varepsilon, Z_\delta(e^{kt}\overline{v}_\varepsilon)),\\ \overline{u}_\varepsilon &= e^{-kt}u_\varepsilon,\end{aligned}$$

and the functional $\rho k\overline{v}_\varepsilon$ is defined by the equality $(\rho k\overline{v}_\varepsilon, h) = \rho k(\overline{v}_\varepsilon, h)_{L_2(\Omega)}$ for $u \in V$.

Let us consider the action of functionals in the left and right hand side of the equality (4.4) on the function $\overline{v}_\varepsilon$:

$$\begin{aligned}\frac{1}{2}\rho\frac{d}{dt}\|\overline{v}_\varepsilon(t)\|_H^2 + \rho k\|\overline{v}_\varepsilon(t)\|_H^2 + (A(\overline{v}_\varepsilon(t)), \overline{v}_\varepsilon(t))\\ - (\overline{K}_\varepsilon(\overline{v}_\varepsilon(t)), \overline{v}_\varepsilon(t)) + (\overline{C}(\overline{v}_\varepsilon, Z_\delta(e^{kt}\overline{v}_\varepsilon))(t), \overline{v}_\varepsilon(t)) = (\overline{u}_\varepsilon(t), \overline{v}_\varepsilon(t)).\end{aligned} \tag{4.5}$$

It is known from [2] that $(\overline{K}_\varepsilon(\overline{v}_\varepsilon(t)), \overline{v}_\varepsilon(t)) = 0$ for all $t \in [0,T]$. So, integrating by part (4.5) in t on the interval $[0,t]$ we obtain

$$\begin{aligned}\frac{1}{2}\rho\|\overline{v}_\varepsilon(t)\|_H^2 + \rho k\|\overline{v}_\varepsilon\|^2_{L_2([0,t];H)} + \mu_0\|\overline{v}_\varepsilon\|^2_{L_2([0,t];V)} - \frac{1}{2}\rho\|\overline{v}^0\|_H^2\\ + \int_0^t(\overline{C}(\overline{v}_\varepsilon, Z_\delta(e^{k\tau}\overline{v}_\varepsilon))(\tau), \overline{v}_\varepsilon(\tau))\,d\tau = \int_0^t(\overline{u}_\varepsilon(\tau), \overline{v}_\varepsilon(\tau))\,d\tau.\end{aligned}$$

Taking $w = 0$ in the inequality (3.4) and applying the Cauchy inequality we can write

$$\begin{aligned}\frac{1}{2}\rho\|\overline{v}_\varepsilon(t)\|_H^2 + \rho k\|\overline{v}_\varepsilon\|^2_{L_2([0,T];H)} + \mu_0\|\overline{v}_\varepsilon\|^2_{L_2([0,T];V)}\\ - \frac{1}{2}\rho\|\overline{v}^0\|_H^2 + \mu_1\sqrt{\frac{T}{2k}}\|\overline{v}_\varepsilon\|_E^2 \le \|\overline{u}_\varepsilon\|_{E^*}\|\overline{v}_\varepsilon\|_E\,.\end{aligned}$$

Taking k sufficiently large so that $\mu_1\sqrt{T/(2k)} < \mu_0/2$ and using again the Cauchy inequality we arrive to

$$\rho\|\overline{v}_\varepsilon\|^2_{L_\infty([0,T];H)} + 2\rho k\|\overline{v}_\varepsilon\|^2_{L_2(Q_T)} + \mu_0\|\overline{v}_\varepsilon\|_E^2 - \rho\|v^0\|_H^2 - \frac{1}{2}\mu_0\|\overline{v}_\varepsilon\|_E^2 \le \frac{2}{\mu_0}\|\overline{u}_\varepsilon\|^2_{E^*}.$$

From the above inequality we get

$$\frac{1}{2}\rho\|\overline{v}_\varepsilon\|^2_{L_\infty([0,T];H)} + 2\rho k\|\overline{v}_\varepsilon\|^2_{L_2(Q_T)} + \frac{1}{2}\mu_0\|\overline{v}_\varepsilon\|_E^2 \le \rho\|v^0\|_H^2 + \frac{2}{\mu_0}\|\overline{u}_\varepsilon\|^2_{E^*}$$

that gives the required estimate (4.1).

To obtain (4.2) let us express, as earlier, the derivative v'_ε from (4.3):

$$v'_\varepsilon = -\frac{1}{\rho}(A(v_\varepsilon) - K_\varepsilon(v_\varepsilon) + C(v_\varepsilon, Z_\delta(v_\varepsilon)) - u_\varepsilon).$$

It follows that

$$\|v'_\varepsilon\|_{E_1^*} \le C_7(\|A(v_\varepsilon)\|_{E^*} + \|K_\varepsilon(v_\varepsilon)\|_{E_1^*} + \|C(v_\varepsilon, Z_\delta(v_\varepsilon))\|_{E^*} + \|u_\varepsilon\|_{E^*}).$$

In the paper [12] the following estimate is obtained:

$$\|K_\varepsilon(v_\varepsilon)\|_{E_1^*} \le C_8 \|v_\varepsilon\|^2_{L_2([0,T];L_4(\Omega)^n)} \le C_9 \|v_\varepsilon\|^2_E. \tag{4.6}$$

This estimate, the boundedness of operators A and C on E and the estimate (4.1) imply (4.2). □

We are now in position to formulate the main result of this section.

Theorem 4.2. *The regularized problem* (2.8)–(2.9) *admits an admissible solution* $(\widetilde{v}, \widetilde{u}) \in W_1 \times E^*$.

Proof. Let us take any sequence of positive numbers $\varepsilon_l \to 0$. From Theorem 3.1 we know that every corresponding approximating problem (3.1)–(3.2) has a solution $(v_l, u_l) \in W_1 \times E^*$, i.e.

$$\rho v_l' + A(v_l) - K_{\varepsilon_l}(v_l) + C(v_l, Z_\delta(v_l)) = u_l \in U(v_l) \tag{4.7}$$

From the estimates (4.1), (4.2) it follows that the sequence $\{v_l\}$ is bounded with respect to the norm $\|\cdot\|_{EL}$ while the sequence $\{v_l'\}$ is bounded with respect to the norm $\|\cdot\|_{E_1^*}$. Then, without loss of generality, we can assume that

$$\begin{aligned} v_l &\rightharpoonup \widetilde{v} && \text{weakly in } E, \\ v_l &\rightharpoonup \widetilde{v} && *\text{-weakly in } L_\infty([0,T];H), \\ v_l &\to \widetilde{v} && \text{strongly in } L_2(Q_T)^n, \\ v_l' &\rightharpoonup v' && \text{in the sense of distributions.} \end{aligned}$$

From conditions (U3), (U4) it follows that we can assume, without loss of generality, that

$$u_l \to \widetilde{u} \in U(\widetilde{v}) \quad \text{strongly on } E^*.$$

Since a bounded linear operator is weakly continuous, we can assume, without loss of generality, that

$$\begin{aligned} A(v_l) &\rightharpoonup A(\widetilde{v}) && \text{weakly in } E^*, \\ \mathcal{E}(v_l)(s,x) &\rightharpoonup \mathcal{E}(\widetilde{v})(s,x) && \text{weakly in } L_2(Q_T)^n. \end{aligned}$$

Following [12] it can be proved that

$$C(v_l, Z_\delta(v_l)) \rightharpoonup C(\widetilde{v}, Z_\delta(\widetilde{v})) \quad \text{weakly in } E^*,$$

and from Lemma 2.2 of [2] it is known that

$$K_{\varepsilon_l}(v_l) \rightharpoonup K(\widetilde{v}) \quad \text{in the sense of distributions.}$$

Now, it remains only to pass to the limit in the sense of distributions in relation (4.7) while $\varepsilon_l \to 0$ to obtain

$$\rho \widetilde{v}' + A(\widetilde{v}) - K(\widetilde{v}) + C(\widetilde{v}, Z_\delta(\widetilde{v})) = \widetilde{u} \in U(\widetilde{v}).$$

Note that $\widetilde{v} \in E$ implies that $\widetilde{v}' \in E_1^*$ and hence $\widetilde{v} \in W_1$. The pair $(\widetilde{v}, \widetilde{u})$ is the required one. □

5. The optimization problem

In this section we consider the problem of existence of an optimal admissible solution $(\overline{v}, \overline{u})$.

We suppose that the given convex cost functional $J: E \times E^* \to \mathbb{R}$ satisfies the following conditions:

(J1) J is lower semicontinuous in the sense that, given a sequence $\{v_l, u_l\} \subset E \times E^*$, conditions $\|v_l - v_0\|_E \to 0$, $\|u_l - u_0\|_{E^*} \to 0$ imply

$$J(v_0, u_0) \leq \underline{\lim}_{l \to \infty} J(v_l, u_l),$$

(J2) J is bounded below, i.e. there exists a constant L such that

$$-\infty < L \leq J(v, u) \leq \infty \quad \text{for all } (v, u) \in E \times E^*.$$

Let us denote by $\Sigma \subset W_1 \times E^*$ the set of all admissible solutions of the regularized control problem (2.8)–(2.9). Our goal is to solve the following optimization problem:

(P) To find an admissible solution $(\overline{v}, \overline{u})$ of (2.8)–(2.9) such that

$$J(\overline{v}, \overline{u}) = \inf_{(v,u) \in \Sigma} J(v, u).$$

THEOREM 5.1. *Under conditions* (J1), (J2) *problem* (P) *has a solution.*

PROOF. From Theorem 4.2 we know that $\Sigma \neq \emptyset$, therefore there exists a minimizing sequence $(v_l, u_l) \in \Sigma$ such that

$$\lim_{l \to \infty} J(v_l, u_l) = \inf_{(v,u) \in \Sigma} J(v, u).$$

Notice that the sequence $\{v_l\}$ is bounded in W_1. In fact, the estimate

$$\|K(v)\|_{E_1^*} \leq C\|v\|_E^2,$$

similar to (4.6) holds for the operator K and we can repeat the same line of reasoning as in the proof of Proposition 4.1. Then, we can assume, as earlier, that without loss of generality,

$$\begin{array}{ll}
v_l \rightharpoonup \overline{v} & \text{weakly in } E, \\
v_l \rightharpoonup \overline{v} & *\text{-weakly in } L_\infty(0, T; H), \\
v_l \to \overline{v} & \text{strongly in } L_2(Q_T), \\
v_l' \rightharpoonup v' & \text{in the sense of distributions}, \\
u_l \to \overline{u} \in U(\overline{v}) & \text{strongly in } E^*.
\end{array}$$

We have also the convergences

$$\begin{aligned} A(v_l) &\rightharpoonup A(\overline{v}) && \text{weakly in } E^*, \\ \mathcal{E}(v_l)(s,x) &\rightharpoonup \mathcal{E}(\overline{v})(s,x) && \text{weakly in } L_2(Q_T)^{n^2}, \\ C(v_l, Z_\delta(v_l)) &\rightharpoonup C(\overline{v}, Z_\delta(\overline{v})) && \text{weakly in } E^*, \\ K(v_l) &\rightharpoonup K(\overline{v}) && \text{in the sense of distributions.} \end{aligned}$$

Now, passing to the limit in the relation

$$\rho v_l' + A(v_l) - K(v_l) + C(v_l, Z_\delta(v_l)) = u_l \in U(v_l)$$

we obtain that $(\overline{v}, \overline{u}) \in \Sigma$.

Since the functional J is lower semicontinuous also with respect to the weak topology, we have that

$$J(\overline{v}, \overline{u}) \leq \inf_{(v,u) \in \Sigma} J(v,u)$$

implying that $(\overline{v}, \overline{u})$ is the desired solution of problem (P). □

As an example of J satisfying (J1) and (J2) we can consider the integral functional of the form

$$J(v,u) = \int_0^T \varphi(v(t,\,\cdot\,), u(t,\,\cdot\,))\, dt$$

where $\varphi: V \times V^* \to \mathbb{R}$ is a given convex, continuous, bounded from below functional.

To present a more particular situation, consider the problem of damping of a viscoelastic fluid supposing that the external forces are chosen in the form of combinations of coercions created by a finite number of sources depending on the state of the system. More precisely, let continuous maps $f_i: W_1 \to E^*$, $i = 1, \ldots, m$ satisfy the following conditions:

- (f1) each f_i is globally bounded and transforms bounded sets into relatively compact ones,
- (f2) each f_i is weakly closed in the sense that $\{v_l\}_{l=1}^\infty \subset W_1$, $v_l \rightharpoonup v_0$, $f_i(v_l) \to u_0$ imply $u_0 = f_i(v_0)$.

Define the feedback multimap $U: W_1 \to Kv(E^*)$ as

$$U(v) = \bigg\{ u = \sum_{i=1}^m \lambda_i f_i(v) : \sum_{i=1}^m \lambda_i = 1 \bigg\}.$$

It is easy to verify that U satisfies conditions (U1)–(U4) and hence we may conclude, by Theorem 11, that there exists an admissible solution $(\overline{v}, \overline{u})$ of problem (2.8)–(2.9) minimizing the functional

$$J(u,v) = \int_0^T (\|v(s,\,\cdot\,)\|_V^2 + \|u(s,\,\cdot\,)\|_{V^*}^2)\, ds.$$

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Manuscript received December 31, 2002

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