Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 29, 2007, 163–180

CONLEY INDEX OVER THE BASE MORPHISM FOR MULTIVALUED DISCRETE DYNAMICAL SYSTEMS

Kinga Stolot

Abstract. We define an index of Conley type for a certain class of uppersemicontinuous multivalued dynamical systems, using techniques introduced by Mrozek, Reineck and Srzednicki [4] for the index over the base. We give the characterisation of the nontrivial index and present an example, proving that our index detects isolated invariant sets that are not detected by Kaczyński and Mrozek's [2] index.

1. Introduction

The classical Conley index is a toplogical invariant defined for flows, which provides information on the existence and structure of isolated invariant sets. The notion of an index pair is crucial for the definition of some quotient space obtained by shrinking of an exit set to a point. The actual index is the homotopy type of this quotient space. It appears that the significant information is lost due to this shrinking procedure. This is overcome in case of flows by introducing a notion of a base [4]. Roughly speaking, instead of "glueing" to a point (i.e. shrinking an exit set to a base point) one is "glueing" to an arbitrary space.

In the basic form the Conley index detects isolated invariant sets, but can not distinguish between the sets which are positioned differently in the space

2000 *Mathematics Subject Classification*. Primary: 37B30; Secondary: 54C60, 55N99.

Key words and phrases. Conley index, isolated invariant set, index pair, multivalued dynamical system.

Partially supported by the Polish Committee for Scientific Research (KBN), grant no. 2/P03A/041/24.

but locally look the same. This last feature can be captured if we use the index defined by Mrozek, Reineck and Srzednicki [4].

Although, there are several extensions of the classical Conley index for the discrete dynamical systems (both singlevalue and multivalued), the generalisation of the Mrozek–Reineck–Srzednicki's index over the base still remains an open problem. However, the definitions and results that are presented here in the multivalued setting lacks the generality of the singlevalued version for flows [4], but still we can present an example that our index detects more isolated invariant sets than the cohomological index defined by Kaczyński and Mrozek [2] for multivalued maps.

To obtain the full generality, as in the single valued case one would have to admit more spaces to act as a "base" (here we admit only a space on which the dynamical system acts) and more "glueing functions of an exit set to the base" (here we glue via identity). The results presented in this paper open the way to further extensions. It seems that the cohomological version of the index over the base for multivalued maps would be easier to obtain than the complete generalisation of the homotopy index.

It is worth mentioning that there is a significant difference between Mrozek, Reineck and Srzednicki's construction for flows and the one presented here for discrete multivalued dynamical system. To pose a correct definition of the index one needs not only to fix a "base space", but also a "map acting on the base space" — this is why we call our index, an "index over the base morphism" (or more specifically "over a dynamical system"), and not just an "index over a base" like in [4].

This approach combined with the use of the Szymczak functor, instead of cohomologies and Leray reductions used by Kaczyński and Mrozek, enables us to give some characterization of a trivial index and by this to prove that our index detects more isolated invariant sets than the one defined in [2].

2. Notation

By $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}^-$, $\mathbb{R}$, I we denote respectively integers, natural numbers (with zero), negative integers with zero, real numbers and an interval $[0,1]$. Let X be a topological space.

By Top we denote the category of topological spaces with continuous functions. $\mathcal{H}$Top stands for a *homotopy category over a category* Top. Morphisms of $\mathcal{H}$Top are homotopy classes denoted by $[f]_{\mathrm{Top}}$, for $f \in \mathrm{Top}(X,Y)$. Composition of the morphisms of this category is denoted by

$$[g]_{\mathrm{Top}} \bullet [f]_{\mathrm{Top}} = [g \circ f]_{\mathrm{Top}},$$

where $f \in \mathrm{Top}(X,Y)$ and $g \in \mathrm{Top}(Y,Z)$.

Let X and Y be topological spaces. Then $F\colon X \multimap Y$ is a *multivalued map* i.e. a map of the values being subsets of Y. For the editorial reasons instead of $\multimap$ we use $\to$ in diagrams. The set

$$\operatorname{graph}(F) = \{(x,y) \in X \times Y : y \in F(x)\}$$

is called a *graph* of the map F. If singlevalued maps appear in the multivalued context then we identify y with $\{y\}$, for $y \in Y$. Therefore we use the term graph in the above sense also for single-valued maps.

For $P = (P_1, P_2)$ by $F(P)$ we mean a pair of sets $(F(P_1), F(P_2))$.

Let Z be also a topological space and $G\colon Y \multimap Z$ be a multivalued map. A *composition* of the maps F and G is a map $G \circ F\colon X \multimap Z$, defined as

$$G \circ F(x) := \bigcup \{G(y) : y \in F(x)\}, \text{for } x \in X. \tag{2.1}$$

For $F\colon X \multimap X$, by F^k, for $k \in \mathbb{N} \setminus \{0\}$ we understand k-times composition according to the formula (2.1).

If $F\colon X \multimap Y$ is a multivalued map between two Hausdorff spaces, it is said to be *upper semicontinuous* if the set

$$F^{*-1}(A) := \{x \in X : F(x) \cap A \neq \emptyset\},$$

called a *large counter image* of the set A is closed for any closed $\emptyset \neq A \subset Y$.

Let us denote by $\mathcal{USC}^c$, the category of Hausdorff spaces with upper semicontinuous maps of compact values. Composition of morphisms is defined by the formula (2.1).

If $f\colon X \to Y$ is a continuous (single-valued) map such that

$$\operatorname{graph}(f) \subset \operatorname{graph}(F),$$

we call it a *selector* of F and we would write then $f \in F$.

Let us quote after [5] the definition of the induced morphism by the multivalued map.

Definition 2.1 [5, Definition 3.2]. We say that $F \in \mathcal{USC}^c(X,Y)$ induces a morphism if F possess a selector and any two selectors of F can be joined by the homotopy in F i.e.

$$\text{for all } f, f' \in F \text{ there exists } h \in \operatorname{Top}(X \times I, Y) \text{ such that } h_0 = f \text{ and } h_1 = f',$$

and moreover, $h_t \in F$, for any $t \in I$.

We call $\widehat{F}$ defined as

$$\widehat{F} := \{f \in \operatorname{Top}(X,Y) : f \in F\} \tag{2.2}$$

a *morphism induced by* F or briefly an *induced morphism.* If $f \in \mathcal{USC}^c(X,Y)$ is a singlevalued map we write f, instead of $\widehat{f}$ as it should be according to (2.2).

The composition of induced morphisms $\widehat{G} \circ \widehat{F}$, for $G \in \mathcal{USC}^c(Y,Z)$ is defined along selectors (for details see [5, (3.2)]).

To simplify notation below we would write x instead of $\{x\}$.

Definition 2.2 ([2, Definition 2.1]). Let (X, d_X) be a locally compact metric space and $\Phi: X \times \mathbb{Z} \multimap X$ an upper semicontinous map of compact values. We call Φ a multivalued dynamical system if

(a) $\Phi(x,0) = x$ for all $x \in X$,
(b) $\Phi(\Phi(x,n),m) = \Phi(x,n+m)$ for all $m,n \in \mathbb{Z}$, $mn > 0$ for all $x \in X$,
(c) $y \in \Phi(x,-1)$ if and only if $x \in \Phi(y,1)$ for all $x,y \in X$.

If X is as in the above definition and $F: X \multimap X$ is upper semicontinous map of compact values, then we can define $\Phi_F: X \times \mathbb{Z} \multimap X$ as

$$\Phi_F(x,n) := \begin{cases} F^n(x) & \text{for } x \in X \text{ and } n > 0, \\ x & \text{for } x \in X \text{ and } n = 0, \\ (F^{*-1})^{-n}(x) & \text{for } x \in X \text{ and } n < 0. \end{cases} \tag{2.3}$$

Obviously Φ_F satisfies conditions from Definition 2.2. We say that F *induces a multivalued dynamical system* (2.3), or briefly we would call F a dynamical system. A *trajectory* (*solution*) for a dynamical system F passing through $x \in X$ is a (singlevalued) map $\sigma: J \to X$, such that $\sigma(n+1) \in F(\sigma(n))$, for $n, n+1 \in J$, and $\sigma(n_0) = x$, for some $n_0 \in J$, where J is an interval in $\mathbb{Z}$. By an *interval* in $\mathbb{Z}$ we understand a trace of a closed interval in $\mathbb{R}$ and denote it by $[m,n]$, for $m,n \in \mathbb{Z}$ or $m = -\infty$ or $n = +\infty$.

Assume $N \subset X$ is a compact subset and $F: X \multimap X$ is a dynamical system. We use the following notation:

$$\begin{aligned} \mathrm{Inv}^+ N &:= \{x \in N : \exists \text{ solution } \sigma: \mathbb{N} \to N \text{ for } F \text{ passing through } x\}, \\ \mathrm{Inv}^- N &:= \{x \in N : \exists \text{ solution } \sigma: \mathbb{Z}^- \to N \text{ for } F \text{ passing through } x\}, \\ \mathrm{Inv} N &:= \{x \in N : \exists \text{ solution } \sigma: \mathbb{Z} \to N \text{ for } F \text{ passing through } x\}. \end{aligned}$$

The sets $\mathrm{Inv}^+ N$, $\mathrm{Inv}^- N$ and $\mathrm{Inv} N$ are called respectively a *positive*, *negative invariant part of* N, and an *invariant part of* N.

A compact set $N \subset X$ is called, after [2], an *isolating neighbourhood* for a dynamical system F if and only if

$$\mathrm{Inv} N \cup F(\mathrm{Inv} N) \subset \mathrm{int}\, N.$$

A compact set $S \subset X$ is called an *isolated invariant set* for a dynamical system F, if there exists an isolating neighbourhood N such that S is it's invariant part.

For our purposes we use a slightly modified definition of an index pair introduced in the multivalued context by Kaczyński and Mrozek, [2].

Definition 2.3. Let N be an isolating neighbourhood for a multivalued dynamical system F. Then the pair $P = (P_1, P_2)$ of compact subsets of N such that $P_1 \setminus P_2 \subset \operatorname{int} N$ is called an index pair in the neighbourhood N for a multivalued dynamical system F if

(a) $F(P_i) \cap N \subset P_i$, $i = 1, 2$,
(b) $F(P_1 \setminus P_2) \subset \operatorname{int} N$,
(c) $\operatorname{Inv}^- N \subset \operatorname{int}_N P_1$ and $\operatorname{Inv}^+ N \subset N \setminus P_2$.

Despite other differences notice that here we admit index pairs that are not topological pairs, i.e. we omit the condition $P_2 \subset P_1$, required in [2]. By $IP(N, F)$ we denote a family of index pairs for F and an isolating neighbourhood N.

Following [4] we put $\mathcal{U}(P) := X \times 0 \cup P_1 \times 1/\sim_P$, where $\sim_P$ is the following equivalence relation

$$(x, i) \sim_P (x', i') \ \Leftrightarrow\ x = x' \wedge (i = i' \vee x \in P_2).$$

In $\mathcal{U}(P)$ we consider a quotient topology induced by the natural projection $q_{\sim_P}: X \times 0 \cup P_1 \times 1 \to \mathcal{U}(P)$.

Let us remind after [5] the definition of index map and joining maps. Assume N and M are isolating neighbourhoods for the same invariant set, $P \in IP(N, F)$ and $Q \in IP(M, F)$. Moreover assume that the following condition holds

$$F(Q) \cap N \subset P.$$

Under the above assumptions we define a map $F_{QP}: \mathcal{U}(Q) \multimap \mathcal{U}(P)$ as

$$F_{QP}([x, i]_{\sim_Q}) := \{[y, i]_{\sim_P} : y \in F(x) \cap N\} \cup \{[y, 0]_{\sim_P} : y \in F(x) \cap (X \setminus \operatorname{int} N)\}.$$

The map F_{QP} is called a joining map. In case $P = Q$ we call it an index map and then denote by F_P instead of F_{PP}.

Let us remind that the Szymczak relation denoted here by $\equiv$ and Szymczak functor denoted by Sz are defined in [8] (one can also find them in [6], where the notation is compatible with the one used in this paper).

3. Objects over the base morphism

Assume $X \in \mathrm{Top}$, and $F \in \mathcal{USC}^c(X, X)$ induce a morphism $\widehat{F} \subset \mathrm{Top}(X, X)$.

Let us define a *category of objects over the base morphism*, which we denote by $\mathcal{H}\mathrm{Top}(X, \widehat{F})$. The objects of $\mathcal{H}\mathrm{Top}(X, \widehat{F})$ are triples (U, r_U, s_U), where $U \in \mathrm{Top}$ and $r_U \in \mathrm{Top}(U, X)$, $s_U \in \mathrm{Top}(X, U)$ are such that $r_U \circ s_U = \mathrm{id}_X$.

Assume that also $(V, r_V, s_V) \in \mathcal{H}\mathrm{Top}(X, \widehat{F})$. We say that

$$\Phi \in \mathcal{H}\mathrm{Top}(X, \widehat{F})((U, r_U, s_U), (V, r_V, s_V))$$

if and only if there exists $\phi \in \mathrm{Top}(U, V)$ such that $\Phi = [\phi]_{\mathrm{Top}}$, and moreover, there exists $k \in \mathbb{N}$ such that the following diagrams of the morphisms of the category $\mathcal{H}\mathrm{Top}$ commute

$$\begin{tikzcd} U \arrow[r, "{[\phi]_{\mathrm{Top}}}"] \arrow[d, "{[r_U]_{\mathrm{Top}}}"'] & V \arrow[d, "{[r_V]_{\mathrm{Top}}}"] \\ X \arrow[r, "{[\widehat{F}]^k_{\mathrm{Top}}}"'] & X \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "{[\phi]_{\mathrm{Top}}}"] & V \\ X \arrow[u, "{[s_U]_{\mathrm{Top}}}"] \arrow[r, "{[\widehat{F}]^k_{\mathrm{Top}}}"'] & X \arrow[u, "{[s_V]_{\mathrm{Top}}}"'] \end{tikzcd}$$

If $k > 0$ then $[\widehat{F}]^k_{\mathrm{Top}}$ denotes k time composition of $[\widehat{F}]_{\mathrm{Top}}$ in the category $\mathcal{H}\mathrm{Top}$. When $k = 0$ then $[\widehat{F}]^0_{\mathrm{Top}} := [\mathrm{id}_X]_{\mathrm{Top}}$.

Composition of the morphisms in the category $\mathcal{H}\mathrm{Top}(X, \widehat{F})$ is defined as

$$\Psi \bullet \Phi := [\widehat{\psi} \circ \widehat{\phi}]_{\mathrm{Top}},$$

where Φ is as above and $\Psi \in \mathcal{H}\mathrm{Top}(X, \widehat{F})((V, r_V, s_V), (W, r_W, s_W))$ and $\Psi = [\widehat{\psi}]_{\mathrm{Top}}$. The identity morphism

$$\mathrm{Id}_{(U, r_U, s_U)} \in \mathrm{Top}(X, \widehat{F})((U, r_U, s_U), (U, r_U, s_U)),$$

is defined as $[\mathrm{id}_U]_{\mathrm{Top}} \in \mathcal{H}\mathrm{Top}(U, U)$.

4. Objects over the dynamical system

Assume that X is a locally compact metric space and $F \in \mathcal{USC}^c(X, X)$ induces a morphism $\widehat{F}$ and satisfies some technical assumption — namely condition $(\mathbb{C})$ from [5].

Recall from [5] that F *satisfies condition* $(\mathbb{C})$ if for any compact set $\emptyset \neq K \subset X$ any

$$s\colon X \to X \text{ and } \overline{s}\colon K \to X \text{ such that } s \in F,\ \overline{s} \in F|_K$$

can be joined by a homotopy $h^{s,\overline{s}}\colon K \times I \to X$, which satisfies the following conditions

$$\begin{gathered} (h^{s,\overline{s}})_0 = s|_K \quad \text{and} \quad (h^{s,\overline{s}})_1 = \overline{s}, \\ (h^{s,\overline{s}})_t \in F|_K, \quad \text{for any } t \in I, \\ (h^{s,\overline{s}})_t(x) = s(x) = \overline{s}(x) \quad \text{for } x \in K \text{ such that } s(x) = \overline{s}(x). \end{gathered}$$

We call s a *full selector* of F and $\overline{s}$ a *partial selector* of a map F.

Condition $(\mathbb{C})$ guarantees that the joining maps generated by the dynamical system F which induces a morphism also induce morphisms.

These assumptions hold from now on to the end of this paper.

The category $\mathcal{H}\mathrm{Top}(X, \widehat{F})$ defined for a dynamical system F satisfying the above assumptions will be called a *category of objects over a dynamical system* F.

Let $P = (P_1, P_2) \in IP(N, F)$, for some isolating neighbourhood N for a dynamical system F. Let us consider a triple

$$(\mathcal{U}(P), r_P, s_P),$$

where $r_P \colon \mathcal{U}(P) \to X$ and $s_P \colon X \to \mathcal{U}(P)$ are defined as

$$\begin{aligned} r_P([x,i]_{\sim_P}) &:= x, \quad \text{for } [x,i]_{\sim_P} \in \mathcal{U}(P), \\ s_P(x) &:= [x,0]_{\sim_P}, \quad \text{for } x \in X. \end{aligned}$$

Note that both r_P and s_P are well defined and continuous.

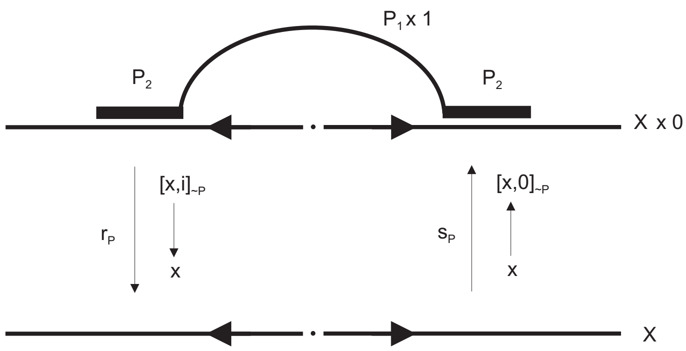

Figure 1. An example of the space $(\mathcal{U}(P), r_P, s_P)$

Theorem 4.1. *Under above assumptions and notation we have for any $P \in IP(N, F)$:*

(a) $(\mathcal{U}(P), r_P, s_P) \in \mathcal{H}\mathrm{Top}(X, \widehat{F})$,

(b) $[\widehat{F_P}]_{\mathrm{Top}} \in \mathcal{H}\mathrm{Top}(X, \widehat{F})((\mathcal{U}(P), r_P, s_P), (\mathcal{U}(P), r_P, s_P))$,

(c) *if in addition there is given $Q \in IP(M, F)$ such that the pairs P and Q satisfy assumptions under which map F_{QP} is defined, then*

$$[\widehat{F_{QP}}]_{\mathrm{Top}} \in \mathcal{H}\mathrm{Top}(X, \widehat{F})((\mathcal{U}(Q), r_Q, s_Q), (\mathcal{U}(P), r_P, s_P)),$$

(d) *if there is given $Q \in IP(M, F)$ such that $i \colon P \subset Q$, then*

$$[\mathcal{U}(i)]_{\mathrm{Top}} \in \mathcal{H}\mathrm{Top}(X, \widehat{F})((\mathcal{U}(P), r_P, s_P), (\mathcal{U}(Q), r_Q, s_Q)),$$

where $\mathcal{U}(i) \colon \mathcal{U}(P) \to \mathcal{U}(Q)$ is defined as $\mathcal{U}(i)([x,i]_{\sim_P}) := [x,i]_{\sim_Q}$, for $[x,i]_{\sim_P} \in \mathcal{U}(P)$.

Proof. To prove (a) it is sufficient to note that $r_P \circ s_P(x) = r_P([x,0]_{\sim_P}) = x$, for any $x \in X$.

Let us prove (c) (note that (b) is a particular case of (c) for $Q = P$ and $M = N$). It is easy to check that diagrams

$$\begin{tikzcd} \mathcal{U}(Q) \arrow[r, "F_{QP}"] \arrow[d, "r_Q"'] & \mathcal{U}(P) \arrow[d, "r_P"] \\ X \arrow[r, "F"'] & X \end{tikzcd} \qquad \begin{tikzcd} \mathcal{U}(Q) \arrow[r, "F_{QP}"] & \mathcal{U}(P) \\ X \arrow[r, "F"'] \arrow[u, "s_Q"] & X \arrow[u, "s_P"'] \end{tikzcd} \tag{4.1}$$

commute in the category $\mathcal{USC}^c$. Obviously the maps r_P, r_Q, s_P and s_Q induce morphisms in the sense of Definition 2.1 and from the assumptions F also induces a morphism. Due to [5, Theorem 4.10] also the map F_{QP} induces a morphism. Therefore for all maps mentioned above the homotopy partial functor $\widehat{Htp}{:}\mathcal{USC}^c \multimap \mathcal{H}\mathrm{Top}$ is defined (for the definition of the functor see [5]).

To prove that $[\widehat{F_{QP}}]_{\mathrm{Top}}$ is a morphism in the category $\mathcal{H}\mathrm{Top}(X, \widehat{F})$ it is enough to show that the following is true

$$[r_P]_{\mathrm{Top}} \bullet [\widehat{F_{QP}}]_{\mathrm{Top}} = [\widehat{F}]_{\mathrm{Top}} \bullet [r_Q]_{\mathrm{Top}}, \tag{4.2}$$

$$[\widehat{F_{QP}}]_{\mathrm{Top}} \bullet [s_Q]_{\mathrm{Top}} = [s_P]_{\mathrm{Top}} \bullet [\widehat{F}]_{\mathrm{Top}}. \tag{4.3}$$

The formulas (4.2) and (4.3) follows from the commutativity of the diagrams (4.1), provided all compositions $r_P \circ F_{QP}$, $F \circ r_Q$, $F_{QP} \circ s_Q$ and $s_P \circ F$ induce morphisms. Indeed both (4.2) and (4.3) holds, because the partial homotopy functor applied to (4.1) preserves the compositions if only all maps under consideration induce morphisms [5, Definition 3.1(b)]. Therefore it remains to show that

(i) $F \circ r_Q$ induces a morphism,
(ii) $s_P \circ F$ induces a morphism.

Note first that both $F \circ r_Q$ and $s_P \circ F$ possess a selector, because F does so. To prove (i) it is enough to show that any two selectors of $F \circ r_Q$ are homotopic within a graph of $F \circ r_Q$. A similar property needs to be shown for selectors of $s_P \circ F$.

Before we write appropriate homotopies we prove some specific properties of selectors of $F \circ r_Q$ and $s_P \circ F$.

$$\text{If } g \in F \circ r_Q, \text{ then } g = g \circ s_Q \circ r_Q. \tag{4.4}$$

$$\text{If } g \in s_P \circ F, \text{ then } g = s_P \circ r_P \circ g. \tag{4.5}$$

Note first that from (4.1) we obtain that

$$g([x,i]_{\sim_Q}) \in F \circ r_Q([x,i]_{\sim_Q}) = r_P \circ F_{QP}([x,i]_{\sim_Q}) = \{y : y \in F(x)\},$$

therefore $g([x,i]_{\sim_Q}) = g([x,0]_{\sim_Q})$ which justifies (4.4).

Similarly to prove (4.5) we exploit (4.1) to obtain that

$$g(x) \in F_{QP} \circ s_Q(x) = s_P \circ F(x) = \{[y,0]_{\sim_P} : y \in F(x)\}.$$

Let us write appropriate homotopies in case of (i). Assume $g^{(k)} \in F \circ r_Q$, for $k = 1,2$. Then $g^{(k)} \circ s_Q \in F \circ r_Q \circ s_Q = F$, and so we can define a map $\widetilde{f^{(k)}}$ as

$$\widetilde{f^{(k)}}([x,i]_{\sim_Q}) := \begin{cases} [g^{(k)} \circ s_Q(x), 1]_{\sim_P} & \text{for } i = 1 \wedge x \in Q_1 \setminus Q_2, \\ [g^{(k)} \circ s_Q(x), 0]_{\sim_P} & \text{for } i = 0 \vee (i = 1 \wedge x \in Q_1 \cap Q_2). \end{cases}$$

From [5, Conclusion 4.12] we obtain that $\widetilde{f^{(k)}} \in F_{QP}$, and from [5, Theorem 4.10] we know that there exists a homotopy $\mathcal{H}: \mathcal{U}(Q) \times I \to \mathcal{U}(P)$, such that

$$\mathcal{H}(\,\cdot\,,0) = \widetilde{f^{(1)}} \text{ and } \mathcal{H}(\,\cdot\,,1) = \widetilde{f^{(2)}},$$

and $\mathcal{H}(\,\cdot\,,t) \in F_{QP}$, for any $t \in I$. Then $r_P \circ \mathcal{H}: \mathcal{U}(Q) \times I \to X$ is a homotopy joining

$$r_P \circ \mathcal{H}(\,\cdot\,,0) = r_P \circ \widetilde{f^{(1)}} = g^{(1)} \circ s_Q \circ r_Q, \tag{4.6}$$

$$r_P \circ \mathcal{H}(\,\cdot\,,1) = r_P \circ \widetilde{f^{(2)}} = g^{(2)} \circ s_Q \circ r_Q, \tag{4.7}$$

and

$$r_P \circ \mathcal{H}(\,\cdot\,,t) \in r_P \circ F_{QP} = F \circ r_Q, \quad \text{for any } t \in I. \tag{4.8}$$

From (4.4), (4.6)–(4.8) we obtain an appropriate homotopy joining

$$r_P \circ \mathcal{H}(\,\cdot\,,0) = g^{(1)} \quad \text{and} \quad r_P \circ \mathcal{H}(\,\cdot\,,1) = g^{(2)}.$$

To prove (ii) let us consider $g^{(k)} \in s_P \circ F$, for $k = 1,2$. Then $r_P \circ g^{(k)} \in r_P \circ s_P \circ F = F$, and from the assumption that F induces a morphism we obtain that there exists a homotopy $h: X \times I \to X$, such that

$$h(\,\cdot\,,0) = r_P \circ g^{(1)} \quad \text{and} \quad h(\,\cdot\,,1) = r_P \circ g^{(2)},$$

and $h(\,\cdot\,,t) \in F$, for any $t \in I$. Therefore by using (4.5) we know that $s_P \circ h: X \times I \to \mathcal{U}(P)$ is a homotopy joining

$$s_P \circ h(\,\cdot\,,0) = s_P \circ r_P \circ g^{(1)} = g^{(1)},$$
$$s_P \circ h(\,\cdot\,,1) = s_P \circ r_P \circ g^{(2)} = g^{(2)},$$

in such a way that $s_P \circ h(\cdot,t) \in s_P \circ F$, for any $t \in I$.

To complete the proof of (c) it is enough to apply the homotopy partial functor to (4.1).

In order to prove (d) it is enough to notice that the following diagrams of singlevalued continuous maps commute

$$
\begin{tikzcd}
\mathcal{U} \arrow[r, "\mathcal{U}(i)"] \arrow[d, "r_P"'] & \mathcal{U}(Q) \arrow[d, "r_Q"] \\
X \arrow[r, "\mathrm{id}_X"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{U} \arrow[r, "\mathcal{U}(i)"] & \mathcal{U}(Q) \\
X \arrow[r, "\mathrm{id}_X"'] \arrow[u, "s_P"] & X \arrow[u, "s_Q"']
\end{tikzcd}
\tag{4.9}
$$

Because all maps appearing in diagrams (4.9) are singlevalued then they trivially induce morphisms. By applying a homotopy partial functor to (4.9), similarly as in the proof of (c) we obtain that

$$
\begin{aligned}
[r_Q]_{\mathrm{Top}} \bullet [\mathcal{U}(i)]_{\mathrm{Top}} &= [\mathrm{id}_X]_{\mathrm{Top}} \bullet [r_P]_{\mathrm{Top}},\\
[\mathcal{U}(i)]_{\mathrm{Top}} \bullet [s_P]_{\mathrm{Top}} &= [s_P]_{\mathrm{Top}} \bullet [\mathrm{id}_X]_{\mathrm{Top}}.
\end{aligned}
\qquad \square
$$

Conclusion 4.2. *Under the assumptions of the previous theorem*

$$
\begin{aligned}
[r_P]_{\mathrm{Top}} \bullet [(\widehat{F_P})^n]_{\mathrm{Top}} &= [(\widehat{F})^n]_{\mathrm{Top}} \bullet [r_P]_{\mathrm{Top}},\\
[(\widehat{F_P})^n]_{\mathrm{Top}} \bullet [s_P]_{\mathrm{Top}} &= [s_P]_{\mathrm{Top}} \bullet [(\widehat{F})^n]_{\mathrm{Top}},
\end{aligned}
$$

for any $n \in \mathbb{N}$.

5. Definition and characterization of the index

The following theorem enables us to pose the definition of the index over the base morphism. Recall that everything that we present here is done under the assumptions stated at the beginning of Section 4.

Theorem 5.1. *Assume S is an isolated invariant set for a multivalued dynamical system F. Then for any isolating neighbourhood N of S and any $P \in IP(N,F)$ the objects in the Szymczak category $\mathrm{Sz}(\mathcal{H}\mathrm{Top}(X,\widehat{F}))$ of the form*

$$((\mathcal{U}(P), r_P, s_P), [\widehat{F_P}]_{\mathrm{Top}}),$$

are isomorphic.

The proof of this theorem is analogous to the proof of [5, Theorem 5.2], with only that difference that now one needs to check additional conditions to guarantee that the maps used in the proof of Theorem 5.1 are actually the morphisms in the category $\mathcal{H}\mathrm{Top}(X,\widehat{F})$. This is proved in the Theorem 4.1, as the only maps used to construct the appropriate isomorphisms in Theorem 5.1 are these coming from inclusions and joining maps.

Theorem 5.1 justifies that the following definition is well posed.

Definition 5.2. The family of all objects in the Szymczak category

$$\mathrm{Sz}(\mathcal{H}\mathrm{Top}(X, \widehat{F}))$$

isomorphic with the object $((\mathcal{U}(P), r_P, s_P), [\widehat{F_P}]_{\mathrm{Top}})$ defined in Theorem 5.1 is called a *homotopy Conley index over the multivalued dynamical system F for an isolated invariant set S* and is denoted by $C_{(X,F)}(S)$.

The proof of the homotopy property of the index over a dynamical system, goes along the same way as the proof given in [7].

For a locally compact metric space X and $F \in \mathcal{USC}^c(X \times I, X)$ which induces a morphism $\widehat{F}$, let us define $F_\mu \in \mathcal{USC}^c(X, X)$, for $\mu \in I$ as follows

$$F_\mu(x) := F(x, \mu), \quad \text{for } x \in X.$$

Assume that each of the maps F_μ for $\mu \in I$ induces a morphism $\widehat{F_\mu}$ and satisfies assumption $(\mathbb{C})$.

By $\mathrm{Inv}(N, \mu)$ we denote an invariant part of the set N under the multivalued dynamical system F_μ. Similarly we introduce notation $\mathrm{Inv}^+(N, \mu)$ and $\mathrm{Inv}^-(N, \mu)$. By $IP(N, \mu)$ we understand a family of index pairs for F_μ in an isolating neighbourhood N.

Under the above assumptions and using the established notation we can state the homotopy property of the index over the base morphism.

Theorem 5.3. *Assume that N is an isolating neighbourhood for F_{λ_0}, for some $\lambda_0 \in I$. Then*

(a) *N is an isolating neighbourhood for F_λ, if λ is sufficiently close to λ_0;*
(b) *if N is an isolating neighbourhood for all $\lambda \in I$, then $C_{(X,F_\lambda)}(\mathrm{Inv}(N, \lambda))$ does not depend on the parameter λ.*

Below we give some characterization of the trivial index over the base.

Let us consider the isolated invariant set $\emptyset$ for the multivalued dynamical system $F\colon X \multimap X$. Obviously $(\emptyset, \emptyset)$ is an index pair for $\emptyset$ and we can write an index map as $F_\emptyset\colon X \times 0 \multimap X \times 0$, as follows

$$F_\emptyset((x, 0)) = (F(x), 0), \quad \text{for } x \in X.$$

Moreover, $r_\emptyset\colon X \times 0 \to X$ and $s_\emptyset\colon X \to X \times 0$ are given by formulas

$$r_\emptyset(x, 0) = x \quad \text{and} \quad s_\emptyset(x) = (x, 0), \text{for } x \in X.$$

Then

$$((X \times 0, r_\emptyset, s_\emptyset), [\widehat{F_\emptyset}]_{\mathrm{Top}}) \in \mathcal{H}\mathrm{Top}(X, \widehat{F}). \tag{5.1}$$

The family of objects isomorphic with (5.1) is denoted by $\overline{0}_{(X,\widehat{F})}$.

The proof of the Ważewski property of the index over a dynamical system, is analogous to the proof given in [5] and [7].

THEOREM 5.4. *If* $C_{(X,F)}(S) \neq \overline{0}_{(X,\widehat{F})}$, *then* $S \neq \emptyset$.

It is straightforward that

REMARK 5.5. Objects $((X \times 0, r_\emptyset, s_\emptyset), [\widehat{F_\emptyset}]_{\mathrm{Top}})$ and $((X, \mathrm{id}_X, \mathrm{id}_X), [\widehat{F}]_{\mathrm{Top}})$ are isomorphic in the category $\mathrm{Endo}(\mathcal{H}\mathrm{Top}(X, \widehat{F}))$.

Let us prove the characterization of the index of the empty set, which enables us to show that our index detects more isolated invariant sets than the one defined by Kaczyński and Mrozek.

THEOREM 5.6. *If* $((\mathcal{U}(P), r_P, s_P), [\widehat{F_P}]_{\mathrm{Top}}) \in \overline{0}_{(X,\widehat{F})}$ *then for some* $m, n \in \mathbb{N}$

$$[(\widehat{F_P})^m]_{\mathrm{Top}} = [s_P \circ (\widehat{F})^n \circ r_P]_{\mathrm{Top}}.$$

PROOF. From the assumptions and from Remark 5.5 we obtain that the following objects are isomorphic in the category $\mathrm{Sz}(\mathcal{H}\mathrm{Top}(X, \widehat{F}))$

$$((\mathcal{U}(P), r_P, s_P), [\widehat{F_P}]_{\mathrm{Top}}) \simeq ((X, \mathrm{id}_X, \mathrm{id}_X), [\widehat{F}]_{\mathrm{Top}}). \tag{5.2}$$

Condition (5.2) is equivalent to the fact that there exists $p, r \in \mathbb{N}$ and morphisms $\phi \in \mathrm{Top}(\mathcal{U}(P), X)$ and $\psi \in \mathrm{Top}(X, \mathcal{U}(P))$ such that

$$\begin{aligned}
[[\phi]_{\mathrm{Top}}, p]_{\equiv} &\colon ((\mathcal{U}(P), r_P, s_P), [\widehat{F_P}]_{\mathrm{Top}}) \to ((X, \mathrm{id}_X, \mathrm{id}_X), [\widehat{F}]_{\mathrm{Top}}), \\
[[\psi]_{\mathrm{Top}}, r]_{\equiv} &\colon ((X, \mathrm{id}_X, \mathrm{id}_X), [\widehat{F}]_{\mathrm{Top}}) \to ((\mathcal{U}(P), r_P, s_P), [\widehat{F_P}]_{\mathrm{Top}})
\end{aligned}$$

are mutually inverse in the Szymczak category $\mathrm{Sz}(\mathcal{H}\mathrm{Top}(X, \widehat{F}))$, i.e.

$$[[\psi \circ \phi]_{\mathrm{Top}}, r + p]_{\equiv} = [[\mathrm{id}_{\mathcal{U}(P)}]_{\mathrm{Top}}, 0]_{\equiv}, \tag{5.3}$$

$$[[\phi \circ \psi]_{\mathrm{Top}}, p + r]_{\equiv} = [[\mathrm{id}_X]_{\mathrm{Top}}, 0]_{\equiv}. \tag{5.4}$$

From the definition of the relation $\equiv$ (see [8]) conditions (5.3) and (5.4) are equivalent to

$$\begin{aligned}
[\psi \circ \phi]_{\mathrm{Top}} \bullet [(\widehat{F_P})^s]_{\mathrm{Top}} &= [(\widehat{F_P})^{s+r+p}]_{\mathrm{Top}}, \\
[\phi \circ \psi]_{\mathrm{Top}} \bullet [(\widehat{F})^t]_{\mathrm{Top}} &= [(\widehat{F})^{t+p+r}]_{\mathrm{Top}}.
\end{aligned}$$

for some $s, t \in \mathbb{N}$. The above two equations are equivalent to

$$[\psi \circ \phi \circ (\widehat{F_P})^s]_{\mathrm{Top}} = [(\widehat{F_P})^{s+r+p}]_{\mathrm{Top}}, \tag{5.5}$$

$$[\phi \circ \psi \circ (\widehat{F})^t]_{\mathrm{Top}} = [(\widehat{F})^{t+p+r}]_{\mathrm{Top}}. \tag{5.6}$$

The following diagrams commute for some $k \in \mathbb{N}$, because

$$[\phi]_{\mathrm{Top}} \in \mathcal{H}\mathrm{Top}(X, \widehat{F})((\mathcal{U}(P), r_P, s_P), (X, \mathrm{id}_X, \mathrm{id}_X)).$$

$$
\begin{tikzcd}
\mathcal{U}(P) \arrow[r, "{[\phi]_{\mathrm{Top}}}"] \arrow[d, "{[r_P]_{\mathrm{Top}}}"'] & X \arrow[d, leftrightarrow, "{[\mathrm{id}_X]_{\mathrm{Top}}}"] \\
X \arrow[r, "{[(\widehat{F})^k]_{\mathrm{Top}}}"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{U}(P) \arrow[r, "{[\phi]_{\mathrm{Top}}}"] & X \arrow[d, leftrightarrow, "{[\mathrm{id}_X]_{\mathrm{Top}}}"] \\
X \arrow[u, "{[s_P]_{\mathrm{Top}}}"] \arrow[r, "{[(\widehat{F})^k]_{\mathrm{Top}}}"'] & X
\end{tikzcd}
\tag{5.7}
$$

Similarly, because $[\psi]_{\mathrm{Top}} \in \mathcal{H}\mathrm{Top}(X, \widehat{F})((X, \mathrm{id}_X, \mathrm{id}_X), (\mathcal{U}(P), r_P, s_P))$, therefore for some $l \in \mathbb{N}$ diagrams below commute

$$
\begin{tikzcd}
X \arrow[r, "{[\psi]_{\mathrm{Top}}}"] \arrow[d, leftrightarrow, "{[\mathrm{id}_X]_{\mathrm{Top}}}"'] & \mathcal{U}(P) \arrow[d, "{[r_P]_{\mathrm{Top}}}"] \\
X \arrow[r, "{[(\widehat{F})^l]_{\mathrm{Top}}}"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "{[\psi]_{\mathrm{Top}}}"] \arrow[d, leftrightarrow, "{[\mathrm{id}_X]_{\mathrm{Top}}}"'] & \mathcal{U}(P) \\
X \arrow[r, "{[(\widehat{F})^l]_{\mathrm{Top}}}"'] & X \arrow[u, "{[s_P]_{\mathrm{Top}}}"']
\end{tikzcd}
\tag{5.8}
$$

From the commutativity of the left diagram in (5.7) and the right diagram in (5.8) we obtain

$$
[\phi]_{\mathrm{Top}} = [(\widehat{F})^k \circ r_P]_{\mathrm{Top}}, \qquad [\psi]_{\mathrm{Top}} = [s_P \circ (\widehat{F})^l]_{\mathrm{Top}}.
$$

Using the above two equations and the fact that $r_P \circ s_P = \mathrm{id}_X$ we learn that (5.5) and (5.6) can be expressed respectively as

$$
[s_P \circ (\widehat{F})^{l+k} \circ r_P \circ (\widehat{F_P})^s]_{\mathrm{Top}} = [(\widehat{F_P})^{s+r+p}]_{\mathrm{Top}}, \tag{5.9}
$$

$$
[(\widehat{F})^{k+l+t}]_{\mathrm{Top}} = [(\widehat{F})^{t+p+r}]_{\mathrm{Top}}. \tag{5.10}
$$

From Conclusion 4.2

$$
[r_P \circ (\widehat{F_P})^s]_{\mathrm{Top}} = [(\widehat{F})^s \circ r_P]_{\mathrm{Top}},
$$

therefore condition (5.9) is equivalent to

$$
[s_P \circ (\widehat{F})^{l+k+s} \circ r_P]_{\mathrm{Top}} = [(\widehat{F_P})^{s+r+p}]_{\mathrm{Top}},
$$

To complete the proof it is enough to put $m := s + r + p$ and $n := l + k + s$. $\square$

6. Index over the base morphism versus the cohomological index

We give an example of the isolated invariant set which can be detected by the index over the base morphism, but can not be detected by the cohomological index of Kaczyński and Mrozek.

Let us consider a space

$$
X := (-\infty, 0] \cup \left\{ \left[\frac{2}{2n+1}, \frac{1}{n} \right] : n \in \mathbb{N} \setminus \{0\} \right\}
$$

and a multivalued dynamical system generated by the map $F\colon X \multimap X$, defined as

$$
(6.1) \quad F(x) := \begin{cases} [-3(n+1), -3n] & \text{for } x \in (-(n+1), -n),\ n \in \mathbb{N}, \\ [-3(n+1), -3(n-1)] & \text{for } x = -n,\ n \in \mathbb{N} \setminus \{0\}, \\ [-3, 0] & \text{for } x = 0, \\ \left[\dfrac{2}{2(n+1)+1}, \dfrac{1}{n+1}\right] & \text{for } x \in \left[\dfrac{2}{2n+1}, \dfrac{1}{n}\right],\ n \in \mathbb{N} \setminus \{0\}. \end{cases}
$$

The upper part of Figure 2 illustrates the graph of F.

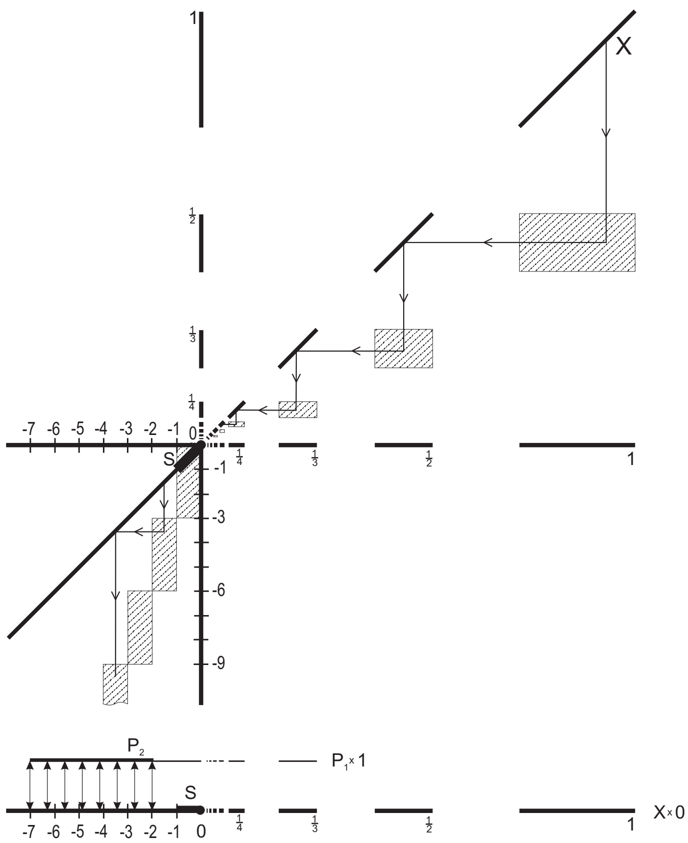

Figure 2

Let us identify the diagonal $\Delta_X = \{(x, x) : x \in X\}$ with the space $X \subset \mathbb{R}$, on which the multivalued dynamical system F acts.

At the upper part of the Figure 2 an example of two trajectories is presented (obviously for a multivalued system there can be more than one trajectory passing through each point).

It is easy to check that $S = [-1,0]$ is an isolated invariant set for $F\colon X \multimap X$. Note that

$$S = \Delta_X \cap \operatorname{graph} F$$

therefore for any $x \in S$ there exists a trajectory $\{x_n\}_{n\in\mathbb{Z}}$ such that $x_n = x$, for any $n \in \mathbb{Z}$, therefore S is an invariant set.

Let us show that $N := [-7,1/2] \cap X$ is an isolating neighbourhood of S. Note that for any $x \in N \setminus S$ all trajectories exit N either for positive times (if $x \in [-7,-1)$), or for negative times (if $x \in (0,1/2] \cap X$). Therefore $\mathrm{Inv} N = S$, because as we have already shown S is an invariant set and $S \subset N$. Moreover,

$$\mathrm{Inv} N \cup F(\mathrm{Inv} N) = [-1,0] \cup [-3,0] \subset \mathrm{int}_X N,$$

and so N is an isolating neighbourhood for S.

Because F has convex values, therefore any two of its selectors can be joined by the homotopy going in the graph of F. Therefore F induces a morphism $\widehat{F}$. Moreover, any partial selector of F can be joined with the full selector by the homotopy which does not change anything on the coincidence points and so condition ($\mathbb{C}$) holds.

Let us put

$$P_1 := [-7,1/3] \cap X \quad \text{and} \quad P_2 := [-7,-2]. \tag{6.2}$$

We shall first check that $P = (P_1, P_2)$ is an index pair in the isolating neighbourhood N. Both sets are compact subsets of X and

$$P_1 \setminus P_2 = (-2,1/3] \cap X \subset \mathrm{int}_X N.$$

Let us check the remaining conditions from Definition 2.3.

(a) $F(P_1) \cap N = ([-21,1/4] \cap X) \cap ([-7,1/2] \cap X) \subset [-7,1/3] \cap X = P_1$;
$F(P_2) \cap N = [-21,-6] \cap ([-7,1/2] \cap X) = [-7,-6] \subset [-7,-2] = P_2$;

(b) $F(P_1 \setminus P_2) = F(([-7,1/3] \cap X) \setminus [-7,-2]) = F((-2,1/3] \cap X) = (-6,1/4] \cap X \subset \mathrm{int}_X N$.

To prove condition (c) note that

$$\mathrm{Inv}^- N = [-7,0] \quad \text{and} \quad \mathrm{Inv}^+ N = [-1,1/2] \cap X. \tag{6.3}$$

Applying (6.3) we obtain that

$$\begin{aligned}
\mathrm{Inv}^- N &= [-7,0] \subset \mathrm{int}_{[-7,1/2]\cap X}([-7,1/3] \cap X) = \mathrm{int}_N P_1,\\
\mathrm{Inv}^+ N &= [-1,1/2] \cap X \subset (-2,1/2] \cap X = ([-7,1/2] \cap X) \setminus [-7,-2] = N \setminus P_2.
\end{aligned}$$

Let us consider $f\colon X \to X$, defined by the formula

$$f(x) := \begin{cases} 3x & \text{for } x \in (-\infty,0],\\ \dfrac{1}{n+1} & \text{for } x \in \left[\dfrac{2}{2n+1}, \dfrac{1}{n}\right],\ n \in \mathbb{N} \setminus \{0\}. \end{cases} \tag{6.4}$$

By comparing formulas (6.1) and (6.4) we obtain immediately that f is a selector of F.

It is equally easy to check that $g\colon \mathcal{U}(P) \to \mathcal{U}(P)$ defined as

$$(6.5)\quad g([x,i]_{\sim_P}) := \begin{cases} [f(x),i]_{\sim_P} & \text{for } x \in (-\infty,0] \text{ such that } F(x) \subset [-7,0], \\ & \text{or } x \in \left[\dfrac{2}{2n+1}, \dfrac{1}{n}\right],\ n \in \mathbb{N}\setminus\{0\}, \\ [f(x),0]_{\sim_P} & \text{for } x \in (-\infty,0] \text{ such that } F(x) \not\subset [-7,0]. \end{cases}$$

is a selector of an index map F_P. The lower part of Figure 2 illustrates a space $\mathcal{U}(P)$ (the vertical arrows indicate the identification of points from "the first level" with the "zero level").

We shall show that for $f \in F$ and $g \in F_P$ defined above the following holds

$$(6.6)\qquad [g^m]_{\mathrm{Top}} \neq [s_P \circ f^n \circ r_P]_{\mathrm{Top}}, \quad \text{for any } m,n \in \mathbb{N},$$

then certainly also

$$[(\widehat{F_P})^m]_{\mathrm{Top}} \neq [s_P \circ (\widehat{F})^n \circ r_P]_{\mathrm{Top}}, \text{for any } m,n \in \mathbb{N}$$

and so by the contraposition of Theorem 5.6 we obtain that $C_{(X,F)}(S) \neq \overline{0}_{(X,\widehat{F})}$.

In order to prove (6.6) let us number the subsequent connected components of X by giving a number n to the component $[2/(2n+1), 1/n]$, for $n \in \mathbb{N}\setminus\{0\}$. The component $(-\infty,0]$ is numbered by 0 and it will not be used later. Let us give the analogous numbers to the components of $\mathcal{U}(P)$, giving number i^0 to the component from the "level zero", placed over the i-th component of X, and number i^1 respectively to the one from the "level one".

Let us consider $x^* = 1/5$ (any $x^* \in (0,1/3) \cap X$ would suit) and any $m,n \in \mathbb{N}$. Note that $[x^*,1]_{\sim_P}$ belongs to the component 5^1 of $\mathcal{U}(P)$. According to the definition (6.5) the point $g^m([x^*,1]_{\sim_P})$ belongs to the component $(5+m)^1$ of $\mathcal{U}(P)$. On the other hand

$$s_P \circ f^n \circ r_P([x^*,1]_{\sim_P}) = s_P \circ f^n(x^*) = [f^n(x^*),0]_{\sim_P}$$

belongs to the component $(5+n)^0$ of $\mathcal{U}(P)$.

Concluding, the maps g^m and $s_P \circ f^n \circ r_P$ are not homotopic, because for $[x^*,1]_{\sim_P} \in \mathcal{U}(P)$ the values $g^m([x^*,1]_{\sim_P})$ and $s_P \circ f^n \circ r_P([x^*,1]_{\sim_P})$ lie in different connected components of $\mathcal{U}(P)$. So we proved that the index over the base morphism of the set S is nontrivial.

Let us show now that Kaczyński and Mrozek's index $KM(S,F)$ of the set S is trivial. Let us briefly remind after [2] definition of $KM(S,F)$.

Note first that $P = (P_1,P_2)$ defined by (6.2) is also an index pair in the sense of Kaczyński and Mrozek [2] (in particular it is a topological pair). We shall use the following notation,

$$T_N(P) := (P_1 \cup (X \setminus \operatorname{int} N), P_2 \cup (X \setminus \operatorname{int} N))$$

and $i_P: P \to T_N(P)$ is an inclusion.

By $I_P^*: H^*(P) \to H^*(P)$ Kaczyński and Mrozek denote an index map defined as

$$I_P^* := H^*(F_{P,T_N(P)}) \circ (H^*(i_P))^{-1},$$

where H^* stands for Aleksander–Spanier's cohomologies functor and

$$F_{P,T_N(P)}: P \multimap T_N(P)$$

is an appropriate restriction of the map F. According to [2] the index of the set S is equal to

$$KM(S,F) = L(H^*(P), I_P^*), \tag{6.7}$$

where L denotes Leray reduction. From the definition of the Leray reduction (for definition see [3]) we can write (6.7) as

$$L(H^*(P), I_P^*) = \{L(H^n(P), I_P^{(n)})\}_{n\in\mathbb{N}} = \{(\mathrm{gim}(I_P^{(n)})', (I_P^{(n)})'')\}_{n\in\mathbb{N}},$$

where

$$\begin{gathered}(I_P^{(n)})': H^n(P)/\mathrm{gker}(I_P^{(n)}) \ni [a] \to [(I_P^{(n)})(a)] \in H^n(P)/\mathrm{gker}(I_P^{(n)}),\\ (I_P^{(n)})'': \mathrm{gim}(I_P^{(n)})' \ni a \to (I_P^{(n)})'(a) \in \mathrm{gim}(I_P^{(n)})'.\end{gathered}$$

Note that in our example only $H^0(P)$ is nontrivial, because $P_1 \setminus P_2 \subset \mathbb{R}$ has infinitely many connected components, but already $H^0(P)/\mathrm{gker} I_P^{(0)}$ is trivial, because

$$\mathrm{gker} I_P^{(0)} = H^0(P). \tag{6.8}$$

To explain (6.8) it is enough to notice that F transforms connected component of X of number n into component $(n+1)$, for $n > 0$. Therefore homomorphism $I_P^{(0)}$ takes generators of a group $H^0(P)$ into subsequent generators with just the order reversed.

References

[1] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis,* Vol.I: *Theory*, Math. Appl., vol. 419, Kluwer Academic Publ., Dordrecht, 1997.

[2] T. Kaczyński and M. Mrozek, *Conley index for discrete multi-valued dynamical systems*, Topology Appl. **65** (1995), 83–96.

[3] M. Mrozek, *Leray functor and cohomological Conley index for discrete dynamical systems*, Trans. Amer. Math. Soc. **318** (1990), no. 1, 149–178.

[4] M. Mrozek, J. F. Reineck and R. Srzednicki, *The Conley index over a base*, Trans. Amer. Math. Soc. **352** (2000), no. 9, 4171–4194.

[5] K. Stolot, *Definition of the homotopy Conley index for discrete multivalued dynamical systems*, Topology Appl. **153** (2006), 3528–3545.

[6] ———, *Homotopy Conley index for multivalued discrete dynamical systems*, Ph. D. Disertation (2003), Jagiellonian University, Kraków. (in Polish)

[7] ———, *Properties of the homotopy Conley index for discrete multivalued dynamical systems*, WMS AGH Preprint 2004/5.

[8] A. Szymczak, *The Conley index for discrete semidynamical systems*, Topology Appl. **66** (1995), 215–240.

Manuscript received June 28, 2005

Kinga Stolot
Faculty of Applied Mathematics
AGH University of Science and Technology
Al. Mickiewicza 30
30-059 Kraków, POLAND

E-mail address: stolot@uci.agh.edu.pl