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# LARGE TIME REGULAR SOLUTIONS TO THE NAVIER–STOKES EQUATIONS IN CYLINDRICAL DOMAINS

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Abstract. We prove the large time existence of solutions to the Navier–Stokes equations with slip boundary conditions in a cylindrical domain. Assuming smallness of $L_2$-norms of derivatives of initial velocity with respect to variable along the axis of the cylinder, we are able to obtain estimate for velocity in $W_2^{2,1}$ without restriction on its magnitude. Then existence follows from the Leray–Schauder fixed point theorem.

## 1. Introduction

We consider the following initial-boundary value problem

$$
\begin{aligned}
&v_t + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) = f && \text{in } \Omega^T = \Omega\times(0,T),\\
&\operatorname{div} v = 0 && \text{in } \Omega^T,\\
&v\cdot\overline{n} = 0 && \text{on } S^T = S\times(0,T),\\
&\overline{n}\cdot\mathbb{T}(v,p)\cdot\overline{\tau}_\alpha = 0,\quad \alpha=1,2, && \text{on } S^T,\\
&v|_{t=0} = v(0) && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where $\Omega\subset\mathbb{R}^3$ is a cylindrical domain, $S=\partial\Omega$, $v$ is the velocity of the fluid motion with $v(x,t)=(v_1(x,t),v_2(x,t),v_3(x,t))\in\mathbb{R}^3$, $p=p(x,t)\in\mathbb{R}^1$ denotes

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the pressure, $f = f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ — the external force field, $\overline{n}$ is the unit outward vector normal to the boundary $S$ and $\overline{\tau}_\alpha$, $\alpha = 1,2$ are tangent vectors to $S$ and $\cdot$ denotes the scalar product in $\mathbb{R}^3$.

We define the stress tensor $\mathbb{T}(v,p)$ as

$$\mathbb{T}(v,p) = \nu\mathbb{D}(v) - p\mathbb{I},$$

where $\nu$ is the constant viscosity coefficient, $\mathbb{I}$ — the unit matrix and $\mathbb{D}(v)$ is the dilatation tensor of the form

$$\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3}.$$

By $x = (x_1, x_2, x_3)$ we denote the Cartesian coordinates. $\Omega \subset \mathbb{R}^3$ is a cylindrical type domain parallel to the axis $x_3$ with arbitrary cross section. We assume that $S = S_1 \cup S_2$ where $S_1$ is the part of the boundary which is parallel to the axis $x_3$ and $S_2$ is perpendicular to $x_3$. Hence

$$S_1 = \{x \in \mathbb{R}^3 : \varphi_0(x_1,x_2) = c_0,\ -a < x_3 < a\},$$

and

$$S_2 = \{x \in \mathbb{R}^3 : \varphi_0(x_1,x_2) < c_0,\ x_3 \text{ is equal to either } -a \text{ or } a\},$$

where $a$, $c_0$ are positive given numbers and $\varphi_0(x_1,x_2) = c_0$ describes a sufficiently smooth closed curve in the plane $x_3 = \text{const}$.

Let us denote $g = f_{,x_3}$, $h = v_{,x_3}$, $\chi = (\operatorname{rot} v)_3$ and define

$$\begin{aligned}
K_1 &= \|f_3\|_{L_2(\Omega^t)} + \|g\|_{L_2(\Omega^t)} + \|F_3\|_{L_2(0,T;L_{6/5}(\Omega))} \\
&\quad + \|h(0)\|_{L_2(\Omega)} + \|\chi(0)\|_{L_2(\Omega)}, \\
K_2 &= K_1 + d_1 + d_2 + \|f\|_{L_2(\Omega^T)} + \|v(0)\|_{H^1(\Omega)}, \\
K_3 &= \|g\|_{L_\sigma(\Omega^T)} + \|h(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)}, \\
d(T) &= \|g\|_{L_2(\Omega^T)} + \|f_3\|_{L_2(S_2^T)} + \|h(0)\|_{L_2(\Omega)},
\end{aligned}$$

where $d_1$, $d_2$ are introduced in lemma 2.3.

We prove the following result:

Theorem 1.1.

(a) *Let* $f \in L_\infty(0,T;L_{6/5}(\Omega)) \cap L_2(\Omega^T)$, $f_3 \in L_2(S_2^T)$, $F_3 = (\operatorname{rot} f)_3 \in L_2(0,T;L_{6/5}(\Omega))$, $g = f_{,x_3} \in L_2(\Omega^T) \cap L_\sigma(\Omega^T)$, $\sigma > 5/3$.

(b) *Assume that* $v(0)$, $h(0) = v_{,x_3}(0)$, $\chi(0) = (\operatorname{rot} v)_3(0)$ *belong to* $L_2(\Omega)$, *and* $v(0) \in H^1(\Omega)$, $h(0) \in W_\sigma^{2-2/\sigma}(\Omega)$, $20/7 < \sigma \le 10/3$.

*Then there exists a solution to problem* (1.1) *such that* $v \in W_2^{2,1}(\Omega^T)$, $\nabla p \in L_2(\Omega^T)$. *Moreover, if* $q = p_{,x_3}$ *and* $5/3 < \sigma < 3$,

$$\begin{aligned}
&\|h\|_{W_\sigma^{2,1}(\Omega^T)} + \|\nabla q\|_{L_\sigma(\Omega^T)} < A, \\
&\|v\|_{W_2^{2,1}(\Omega^T)} + \|\nabla p\|_{L_2(\Omega^T)} \le \varphi(A, K_2),
\end{aligned} \tag{1.2}$$

*where $A$ is a constant chosen for a given $T$ so that, for an increasing function $\varphi$, sufficiently small constant $d(T)$ and some constants $K_i$ involving the above norms,*

$$\varphi(A,K_2)d(T)+cK_3\le A \quad \text{and} \quad A>cK_3,$$

*where an absolute constant $c$ depends on imbedding only.*

The main goal of the paper is to simplify the proof of [5]. Namely, the result of [5] is generalized by weakening its assumptions. In [5], the existence of solutions to problem (1.1) has been proved in Besov spaces. Therefore, we needed much more complicated techniques and estimates, i.e. the solvability of the Stokes problem in Besov spaces and also different imbeddings and interpolation in Besov spaces.

## 2. Preliminaries

This part of the paper is devoted to the results that have been previously shown in [5]. For the convenience of the reader, we quote them, splitting the considerations into propositions on basic estimates on the weak solutions and then examining some useful quantities.

**2.1. Notation.** The following function spaces will be used in the sequel:

- isotropic and anisotropic Lebesgue spaces:

$$\begin{aligned} &L_p(Q), && Q\in\{\Omega^T,S^T,\Omega,S\}, && p\in[1,\infty],\\ &L_q(0,T;L_p(Q)), && Q\in\{\Omega,S\}, && p,q\in[1,\infty]; \end{aligned}$$

- Sobolev spaces:

$$W_q^{s,s/2}(Q^T),\quad Q\in\{\Omega,S\},\quad s\in\mathbb{Z}_+\cup\{0\},\ q\in[1,\infty],$$

with the norm

$$\|u\|_{W_q^{s,s/2}(Q^T)}=\bigg(\sum_{|\alpha|+2a\le s}\int_{Q^T}|D_x^\alpha\partial_t^a u|^q\,dx\,dt\bigg)^{1/q},$$

where

$$D_x^\alpha=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3},\quad |\alpha|=\alpha_1+\alpha_2+\alpha_3,\quad a,\alpha_i\in\mathbb{Z}_+\cup\{0\}.$$

In the special case $q=2$,

$$H^s(Q)=W_2^s(Q),\quad Q\in\{\Omega,S\},\quad s\in\mathbb{Z}_+\cup\{0\},\quad q\in[1,\infty]$$

with the norm

$$\|u\|_{H^s(Q)}=\bigg(\sum_{|\alpha|\le s}\int_Q|D_x^\alpha u|^2\,dx\bigg)^{1/2}.$$

We define a space natural for the study of the weak solutions to the Navier–Stokes equations:

$$V_2^k(\Omega^T) = \left\{ u : ||u||_{V_2^k(\Omega^T)} = \operatorname{ess}\sup_{t\in(0,T)} ||u||_{H^k(\Omega)} + \left( \int_0^T ||\nabla u||^2_{H^k(\Omega)}\, dt \right)^{1/2} < \infty \right\}$$

with $k \in \mathbb{N}$ and $L_2$ replacing $H^0$ in definition of $V_2^0$.

### 2.2. Weak solutions.

Definition 2.1. By a weak solution to problem (1.1) we mean $v \in V_2^0(\Omega^T)$ such that $\operatorname{div} v = 0$, $v \cdot \overline{n}|_S = 0$, satisfying the integral identity

$$\int_{\Omega^T} (-v \cdot \varphi_{,t} + \nu \mathbb{D}(v) \cdot \mathbb{D}(\varphi) + v \cdot \nabla v \cdot \varphi)\, dx\, dt + \int_\Omega v \cdot \varphi|_{t=T}\, dx - \int_\Omega v(0) \cdot \varphi|_{t=0}\, dx = \int_{\Omega^T} f \cdot \varphi\, dx\, dt,$$

which holds for any $\varphi \in W_2^{1,1}(\Omega^T)$ such that $\operatorname{div}\varphi = 0$, $\varphi \cdot \overline{n}\big|_S = 0$.

For the weak solutions we have the Korn inequality.

Lemma 2.2. *Assume that*

$$E_\Omega(v) = |\mathbb{D}(v)|^2_{L_2(\Omega)} < \infty, \quad v \cdot \overline{n}|_S = 0, \quad \operatorname{div} v = 0.$$

*Assume that $\Omega$ is not axially symmetric. Then there exists a constant $c_1$ such that*

$$\|v\|^2_{H^1(\Omega)} \le c_1 E_\Omega(v).$$

*If $\Omega$ is axially symmetric, $\eta = (-x_2, x_1, 0)$, $\alpha = \int_\Omega v \cdot \eta$, then there exists a constant $c_2$ such that*

$$\|v\|^2_{H^1(\Omega)} \le c_2(E_\Omega(v) + |\alpha|^2).$$

Now we formulate energy type estimates for weak solutions of (1.1).

Lemma 2.3 (see [4]). *Let $f \in L_\infty(0,\infty; L_{6/5}(\Omega))$, $\int_{\Omega^t} f \cdot \eta\, dx\, dt' \in L_\infty(0,\infty)$, $v(0) \in L_2(\Omega)$. Let $T > 0$ be given. Assume that there exist constants $a_1$, $a_2$ such that*

$$a_1 \equiv \sup_t |f(t)|_{L_{6/5}(\Omega)} < \infty, \quad a_2 \equiv \sup_t \left| \int_{\Omega^t} f \cdot \eta\, dx\, dt' \right| < \infty.$$

*Then there exist constants*

$$\begin{aligned} d_1^2 &= \frac{c}{\nu_1}a_1^2 + |v(0)|^2_{L_2(\Omega)},\\ d_2^2 &= (\min(1,\nu_2))^{-1}e^{\nu_1 T}\left(\frac{c}{\nu_1}a_1^2 + d_1^2\right),\\ d_3^2 &= \frac{c}{\nu_1}(a_1^2 + a_2^2 + |\alpha^2(0)|) + |v(0)|^2_{L_2(\Omega)},\\ d_4^2 &= (\min(1,\nu_2))^{-1}e^{\nu_1 T}\left[\frac{c}{\nu_1}(a_1^2 + a_2^2 + |\alpha^2(0)|) + d_3^2\right], \end{aligned}$$

*which do not depend on* $k_0 = kT$, $k \in \mathbb{N}$, *and* $\nu = \nu_1 + \nu_2$ *such that in the non-axially symmetric case we have*

$$\begin{aligned} |v(t)|_{L_2(\Omega)} &\le d_1 \quad \text{for any } t \ge 0,\\ \|v\|_{V_2^0(\Omega\times(kT,t))} &\le d_2 \quad \text{for } t \in (kT,(k+1)T),\ k \in \mathbb{N}, \end{aligned} \tag{2.1}$$

*and in the axially symmetric case*

$$\begin{aligned} |v(t)|_{L_2(\Omega)} &\le d_3 \quad \text{for any } t \ge 0,\\ \|v\|_{V_2^0(\Omega\times(kT,t))} &\le d_4 \quad \text{for } t \in (kT,(k+1)T),\ k \in \mathbb{N}. \end{aligned}$$

From the above lemma by an application of the Galerkin method and the considerations from [2, Chapter 6] we have

LEMMA 2.4. *Let the assumptions of Lemma* 2.3 *hold. Then there exists a weak solution to problem* (1.1) *in any interval* $(kT,(k+1)T)$, $k \in \mathbb{N}$, *satisfying*

$$\|v\|_{V_2^0(\Omega\times(kT,(k+1)T)} \le d_i,$$

*where* $i = 2$ *for non-axially symmetric and* $i = 4$ *for axially symmetric domain.*

**2.3. Auxiliary problems.** We note that in the paper the non-axially symmetric case is examined. We distinguish the direction $x_3$. In order to derive estimates for derivatives in direction $x_3$ we introduce the quantities

$$h = v_{,x_3}, \quad q = p_{,x_3}, \quad g = f_{,x_3}.$$

These functions are solutions to the problems that we list in the section.

LEMMA 2.5 (see [5]). *The pair of functions* $(h,q)$ *is a solution to the problem*

$$\begin{aligned} & h_{,t} - \operatorname{div}\mathbb{T}(h,q) = -v\cdot\nabla h - h\cdot\nabla v + g && \text{in } \Omega^T,\\ & \operatorname{div} h = 0 && \text{in } \Omega^T,\\ & h\cdot\overline{n} = 0, \quad \overline{n}\cdot\mathbb{D}(h)\cdot\overline{\tau}_\alpha = 0, \quad \alpha = 1,2 && \text{on } S_1^T,\\ & h_i = 0, \quad i = 1,2, \quad h_{3,x_3} = 0 && \text{on } S_2^T,\\ & h|_{t=0} = h(0) && \text{in } \Omega. \end{aligned} \tag{2.2}$$

We will use the following estimates for $h$ obtained in [5] and [6]:

Lemma 2.6. *Assume that $v$ is a weak solution to problem* (1.1) *satisfying* (2.1). *Assume that $h \in L_\infty(0,T;L_3(\Omega))$, $g \in L_2(\Omega^T)$, $f_3 \in L_2(S_2^T)$, $h(0) \in L_2(\Omega)$. Then*

$$\|h(t)\|^2_{V_2^0(\Omega^t)} \le cd_2^2\|h\|^2_{L_\infty(0,t;L_3(\Omega))} + c(|f_3|^2_{L_2(S_2^t)} + |g|^2_{L_2(\Omega^t)} + |h(0)|^2_{L_2(\Omega)}), \tag{2.3}$$

*where $t \le T$.*

Lemma 2.7. *With $g$, $f_3$, $h(0)$ as in the previous lemma and $\nabla v \in L_2(0,t;L_3(\Omega))$, for the weak solution to* (1.1)

$$\|h(t)\|_{L_2(\Omega)} \le c\exp(c\|\nabla v\|^2_{L_2(0,t;L_3(\Omega))})[\|g\|_{L_2(\Omega^t)} + \|f_3\|_{L_2(S_2^t)} + \|h(0)\|_{L_2(\Omega)}],$$

*for $t \le T$, and*

$$\begin{aligned}\|h\|_{L_2(\Omega^t)} \le c[\|\nabla v\|_{L_2(0,t;L_3(\Omega))}\exp(c\|\nabla v\|^2_{L_2(0,t;L_3(\Omega))}) + 1]\\ \cdot [\|g\|_{L_2(\Omega^t)} + \|f_3\|_{L_2(S_2^t)} + \|h(0)\|_{L_2(\Omega)}],\end{aligned} \tag{2.4}$$

*for $t \le T$, hold.*

Lemma 2.8. *Let $q$ and $f_3$ be given. Then $w = v_3$ is a solution to the problem*

$$\begin{aligned} &w_{,t} + v\cdot\nabla w - \nu\Delta w = q + f_3 && \text{in } \Omega^T,\\ &w_{,n} = 0 && \text{on } S_1^T,\\ &w = 0 && \text{on } S_2^T,\\ &w|_{t=0} = w(0) && \text{in } \Omega,\end{aligned}$$

*where $\partial_n = \overline{n}\cdot\nabla$ and $\overline{n}$ is the normal vector to $S_1$.*

Lemma 2.9. *Let $F_3 = (\operatorname{rot} f)_3$, $h$, $v$ and $w$ be given. Then $\chi = (\operatorname{rot} v)_3$ is a solution to the problem*

$$\begin{aligned} &\chi_{,t} + v\cdot\nabla\chi - h_3\chi + h_2 w_{,x_1} - h_1 w_{,x_2} - \nu\Delta\chi = F_3 && \text{in } \Omega^T,\\ &\chi = v_i(n_{i,x_j}\tau_{1j} + \tau_{1i,x_j}n_j) + v\cdot\overline{\tau}_1(\tau_{12,x_1} - \tau_{11,x_2}) \equiv \chi_* && \text{on } S_1^T,\\ &\chi_{,x_3} = 0 && \text{on } S_2^T,\\ &\chi|_{t=0} = \chi(0) && \text{in } \Omega,\end{aligned} \tag{2.5}$$

*where tangent and normal vectors to $S_1$ are defined as follows*

$$\begin{aligned} \overline{n}|_{S_1} &= \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{1}{|\nabla\varphi|}(\varphi_{,x_1}, \varphi_{,x_2}, 0),\\ \overline{\tau}_1|_{S_1} &= \frac{\nabla^\perp\varphi}{|\nabla\varphi|} = \frac{1}{|\nabla\varphi|}(-\varphi_{,x_2}, \varphi_{,x_1}, 0), \quad \overline{\tau}_2|_{S_1} = (0,0,1),\\ \overline{n}|_{S_2} &= (0,0,1), \qquad \overline{\tau}_1|_{S_2} = (1,0,0), \quad \overline{\tau}_2|_{S_2} = (0,1,0).\end{aligned}$$

## 3. Estimates

Let us introduce the function $\widetilde{\chi}$ as a solution to the problem

$$\begin{aligned} &\widetilde{\chi}_{,t} - \nu\Delta\widetilde{\chi} = 0 && \text{in } \Omega^T, \\ &\widetilde{\chi} = \chi_* && \text{on } S_1^T, \\ &\widetilde{\chi}_{,x_3} = 0 && \text{on } S_2^T, \\ &\widetilde{\chi}|_{t=0} = 0 && \text{in } \Omega. \end{aligned}$$

Then the new function $\chi' = \chi - \widetilde{\chi}$, is a solution to the following problem

$$\tag{3.1} \begin{aligned} &\chi'_{,t} + v\cdot\nabla\chi' - h_3\chi' + h_2 w_{,x_1} - h_1 w_{,x_2} - \nu\Delta\chi' \\ &\qquad\qquad = F_3 - v\cdot\nabla\widetilde{\chi} + h_3\widetilde{\chi} && \text{in } \Omega^T, \\ &\chi' = 0 && \text{on } S_1^T, \\ &\chi'_{,x_3} = 0 && \text{on } S_2^T, \\ &\chi'|_{t=0} = \chi(0) && \text{in } \Omega. \end{aligned}$$

LEMMA 3.1. *Assume that* $h \in L_\infty(0,t;L_3(\Omega)), \chi(0) \in L_2(\Omega)$, $v' = (v_1, v_2) \in L_\infty(0,t;W^1_{9/5}(\Omega)\cap H^{1/2+\varepsilon}(\Omega)) \cap W_2^{1,1/2}(\Omega^t)$, $F_3 \in L_2(0,t;L_{6/5}(\Omega))$. *Let the assumptions of Lemma* 2.3 *be satisfied. Then solutions of problem* (2.5) *satisfy*

$$\tag{3.2} \begin{aligned} \|\chi\|^2_{V_2^0(\Omega^t)} \le{}& cd_2^2(\|h\|^2_{L_\infty(0,t;L_3(\Omega))} + \|v'\|^2_{L_\infty(0,t;W^1_{9/5}(\Omega))}) \\ &+ c(\|F_3\|^2_{L_2(0,t:L_{6/5}(\Omega))} + \|\chi(0)\|^2_{L_2(\Omega)} \\ &+ \|v'\|^2_{L_\infty(0,t;H^{1/2+\varepsilon}(\Omega))} + \|v'\|^2_{W_2^{1,1/2}(\Omega^t)}), \end{aligned}$$

*for* $t \le T$, *where* $\varepsilon > 0$.

PROOF. Multiplying $(3.1)_1$ by $\chi'$, integrating over $\Omega$, using the boundary conditions $(3.1)_{2,3}$ and $(1.1)_3$ we obtain

$$\tag{3.3} \begin{aligned} \frac{1}{2}\frac{d}{dt}\|\chi'\|^2_{L_2(\Omega)} + \nu\|\nabla\chi'\|^2_{L_2(\Omega)} ={}& \int_\Omega h_3\chi'^2\,dx - \int_\Omega (h_2 w_{,x_1} - h_1 w_{,x_2})\chi'\,dx \\ &+ \int_\Omega F_3\chi'\,dx - \int_\Omega v\cdot\nabla\widetilde{\chi}\chi'\,dx + \int_\Omega h_3\widetilde{\chi}\chi'\,dx. \end{aligned}$$

Now we estimate the terms on the r.h.s. of the above inequality. The first term can be bounded by

$$\begin{aligned} \int_\Omega h_3\chi'^2\,dx &\le \varepsilon_1\|\chi'\|^2_{L_6(\Omega)} + \frac{c}{\varepsilon_1}\|\chi'\|^2_{L_2(\Omega)}\|h\|^2_{L_3(\Omega)} \\ &\le \varepsilon_1\|\chi'\|^2_{L_6(\Omega)} + \frac{c}{\varepsilon_1}(\|\chi\|^2_{L_2(\Omega)} + \|\widetilde{\chi}\|^2_{L_2(\Omega)})\|h\|^2_{L_3(\Omega)}. \end{aligned}$$

The second term on the r.h.s. of (3.3) can be estimated by

$$\frac{\varepsilon_2}{2}\|\chi'\|^2_{L_6(\Omega)} + \frac{1}{2\varepsilon_2}\|h\|^2_{L_3(\Omega)}\|w_{,x'}\|^2_{L_2(\Omega)},$$

the third by:

$$\frac{\varepsilon_3}{2}\|\chi'\|^2_{L_6(\Omega)} + \frac{1}{2\varepsilon_3}\|F_3\|^2_{L_{6/5}(\Omega)},$$

and the fourth we express in the form

$$\int_\Omega v\cdot\nabla\chi'\widetilde{\chi}\,dx,$$

and estimate as follows

$$\frac{\varepsilon_4}{2}\|\nabla\chi'\|^2_{L_2(\Omega)} + \frac{1}{2\varepsilon_4}\int_\Omega v^2|\widetilde{\chi}|^2\,dx \le \frac{\varepsilon_4}{2}\|\nabla\chi'\|^2_{L_2(\Omega)} + \frac{1}{2\varepsilon_4}\|v\|^2_{L_6(\Omega)}\|\widetilde{\chi}\|^2_{L_3(\Omega)}.$$

Finally, the last term on the r.h.s. of (3.3) can be bounded by

$$\frac{\varepsilon_5}{2}\|\chi'\|^2_{L_6(\Omega)} + \frac{1}{2\varepsilon_5}\|h\|^2_{L_{12/7}(\Omega)}\|\widetilde{\chi}\|^2_{L_4(\Omega)}.$$

Using the above estimates in (3.3), assuming that $\varepsilon_1,\ldots,\varepsilon_5$ are sufficiently small and integrating the result with respect to time we obtain

$$\begin{aligned}\|\chi'(t)\|^2_{L_2(\Omega)} &+ \nu\int_0^t\|\nabla\chi'(t')\|^2_{L_2(\Omega)}\,dt' \le c\int_0^t dt'\|\chi'\|^2_{L_2(\Omega)}\sup_t\|h\|^2_{L_3(\Omega)}\\ &+ c\sup_t\|h\|^2_{L_3(\Omega)}\int_0^t\|w_{,x'}\|^2_{L_2(\Omega)}\,dt' + c\sup_t\|\widetilde{\chi}\|^2_{L_3(\Omega)}\int_0^t\|v(t')\|^2_{L_6(\Omega)}\,dt'\\ &+ c\sup_t\|h(t)\|^2_{L_{12/7}(\Omega)}\int_0^t\|\widetilde{\chi}\|^2_{L_4(\Omega)}\,dt' + c\|F_3\|^2_{L_2(0,t;L_{6/5}(\Omega))} + \|\chi(0)\|^2_{L_2(\Omega)}.\end{aligned}$$

Now, applying the energy estimate (2.1) we have

$$\begin{aligned}\|\chi'\|^2_{V_2^0(\Omega^T)} \le{}& c\sup_t\|h\|^2_{L_3(\Omega)}\int_0^t\|\widetilde{\chi}\|^2_{L_2(\Omega)}\\ &+ cd_2^2\big(\sup_t\|h(t)\|^2_{L_3(\Omega)} + \sup_t\|\widetilde{\chi}(t)\|^2_{L_3(\Omega)}\big)\\ &+ c\sup_t\|h(t)\|^2_{L_{12/7}(\Omega)}\int_0^t\|\widetilde{\chi}\|^2_{L_4(\Omega)}\,dt'\\ &+ c\|F_3\|^2_{L_2(0,t;L_{6/5}(\Omega))} + \|\chi(0)\|^2_{L_2(\Omega)}.\end{aligned}\tag{3.4}$$

Next, we will use the following relations

$$\begin{aligned}\int_0^t\|\widetilde{\chi}\|^2_{L_4(\Omega)}\,dt' &\le c\int_0^t\|v'\|^2_{L_4(S_1)}\,dt' \le c\int_0^t\|v'\|^2_{H^1(\Omega)}\,dt' \le cd_2^2,\\ \|\widetilde{\chi}\|_{L_\infty(0,t;L_3(\Omega))} &\le c\|v'\|_{L_\infty(0,t;L_3(S_1))} \le c\|v'\|_{L_\infty(0,t;W^1_{9/5}(\Omega))},\\ \int_0^t\|\widetilde{\chi}\|^2_{L_2(\Omega)}\,dt' &\le c\int_0^t\|v\|^2_{W^1_2(\Omega)}\,dt' \le cd_2^2,\end{aligned}$$

and the transformation $\chi' = \chi - \widetilde{\chi}$ to obtain from (3.4) the inequality

$$\begin{aligned}\|\chi\|^2_{V_2^0(\Omega^T)} \le{}& cd_2^2(\|h\|^2_{L_\infty(0,t;L_3(\Omega))} + \|v'\|^2_{L_\infty(0,t;W^1_{9/5}(\Omega))})\\ &+ c\|F_3\|^2_{L_2(0,t;L_{6/5}(\Omega))} + \|\chi(0)\|^2_{L_2(\Omega)} + \|\widetilde{\chi}\|^2_{V_2^0(\Omega^t)},\end{aligned}\tag{3.5}$$

where

$$\|\widetilde{\chi}\|^2_{V_2^0(\Omega^t)} \le \|\widetilde{\chi}\|^2_{L_\infty(0,t;L_2(\Omega))} + \int_0^t \|\widetilde{\chi}\|^2_{H^1(\Omega)}\,dt' \\ \le c\|v'\|^2_{L_\infty(0,t;H^{1/2+\varepsilon})} + c\|v'\|^2_{W_2^{1,1/2}(\Omega^t)}.$$

Therefore, we obtain from (3.5) the inequality (3.2). This concludes the proof.□

Let us consider the problem

$$\begin{aligned} & v_{1,x_2} - v_{2,x_1} = \chi && \text{in } \Omega', \\ & v_{1,x_1} + v_{2,x_2} = -h_3 && \text{in } \Omega', \\ & v' \cdot \overline{n}' = 0 && \text{on } S_1', \end{aligned} \tag{3.6}$$

where $\Omega' = \Omega \cap \{\text{plane} : x_3 = \text{const} \in (-a,a)\}$, $S_1' = S_1 \cap \{\text{plane } x_3 = \text{const} \in (-a,a)\}$, and $x_3$, $t$ are treated as parameters.

Lemma 3.2. *Let the assumptions of Lemmas* 3.1 *and* 2.6 *be satisfied. Let*

$$K_1(t) = \|f_3\|_{L_2(\Omega^t)} + \|g\|_{L_2(\Omega^t)} + \|F_3\|_{L_2(0,t;L_{6/5}(\Omega))} \\ + \|h(0)\|_{L_2(\Omega)} + \|\chi(0)\|_{L_2(\Omega)}.$$

*Then*

$$\|v'\|^2_{V_2^1(\Omega^t)} \le c\|v'\|^2_{L_2(\Omega;H^{1/2}(0,T))} + cd_2^2\|h\|^2_{L_\infty(0,t;L_3(\Omega))} + cK_1^2 + c(d_1^2+d_2^2). \tag{3.7}$$

Proof. In view of (3.2) and (2.3) we obtain for solutions to problem (3.6) the estimate

$$\|v'\|^2_{V_2^1(\Omega^t)} \le c(d_2^2\|v'\|^2_{L_\infty(0,t;W^1_{9/5}(\Omega))} + \|v'\|^2_{L_\infty(0,t;H^{1/2+\varepsilon}(\Omega))} \\ + \|v'\|^2_{W_2^{1,1/2}(\Omega^t)}) + cd_2^2\|h\|^2_{L_\infty(0,t;L_3(\Omega))} + cK_1^2. \tag{3.8}$$

By interpolation inequalities we have

$$\begin{aligned} \|v'\|^2_{L_\infty(0,t;H^{1/2+\varepsilon}(\Omega))} &\le \varepsilon\|v'\|^2_{L_\infty(0,t;H^1(\Omega))} + c(1/\varepsilon)d_1^2, \\ \|v'\|^2_{L_2(0,t;H^1(\Omega))} &\le \varepsilon\|v'\|^2_{L_2(0,t;H^2(\Omega))} + c(1/\varepsilon)d_2^2 \end{aligned} \tag{3.9}$$

and

$$\|v'\|_{L_\infty(0,t;L_3(s_1))} \le \varepsilon\|v'\|_{L_\infty(0,t;H^1(\Omega))} + c(1/\varepsilon)d_2. \tag{3.10}$$

Assuming that $\varepsilon$ is sufficiently small we obtain from (3.8)–(3.10) the inequality (3.7). This concludes the proof. □

Let us consider problem (1.1) in the form

$$\begin{aligned} & v_{,t} - \operatorname{div}\mathbb{T}(v,p) = -v'\cdot\nabla v - wh + f && \text{in } \Omega^T, \\ & \operatorname{div} v = 0 && \text{in } \Omega^T, \\ & v\cdot\overline{n} = 0, \quad \overline{n}\cdot\mathbb{T}(v,p)\cdot\overline{\tau}_\alpha = 0, \quad \alpha = 1,2 && \text{on } S^T, \\ & v|_{t=0} = v(0) && \text{in } \Omega. \end{aligned} \tag{3.11}$$

Lemma 3.3. *Let the assumptions of Lemmas* 3.1, 3.2 *and* 2.6 *be satisfied. Let* $h \in L_{10/3}(\Omega^T)$, $f \in L_2(\Omega^T)$, $v(0) \in H^1(\Omega)$. *Then for solutions of* (3.11) *we obtain the inequality*

$$\|v\|_{W_2^{2,1}(\Omega^t)} + \|\nabla p\|_{L_2(\Omega^t)} \le c(d_2 H_1 + K_2)^2 + c(\|f\|_{L_2(\Omega^t)} + \|v(0)\|_{H^1(\Omega)}), \tag{3.12}$$

*for* $t \le T$, *where* $K_2$ *and* $H_1$ *are defined by* (3.15)–(3.16) *below.*

Proof. In view of [6, Lemma 3.7] we have that

$$\|v'\|_{L_{10}(\Omega^T)} \le c\|v'\|_{V_2^1(\Omega^T)}.$$

Hence

$$\begin{aligned} \|v'\nabla v\|_{L_{5/3}(\Omega^T)} &\le \|v'\|_{L_{10}(\Omega^T)}\|\nabla v\|_{L_2(\Omega^T)} \le d_2\|v'\|_{L_{10}(\Omega^T)} \le cd_2\|v'\|_{V_2^1(\Omega^T)}, \\ \|wh\|_{L_{5/3}(\Omega^T)} &\le \|w\|_{L_{10/3}(\Omega^T)}\|h\|_{L_{10/3}(\Omega^T)} \le cd_2\|h\|_{L_{10/3}(\Omega^T)}. \end{aligned}$$

Summarizing the above estimates we have

$$\begin{aligned} \|v\|_{W_{5/3}^{2,1}(\Omega^T)} \le cd_2(\|v'\|_{V_2^1(\Omega^T)} + \|h\|_{L_{10/3}(\Omega^T)}) \\ + c(\|f\|_{L_{5/3}(\Omega^T)} + \|v(0)\|_{W_{5/3}^{4/5}(\Omega)}). \end{aligned} \tag{3.13}$$

Applying (3.7) in (3.13) and using the interpolation

$$\|v'\|_{L_2(\Omega;H^{1/2}(0,T))} \le \varepsilon\|v'\|_{W_{5/3}^{2,1}(\Omega^T)} + c(1/\varepsilon)d_2,$$

we obtain

$$\|v\|_{W_{5/3}^{2,1}(\Omega^T)} \le cd_2(\|h\|_{L_\infty(0,T;L_3(\Omega))} + \|h\|_{L_{10/3}(\Omega^T)}) + cK_2, \tag{3.14}$$

where

$$K_2 = K_1 + d_1 + d_2 + \|f\|_{L_2(\Omega^T)} + \|v(0)\|_{H^1(\Omega)}. \tag{3.15}$$

Let

$$H_1 = \|h\|_{L_\infty(0,t;L_3(\Omega))} + \|h\|_{L_{10/3}(\Omega^t)}. \tag{3.16}$$

Then (3.14) and (3.7) take the form

$$\|v\|_{W_{5/3}^{2,1}(\Omega^T)} + \|v'\|_{V_2^1(\Omega^t)} \le c(d_2 H_1 + K_2)^2$$

since

$$\begin{aligned} \|v'\nabla v\|_{L_2(\Omega^T)} &\le \|v'\|_{L_{10}(\Omega^T)}\|\nabla v\|_{L_{5/2}(\Omega^T)} \\ &\le \|v'\|_{V_2^1(\Omega^T)}\|v\|_{W_{5/3}^{2,1}(\Omega^T)} \le c(d_2 H_1 + K_2)^2, \\ \|wh\|_{L_2(\Omega^T)} &\le \|w\|_{L_5(\Omega^T)}\|h\|_{L_{10/3}(\Omega^T)} \\ &\le c\|v\|_{W_{5/3}^{2,1}(\Omega^T)}\|h\|_{L_{10/3}(\Omega^T)} \le c(d_2 H_1 + K_2)H_1. \end{aligned}$$

In view of the above estimates we obtain for solutions to problem (3.11) the inequality (3.12). This concludes the proof. □

Let us consider now the problem (2.2).

LEMMA 3.4. *Assume that* $v \in W_2^{2,1}(\Omega^T))$, $h \in L_2(\Omega^T)$, $g \in L_\sigma(\Omega^T)$ *and* $h(0) \in W_\sigma^{2-2/\sigma}(\Omega)$. *Then for solutions of problem* (2.2) *the following inequality holds*

$$\begin{aligned}(3.17)\quad \|h\|_{W_\sigma^{2,1}(\Omega^T)} + \|\nabla q\|_{L_\sigma(\Omega^T)} &\le \varphi(\|v\|_{W_2^{2,1}(\Omega^T)})\|h\|_{L_2(\Omega^T)} \\ &\quad + c(\|g\|_{L_\sigma(\Omega^T)} + \|h(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)}),\end{aligned}$$

*with* $\varphi(a) = ca^4$.

PROOF. For solutions of problem (2.2) we have the inequality

$$\begin{aligned}(3.18)\quad \|h\|_{W_\sigma^{2,1}(\Omega^T)} + \|\nabla q\|_{L_\sigma(\Omega^T)} &\le c(\|v\nabla h\|_{L_\sigma(\Omega^T)} + \|h\cdot\nabla v\|_{L_\sigma(\Omega^T)} \\ &\quad + \|g\|_{L_\sigma(\Omega^T)} + \|h(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)}).\end{aligned}$$

Let us use the interpolation results

$$\begin{aligned}\|v\nabla h\|_{L_\sigma(\Omega^T)} &\le \|v\|_{L_{\sigma\lambda_1}(\Omega^T)}\|\nabla h\|_{L_{\sigma\lambda_2}(\Omega^T)} \\ &\le \|v\|_{L_{10}(\Omega^T)}(\varepsilon_1^{1-\kappa_1}\|h\|_{W_\sigma^{2,1}(\Omega^T)} + c\varepsilon_1^{-\kappa_1}\|h\|_{L_2(\Omega^T)}) \equiv I_1,\end{aligned}$$

where

$$\kappa_1 = \left(\frac{5}{\sigma} - \frac{5}{\sigma\lambda_2} + 1\right)\frac{1}{2} = \left(\frac{5}{\sigma\lambda_1} + 1\right)\frac{1}{2} = \frac{3}{4} \quad \text{because } \sigma\lambda_1 = 10.$$

Hence

$$I_1 \le \varepsilon_2^{1/4}\|h\|_{W_\sigma^{2,1}(\Omega^T)} + c\varepsilon_2^{-3/4}\|v\|_{L_{10}(\Omega^T)}^4\|h\|_{L_2(\Omega^T)}.$$

Similarly

$$\begin{aligned}\|h\nabla v\|_{L_\sigma(\Omega^T)} &\le \|h\|_{L_{\sigma\lambda_1}(\Omega^T)}\|\nabla v\|_{L_{\sigma\lambda_2}(\Omega^T)} \\ &\le \|\nabla v\|_{L_{10/3}(\Omega^T)}(\varepsilon_3^{1-\kappa_2}\|h\|_{W_\sigma^{2,1}(\Omega^T)} + c\varepsilon_3^{-\kappa_2}\|h\|_{L_2(\Omega^T)}) \equiv I_2,\end{aligned}$$

where

$$\kappa_2 = \left(\frac{5}{\sigma} - \frac{5}{\sigma\lambda_1}\right)\frac{1}{2} = \frac{5}{2\sigma\lambda_2} = \frac{3}{4} \quad \text{because } \sigma\lambda_2 = \frac{10}{3}.$$

Hence

$$I_2 \le \varepsilon_4^{1/4}\|h\|_{W_\sigma^{2,1}(\Omega^T)} + c\varepsilon_4^{-3/4}\|\nabla v\|_{L_{10/3}(\Omega^T)}^4\|h\|_{L_2(\Omega^T)}$$

holds. In view of the above estimates we obtain from (3.18) the inequality (3.17). This concludes the proof. □

LEMMA 3.5. *With the assumptions of the Lemma 3.4, for $5/3 < \sigma < 3$, there exists a sufficiently large constant $A$ such that*

$$\|h\|_{W_\sigma^{2,1}(\Omega^T)} + \|\nabla q\|_{L_\sigma(\Omega^T)} \le A. \tag{3.19}$$

PROOF. Since

$$\|\nabla v\|_{L_2(0,T;L_3(\Omega))} \le c\|v\|_{W_2^{2,1}(\Omega^T)}$$

and by imbedding

$$H_1 \le c\|h\|_{W_\sigma^{2,1}(\Omega^T)} \quad \text{for } \sigma > \frac{5}{3}, \tag{3.20}$$

we obtain from the inequalities (2.4), (3.12), (3.17) and (3.20)

$$\|h\|_{W_\sigma^{2,1}(\Omega^T)} + \|\nabla q\|_{L_\sigma(\Omega^T)} \le \varphi(\|h\|_{W_\sigma^{2,1}(\Omega^T)}, K_2)d(T) + cK_3,$$

where

$$\begin{aligned} d(T) &= \|g\|_{L_2(\Omega^T)} + \|f_3\|_{L_2(S_2^T)} + \|h(0)\|_{L_2(\Omega)}, \\ K_3 &= \|g\|_{L_\sigma(\Omega^T)} + \|h(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)}. \end{aligned}$$

For sufficiently small $d(T)$ there exists a constant $A$ such that

$$\varphi(A, K_2)d(T) + cK_3 \le A \quad \text{and} \quad A > cK_3.$$

Hence the estimate (3.19) holds. □

## 4. Existence

To prove the existence of solutions we consider the problem

$$\begin{aligned} & h_t - \operatorname{div} \mathbb{T}(h,q) = -\lambda[v(\widetilde{h},\overline{v}) \cdot \nabla\widetilde{h} + \widetilde{h} \cdot \nabla v(\widetilde{h},\overline{v})] + g && \text{in } \Omega^T, \\ & \operatorname{div} h = 0 && \text{in } \Omega^T, \\ & h \cdot \overline{n} = 0, \quad \overline{n} \cdot \mathbb{D}(h) \cdot \overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_1^T, \\ & h_i = 0, \quad i = 1,2, \quad h_{3,x_3} = 0 && \text{on } S_2^T, \\ & h|_{t=0} = h(0) && \text{in } \Omega, \end{aligned} \tag{4.1}$$

where $\lambda \in [0,1]$. Let $\mathfrak{M}(\Omega^T) = \{h : \|h\|_{L_\infty(0,T;W_\eta^1(\Omega))} < \infty\}$.

The problem (4.1) implies the mapping $\Phi: \mathfrak{M}(\Omega^T) \to W_\sigma^{2,1}(\Omega^T) \hookrightarrow \mathfrak{M}(\Omega^T)$ where the last imbedding and so the mapping $\Phi$ is compact for $20/7 < \sigma < 10/3$, $\eta > 4$. We show the continuity of the mapping $\Phi$.

LEMMA 4.1. *The mapping $\Phi$ is uniformly continuous in the product $\mathfrak{M}(\Omega^T) \times [0,1]$ where $\mathfrak{M}(\Omega^T)$ is defined as above and $20/7 < \sigma \le 10/3$, $\eta > 4$.*

PROOF. Uniform continuity with respect to $\lambda \in [0,1]$ is evident. Therefore we examine the uniform continuity with respect to elements of $\mathfrak{M}(\Omega^T)$ for any

$\lambda \in [0,1]$. Since dependence on $\lambda$ is very simple we omit $\lambda$ in the considerations below because it does not change the proof.

To have compact $\Phi$ we need compactness of imbedding

$$\text{if}\quad W^{2,1}_\sigma(\Omega^T) \hookrightarrow L_\infty(0,T;W^1_\eta(\Omega)) \quad\text{then}\quad \frac{5}{\sigma}-\frac{3}{\eta}-\frac{2}{\infty}<1,\quad \sigma<\eta.$$

Let $\widetilde{h}_s \in \mathfrak{M}(\Omega^T)$, $s=1,2$, $i=1,2$, be two elements. Therefore, we consider the following problems

$$\begin{aligned}
& h_{s,t} - \operatorname{div}\mathbb{T}(h_s,q_s) = -v_s\cdot\nabla\widetilde{h}_s - \widetilde{h}_s\cdot\nabla v_s + g && \text{in } \Omega^T,\\
& \operatorname{div} h_s = 0 && \text{in } \Omega^T,\\
& h_s\cdot\overline{n} = 0,\quad \overline{n}\cdot\mathbb{D}(h_s)\cdot\overline{\tau}_\alpha = 0,\quad \alpha=1,2 && \text{on } S_1^T,\\
& h_{si} = 0,\quad i=1,2,\quad h_{s3,x_3} = 0 && \text{on } S_2^T,\\
& h_s|_{t=0} = h(0) && \text{in } \Omega,
\end{aligned}\tag{4.2}$$

where $s=1,2$;

$$\begin{aligned}
& \chi_{s,t} + v_s\cdot\nabla\chi_s - \widetilde{h}_{s3}\chi_s + \widetilde{h}_{s2}w_{s,x_1} - \widetilde{h}_{s1}w_{s,x_2} - \nu\Delta\chi_s = F_3 && \text{in } \Omega^T,\\
& \chi_s = \chi_{s*} && \text{on } S_1^T,\\
& \chi_s = 0 && \text{on } S_2^T,\\
& \chi_s|_{t=0} = \chi(0) && \text{on } \Omega,
\end{aligned}$$

where $s=1,2$, and $\chi_{s*}$ is defined as in (2.5);

$$\begin{aligned}
& v_{s2,x_1} - v_{s1,x_2} = \chi_s && \text{in } \Omega',\\
& v_{s1,x_1} + v_{s2,x_2} = -h_{s3} && \text{in } \Omega',\\
& v_s'\cdot\overline{n}' = 0 && \text{on } S_1',
\end{aligned}$$

where $s=1,2$, $\Omega'$ nad $S_1'$ are cross-sections of $\Omega$ and $S_1$ with a plane perpendicular to axis $x_3$.

First we examine the problem on $\chi$. Let us introduce the function $\widetilde{\chi}_s$ as a solution to the problem

$$\begin{aligned}
& \widetilde{\chi}_{s,t} - \nu\Delta\widetilde{\chi}_s = 0 && \text{in } \Omega^T,\\
& \widetilde{\chi}_s = \chi_{s*} && \text{on } S_1^T,\\
& \widetilde{\chi}_{s,x_3} = 0 && \text{on } S_2^T,\\
& \widetilde{\chi}_s|_{t=0} = 0 && \text{in } \Omega,
\end{aligned}$$

where $s = 1, 2$. Introducing the new function $\chi'_s = \chi_s - \widetilde{\chi}_s$, $s = 1, 2$, we see that it is a solution to the problem

$$\begin{aligned} &\chi'_{s,t} + v_s \cdot \nabla \chi'_s - \widetilde{h}_{s3}\chi'_s + \widetilde{h}_{s2} w_{s,x_1} - \widetilde{h}_{s1} w_{s,x_2} \\ &\qquad\qquad - \nu\Delta\chi'_s = F_3 - v_s \cdot \nabla\widetilde{\chi}_s + \widetilde{h}_{s3}\widetilde{\chi}_s && \text{in } \Omega^T, \\ &\chi'_s = 0 && \text{on } S_1^T, \\ &\chi'_{s,x_3} = 0 && \text{on } S_2^T, \\ &\chi'_s|_{t=0} = \chi_s(0) && \text{in } \Omega. \end{aligned}$$

The problem for $v_s$ reads:

$$\begin{aligned} &v_{s,t} - \operatorname{div} \mathbb{T}(v_s, p_s) = -v'_s \cdot \nabla' v_s - w_s\widetilde{h}_s + f && \text{in } \Omega^T, \\ &\operatorname{div} v_s = 0 && \text{in } \Omega^T, \\ &v_s \cdot \overline{n} = 0, \quad \overline{n} \cdot \mathbb{T}(v_s, p_s) \cdot \overline{\tau}_\alpha = 0, \quad \alpha = 1, 2 && \text{on } S^T, \\ &v_s|_{t=0} = v(0) && \text{in } \Omega. \end{aligned} \tag{4.3}$$

For $v_s$ we have the estimate of the form (3.12), i.e.

$$\|v_s\|_{W_2^{2,1}(\Omega^t)} \le c(d_2 H_1 + K_2)^2 + c(\|f\|_{L_2(\Omega^t)} + \|v(0)\|_{H^1(\Omega)})$$

with $H_1$, $K_2$ defined as in (3.15)–(3.16) as dependent on $\widetilde{h}_s$ instead of $h_s$. Therefore, since $\mathfrak{M}(\Omega^T) \hookrightarrow L_\infty(0, T; L_3(\Omega))$ and $\mathfrak{M}(\Omega^T) \hookrightarrow L_{10/3}(\Omega^T)$, we can replace this relation with

$$\|v_s\|_{W_2^{2,1}(\Omega^t)} \le c(d_2\|\widetilde{h}_s\|_{\mathfrak{M}(\Omega^t)} + K_2)^2 + c(\|f\|_{L_2(\Omega^t)} + \|v(0)\|_{H^1(\Omega)}). \tag{4.4}$$

For problem (4.2) and the functions $h_s$ we have

$$\begin{aligned} &\|h_s\|_{W_\sigma^{2,1}(\Omega^T)} + \|\nabla q_s\|_{L_\sigma(\Omega^T)} \\ &\quad \le c\|v_s\nabla\tilde{h}_s\|_{L_\sigma(\Omega^t)} + \|\tilde{h}_s\nabla v_s\|_{L_\sigma(\Omega^t)} + \|g\|_{L_\sigma(\Omega^t)} + \|h_s(0)\|_{L_\sigma(\Omega^t)} \\ &\qquad\qquad \equiv I_1 + I_2 + \|g\|_{L_\sigma(\Omega^t)} + \|h_s(0)\|_{L_\sigma(\Omega^t)}. \end{aligned} \tag{4.5}$$

Note, that we can not apply directly the results analogous to Lemma 3.4 and instead, we need to estimate the r.h.s. of (4.5) in different way.

The first term on the r.h.s. of (4.5) we split into:

$$I_1 \equiv \|v_s\nabla\widetilde{h}_s\|_{L_\sigma(\Omega^t)} \le \|v_s\|_{L_{\sigma\lambda_1}(\Omega^t)}\|\nabla\widetilde{h}_s\|_{L_{\sigma\lambda_2}(\Omega^t)}$$

with $1/\lambda_1 + 1/\lambda_2 = 1$.

We estimate $I_1$ under assumptions that $v \in W_2^{2,1}(\Omega^t) \hookrightarrow L_{\sigma\lambda_1}(\Omega^t)$ and $\nabla\widetilde{h}_s \in L_\eta(\Omega)$. Therefore, we have the following relations:

$$\frac{5}{2} - \frac{5}{\sigma\lambda_1} \le 2, \quad \sigma\lambda_2 \le \eta.$$

Let $\sigma\lambda_2 = \eta$. Then

$$\frac{1}{2} \le \frac{5}{\sigma} - \frac{5}{\eta}.$$

We combine this relations with the compactness condition to get

$$\frac{1}{2} + \frac{2}{\eta} \le \frac{5}{\sigma} - \frac{3}{\eta} < 1$$

and we deduce $\eta > 4$ and $\sigma > 20/7$.

The second term on the r.h.s. of (4.5) is estimated by

$$I_2 \equiv \|\nabla v_s \tilde{h}_s\|_{L_\sigma(\Omega^t)} \le \|\nabla v_s\|_{L_{\sigma\mu_1}(\Omega^t)} \|\widetilde{h}_s\|_{L_{\sigma\mu_2}(\Omega^t)}$$

with $1/\mu_1 + 1/\mu_2 = 1$. Since $\widetilde{h}_s \in L_\infty(0,T;W^1_\eta(\Omega))$ with $\eta > 4$ we have $\widetilde{h}_s \in L_\infty(0,T;L_\rho(\Omega))$ with arbitrary $\rho \le \infty$. Then we set $\mu_2 = \infty$ and then $\mu_1 = 1$. Consequently, for $v_s \in W^{2,1}_2(\Omega^T) \hookrightarrow L_\sigma(0,T;W^1_\sigma(\Omega))$ we have the relation

$$\frac{5}{2} - \frac{5}{\sigma} \le 1.$$

Hence $\sigma \le 10/3$.

Summarizing estimates for $I_1$ and $I_2$ and applying to (4.5) we infer

$$\begin{aligned} &\|h_s\|_{W^{2,1}_\sigma(\Omega^T)} + \|\nabla q_s\|_{L_\sigma(\Omega^T)} \\ &\qquad \le c\|v_s\|_{W^{2,1}_2(\Omega^t)} + c\|\widetilde{h}_s\|_{\mathfrak{M}(\Omega^t)} + c(\|g\|_{L_\sigma(\Omega^t)} + \|h(0)\|_{W^{2-2/\sigma}_\sigma(\Omega)}), \end{aligned} \tag{4.6}$$

Next, we use also the estimate on $v_s$, i.e. (4.4) to infer the inequality

$$\|h_s\|_{\mathfrak{M}(\Omega^T)} \le \varphi(\|\widetilde{h}_s\|_{\mathfrak{M}(\Omega^t)}, K_4) + cK_3 \tag{4.7}$$

where $\varphi$ is an increasing positive function and $K_4 = K_2 + d_2 + \|f\|_{L_2(\Omega^t)} + \|v(0)\|_{H^1(\Omega)}$.

This proves that bounded sets in $\mathfrak{M}(\Omega^t)$ are transformed into bounded sets in $\mathfrak{M}(\Omega^t)$.

To show the continuity, we formulate the problems for the differences:

$$H = h_1 - h_2, \quad Q = q_1 - q_2, \quad V = v_1 - v_2, \quad i = 1,2.$$

Thus, $H$ satisfies

$$\begin{aligned} &H_{,t} - \operatorname{div}\mathbb{T}(H,Q) = -V\cdot\nabla\widetilde{h}_1 - v_2\cdot\nabla\widetilde{H} - \widetilde{H}\cdot\nabla v_1 - \widetilde{h}_2\cdot\nabla V && \text{in } \Omega^T, \\ &\operatorname{div} H = 0 && \text{in } \Omega^T, \\ &H\cdot\overline{n} = 0, \quad \overline{n}\cdot\mathbb{D}(H)\cdot\overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S^T_1, \\ &H_i = 0, \quad i = 1,2, \quad H_{3,x_3} = 0 && \text{on } S^T_2, \\ &H|_{t=0} = 0 && \text{in } \Omega. \end{aligned} \tag{4.8}$$

For solutions of (4.8) we have

$$\|H\|_{W^{2,1}_\sigma(\Omega^t)} + \|\nabla Q\|_{L_\sigma(\Omega^t)} \le c(\|V\cdot\nabla\widetilde{h}_1\|_{L_\sigma(\Omega^t)} + \|v_2\cdot\nabla\widetilde{H}\|_{L_\sigma(\Omega^t)} + \|\widetilde{H}\cdot\nabla v_1\|_{L_\sigma(\Omega^t)} + \|\widetilde{h}_2\cdot\nabla V\|_{L_\sigma(\Omega^t)}).$$

This we can estimate with

$$\begin{aligned}(4.9)\quad \|H\|_{W^{2,1}_\sigma(\Omega^t)} &+ \|\nabla Q\|_{L_\sigma(\Omega^t)} \\ &\le c(\|V\|_{L_{\sigma\alpha_1}(\Omega^t)}\|\nabla\widetilde{h}_1\|_{L_{\sigma\alpha_2}(\Omega^t)} + \|v_2\|_{L_{\sigma\beta_1}(\Omega^t)}\|\nabla\widetilde{H}\|_{L_{\sigma\beta_2}(\Omega^t)} \\ &\quad + \|\widetilde{H}\|_{L_{\sigma\gamma_1}(\Omega^t)}\|\nabla v_1\|_{L_{\sigma\gamma_2}(\Omega^t)} + \|\widetilde{h}_2\|_{L_{\sigma\delta_1}(\Omega^t)}\|\nabla V\|_{L_{\sigma\delta_2}(\Omega^t)})\end{aligned}$$

Note that first two terms on the r.h.s. of (4.9) can be estimated similarly as $I_1$ in (4.5) while third and fourth — with use of imbeddings applied to $I_2$. Then, with $20/7<\sigma<10/3$, $\eta>4$ we obtain

$$\|H\|_{W^{2,1}_\sigma(\Omega^t)} + \|\nabla Q\|_{L_\sigma(\Omega^t)} \le \|V\|_{W^{2,1}_2(\Omega^t)}\|\widetilde{h}\|_{\mathfrak{M}(\Omega^t)} + \|v\|_{W^{2,1}_2(\Omega^t)}\|\widetilde{H}\|_{\mathfrak{M}(\Omega^t)}.$$

Assume that $\widetilde{h}_s$, $s=1,2$, belong to a bounded set in $\mathfrak{M}(\Omega^T)$. Hence, there exists a constant $A$ such that

$$(4.10)\qquad \|\widetilde{h}_s\|_{\mathfrak{M}(\Omega^T)} \le A, \quad \|v_s\|_{W^{2,1}_2(\Omega^T)} \le \varphi(A).$$

Therefore

$$(4.11)\qquad \|H\|_{W^{2,1}_\sigma(\Omega^t)} + \|\nabla Q\|_{L_\sigma(\Omega^t)} \le c(A)\|V\|_{W^{2,1}_2(\Omega^t)} + \varphi(A)\|\tilde{H}\|_{\mathfrak{M}(\Omega^t)}.$$

Thus, to show the continuity of the transformation $\Phi$ we should find an estimate for $\|V\|_{W^{2,1}_2(\Omega^t)}$. For this purpose we consider the problem

$$(4.12)\qquad \begin{aligned} &V_{,t} - \operatorname{div}\mathbb{T}(V,Q) = -V'\cdot\nabla v_1 - v_2'\cdot\nabla V - Wh_1 - w_2H && \text{in } \Omega^T,\\ &\operatorname{div} V = 0 && \text{in } \Omega^T,\\ &V\cdot\overline{n} = 0, \quad \overline{n}\cdot\mathbb{T}(V,Q)\cdot\overline{\tau}_\alpha = 0, \quad \alpha=1,2, && \text{on } S^T,\\ &V|_{t=0} = 0 && \text{in } \Omega,\end{aligned}$$

where $V'=(V_1,V_2)$, $W=V_3$, $v_s'=(v_{s1},v_{s2})$, $w_s=v_{s3}$.

For solutions of (4.12) we have

$$\begin{aligned}(4.13)\quad \|V\|_{W^{2,1}_2(\Omega^t)} + \|\nabla Q\|_{L_2(\Omega^t)} &\le c(\|V'\cdot\nabla v_1\|_{L_2(\Omega^t)} \\ &\quad + \|v_2'\cdot\nabla V\|_{L_2(\Omega^t)} + \|Wh_1\|_{L_2(\Omega^t)} + \|w_2H\|_{L_2(\Omega^t)}).\end{aligned}$$

We bound the first term on the r.h.s. of (4.13) by

$$c\|V\|_{L_5(\Omega^t)}\|v_1\|_{W^{2,1}_2(\Omega^t)} \equiv I_1.$$

By interpolation we get

$$I_1 \le \varepsilon_1\|V\|_{W^{2,1}_2(\Omega^t)} + c(1/\varepsilon_1)\varphi(\|v_1\|_{W^{2,1}_2(\Omega^t)})\|V\|_{L_2(\Omega^t)}.$$

Similarly, we estimate the second term on the r.h.s. of (4.13) by

$$c\|\nabla V\|_{L_{5/2}(\Omega^t)}\|v_2\|_{W_2^{2,1}(\Omega^t)}\equiv I_2$$

and

$$I_2\le\varepsilon_2\|V\|_{W_2^{2,1}(\Omega^t)}+c(1/\varepsilon_2)\varphi(\|v_2\|_{W_2^{2,1}(\Omega^t)})\|V\|_{L_2(\Omega^t)}.$$

By the Hölder inequality the third term on the r.h.s. of (4.13) is bounded by

$$c\|W\|_{L_{\sigma_1}(\Omega^t)}\|h_1\|_{L_{\sigma_2}(\Omega^t)}\equiv I_3,$$

where $5/2-5/\sigma_1\le 2$, $\sigma_2\le\infty$, $1/\sigma_1+1/\sigma_2=1/2$, which are satisfied for $\sigma_1<10$. Since $5/2-5/\sigma_1<2$, we apply the interpolation inequality to the first factor in $I_3$. Hence we get

$$I_3\le\varepsilon_3\|V\|_{W_2^{2,1}(\Omega^t)}+c(1/\varepsilon_3)\varphi(\|h_1\|_{\mathfrak{M}(\Omega^t)})\|V\|_{L_2(\Omega^t)}.$$

Finally, by the Hölder inequality, the fourth term on the r.h.s. of (4.13) is estimated by

$$c\|w_2\|_{L_{\varrho_1}(\Omega^t)}\|H\|_{L_{\varrho_2}(\Omega^t)}\equiv I_4,$$

where $1/\varrho_1+1/\varrho_2=1/2$, $5/2-5/\varrho_1\le 2$, so we can take $\varrho_2=5/2$. Hence,

$$I_4\le c\|v_2\|_{W_2^{2,1}(\Omega^t)}\|H\|_{L_{5/2}(\Omega^t)}.$$

Utilizing the above estimates in (4.13) and assuming that $\varepsilon_1,\dots,\varepsilon_3$ are sufficiently small we obtain

$$\begin{aligned}\text{(4.14)}\quad &\|V\|_{W_2^{2,1}(\Omega^t)}+\|\nabla Q\|_{L_2(\Omega^t)}\\&\le\varphi(\|v_1,v_2\|_{W_2^{2,1}(\Omega^t)},\|h_1\|_{W_\sigma^{2,1}(\Omega^t)})\cdot(\|V\|_{L_2(\Omega^t)}+\|H\|_{L_{5/2}(\Omega^t)}).\end{aligned}$$

Utilizing (4.4), (4.7) and (4.10) in (4.14) implies

$$\text{(4.15)}\qquad \|V\|_{W_2^{2,1}(\Omega^t)}+\|\nabla Q\|_{L_2(\Omega^t)}\le\varphi(A)(\|V\|_{L_2(\Omega^t)}+\|H\|_{L_{5/2}(\Omega^t)}).$$

Finally we estimate the r.h.s. of (4.15). We multiply $(4.8)_1$ by $H$ and integrate over $\Omega$. In particular,

$$\int_\Omega v_2\cdot\nabla\widetilde H\cdot H\,dx=-\int_\Omega v_2\nabla H\cdot\widetilde H\,dx\le\|v_2\|_{L_6(\Omega)}\|\widetilde H\|_{L_3(\Omega)}\|\nabla H\|_{L_2(\Omega)}.$$

Then $(4.8)_1$ yields

$$\begin{aligned}\frac{d}{dt}\|H\|^2_{L_2(\Omega)}+\nu\|H\|^2_{H^1(\Omega)}&\le c(\|V\cdot\nabla\widetilde h_1\|^2_{L_{6/5}(\Omega)}\\&+\|v_2\|^2_{L_6(\Omega)}\|\widetilde H\|^2_{L_3(\Omega)}+\|\widetilde H\cdot\nabla v_1\|^2_{L_{6/5}(\Omega)}+\|\widetilde h_2\cdot\nabla V\|^2_{L_{6/5}(\Omega)}).\end{aligned}$$

By the Hölder inequality, this implies

$$\begin{aligned}&\frac{d}{dt}\|H\|^2_{L_2(\Omega)}+\nu\|H\|^2_{H^1(\Omega)}\le c(\|V\|^2_{L_2(\Omega)}\|\nabla\widetilde h_1\|^2_{L_3(\Omega)}\\&+\|v_2\|^2_{L_6(\Omega)}\|\widetilde H\|^2_{L_3(\Omega)}+\sup_t\|\widetilde h_2\|^2_{L_3(\Omega)}\|\nabla V\|^2_{L_2(\Omega)})+\|\widetilde H\|^2_{L_2(\Omega)}\|\nabla v_1\|^2_{L_3(\Omega)}.\end{aligned}$$

Using that, in view of (4.10), the third expression on the r.h.s. of the above inequality is estimated by $c\varphi(A)\|V\|^2_{H^1(\Omega)}$ we obtain

$$\begin{aligned}(4.16)\quad \frac{d}{dt}\|H\|^2_{L_2(\Omega)}+\nu\|H\|^2_{H^1(\Omega)} &\le c(\varphi(A)\|V\|^2_{H^1(\Omega)}+\|v_2\|^2_{L_6(\Omega)}\|\widetilde{H}\|^2_{L_3(\Omega)}\\ &\quad+\|V\|^2_{L_2(\Omega)}\|\nabla\widetilde{h}_1\|^2_{L_3(\Omega)}+\|\widetilde{H}\|^2_{L_2(\Omega)}\|\nabla v_1\|^2_{L_3(\Omega)}).\end{aligned}$$

Multiplying $(4.12)_1$ by $V$ and integrating over $\Omega$, it follows that

$$\begin{aligned}(4.17)\quad &\frac{d}{dt}\|V\|^2_{L_2(\Omega)}+\nu\|V\|^2_{H^1(\Omega)}\\ &\qquad\le c\|V\|^2_{L_2(\Omega)}(\|\nabla v_1\|^2_{L_3(\Omega)}+\|h_1\|^2_{L_3(\Omega)})+c\|w_2\|^2_{L_3(\Omega)}\|H\|^2_{L_2(\Omega)}.\end{aligned}$$

Multiplying (4.17) by a constant $c_*$ such that $\nu c_*-c\varphi(A)\ge\nu$ and adding to (4.16), we get

$$\begin{aligned}&\frac{d}{dt}(c_*\|V\|^2_{L_2(\Omega)}+\|H\|^2_{L_2(\Omega)})+\nu(\|V\|^2_{H^1(\Omega)}+\|H\|^2_{H^1(\Omega)})\\ &\qquad\le cc_*\|V\|^2_{L_2(\Omega)}(\|\nabla v_1\|^2_{L_3(\Omega)}+\|h_1\|^2_{L_3(\Omega)})+cc_*\|w_2\|^2_{L_3(\Omega)}\|H\|^2_{L_2(\Omega)}\\ &\qquad+c(\|v_2\|^2_{L_6(\Omega)}\|\widetilde{H}\|^2_{L_3(\Omega)}+\|\nabla v_1\|^2_{L_3(\Omega)}\|\widetilde{H}\|^2_{L_2(\Omega)}+\|V\|^2_{L_2(\Omega)}\|\nabla\widetilde{h}_1\|^2_{L_3(\Omega)}).\end{aligned}$$

Integrating this inequality with respect to time yields

$$\begin{aligned}(4.18)\quad &\|V(t)\|^2_{L_2(\Omega)}+\|H(t)\|^2_{L_2(\Omega)}+\nu\int_0^t(\|V(t')\|^2_{H^1(\Omega)}+\|H(t')\|^2_{H^1(\Omega)})\,dt'\\ &\le c\exp c\int_0^t(\|\nabla v_1(t')\|^2_{L_3(\Omega)}+\|h_1(t')\|^2_{L_3(\Omega)}+\|w_2(t')\|^2_{L_3(\Omega)}+\|\nabla\widetilde{h}_1(t')\|^2_{L_3(\Omega)})\,dt',\\ &(\|v_2\|^2_{L_2(0,t;L_6(\Omega))}\|\widetilde{H}\|^2_{L_\infty(0,t;L_3(\Omega))}+\|\nabla v_1\|^2_{L_2(0,T;L_3(\Omega))}\|\widetilde{H}\|_{L_\infty(0,T;L_2(\Omega))})\equiv J.\end{aligned}$$

By the imbedding results we get

$$\begin{aligned}J\le c\exp c(\|v_1\|^2_{W^{2,1}_r(\Omega^t)}+\|h\|^2_{W^{2,1}_\delta(\Omega^t)})&(\|v_2\|^2_{L_2(0,t;L_6(\Omega))}\|\widetilde{H}\|^2_{L_\infty(0,t;L_3(\Omega))}\\ &+\|\nabla v_1\|^2_{L_2(0,T;L_3(\Omega))}\|\widetilde{H}\|_{L_\infty(0,T;L_2(\Omega))})\equiv J_1.\end{aligned}$$

By (4.4), (4.7) and (4.10) we obtain

$$J_1\le\varphi(A)(\|\widetilde{H}\|^2_{L_\infty(0,T;L_3(\Omega))}+\|\widetilde{H}\|^2_{L_\infty(0,T;L_2(\Omega))}).$$

Therefore, (4.18) takes the form

$$(4.19)\quad \|V\|_{V_2^0(\Omega^t)}+\|H\|_{V_2^0(\Omega^t)}\le\varphi(A)(\|\widetilde{H}\|_{L_\infty(0,T;L_3(\Omega))}+\|\widetilde{H}\|_{L_\infty(0,T;L_2(\Omega))}).$$

Utilizing (4.19) in (4.15) and the result in (4.11) we obtain

$$\|H\|_{\mathfrak{M}(\Omega^T)}\le\varphi(A)\|\widetilde{H}\|_{\mathfrak{M}(\Omega^T)},$$

which implies the uniform continuity of mapping $\Phi$ and ends the proof. $\square$

Proof of Theorem 1.1. Since $\Phi$ is uniformly continuous and compact for $20/7 < \sigma \leq 10/3$, the Leray–Schauder fixed point theorem yields the existence result. Moreover, for $5/3 < \sigma < 3$, by (3.20), Lemmas 3.5 and 3.3, we have estimates of the form (1.2). This concludes the proof. $\square$

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