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MONOTONE ITERATIVE METHOD FOR INFINITE SYSTEMS OF PARABOLIC FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH NONLOCAL INITIAL CONDITIONS

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Abstract. The nonlocal initial value problem for an infinite system of parabolic semilinear functional-differential equations is studied. General operators of parabolic type of second order with variable coefficients are considered and the system is weakly coupled. We prove a theorem on existence of a classical solution in the class of continuous bounded functions and in the class of continuous functions satisfying a certain growth condition. Partial uniqueness result is obtained as well.

1. Introduction

In this paper we prove theorems on the existence and uniqueness of a classical solutions to infinite semilinear nonlocal initial-value parabolic functional-differential problems in the class of continuous, bounded functions and in the class of functions satisfying a certain growth condition. The author continue studying of infinite systems of functional-differential equations of parabolic type

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(cf. [10], [11]). The result obtained in this paper is a generalization of that reported in [10], namely it extends the previously results in direction of nonlocal initial condition.

Conditions of this type are considered for example when the precise measurement of the quantity $u(x,0)$ is imposible and the measurement of the quantity

$$u(0,x)+\sum_{j=1}^{r}h_j(x)g_j(x,u|_{Z_j}) \tag{1.1}$$

can be more precise. Situation like this takes place for instance while measuring a small amount of diffusing gas with not enough precise instrument. When in such situation the difussion is observed in time intervals the nonlocal conditions of the form

$$g_j(x,u|_{Z_j})=\frac{1}{T_{2j}-T_{2j-1}}\int_{T_{2j-1}}^{T_{2j}}u(x,\tau)\,d\tau,$$

are considered. As another example can serve the investigating of emission spectra of stars that makes it possible to calculate their temperature. Thus, the problem of calculating the temperature of star redudes to studying of light impulses of star that reach the observer in the Earth. Since optical density of medium in outer space changes in time, the average velocity of light between star and the observer in the Earth changes in time as well. And in consequence the observer in the Earth can receive the light impulses $u(x,t_0), h_1(x)u(x,t_1),\dots,h_r(x)u(x,t_r)$ emitted in diffrent moments of time $t_0,\dots,t_r$ simultaneously. This leads to the condition in the form (1.1) with functions $g_j(x,u|_{Z_j})=u(x,t_j)$.

Existence and uniqueness of differential-functional parabolic systems with these kind of nonlocal conditions in bounded domain were investigated by Byszewski among others in [3], [4].

To obtain the solution of considered problem monotone iterative method (cf. [8]) are used. Monotone iterative technique coupled with the method of sub- and supersolutions, provides an effective mechanism ensuring constructive existence results for nonlinear problems. The lower and upper functions serve as bounds for solutions which are improved by a monotone iterative process. In this process we construct two sequences which approximate the desired solution uniformly and monotonically. We use some results on differential inequalities to show that sequences obtained by monotone iteration consist of sub- and supersolutions, as well as to get their uniform convergence of these sequences.

The first classical initial-boundary value problem for infinite systems of weakly coupled differential-functional equations of parabolic type was dealt with using the same monotone iterative technique in [2].

This paper is organized as follows. In the next section we formulate the problem under consideration. In Section 3 necessary notations and definitions are introduced. We also formulate some general assumptions. Section 4 contains the theorem on the existence and partial result on uniqueness of bounded solutions. In the last section we state and prove result analogous to that from Section 4 but for unbounded continuous solutions satisfying a certain growth condition.

2. Problem statement

Let S be an infinite set of indices. Let $T > 0$ and $\Omega = \{(t, x) : t \in (0, T],\ x \in \mathbb{R}^m\}$. Moreover, let t_k, $k = 1, \dots, 2r$ be real numbers such that $0 < t_1 < \dots < t_{2r} \le T$ and let $Z_k := [t_{2k-1}, t_{2k}] \times \mathbb{R}^m$, where $k = 1, \dots, r$.

Let $B(S)$ be the space of bounded mappings

$$\xi \colon S \ni i \to \xi^i \in \mathbb{R} \quad \text{such that } \sup\{|\xi^i| : i \in S\} < \infty$$

endowed with the supremum norm

$$\|\xi\|_{B(S)} := \sup\{|\xi^i| : i \in S\}.$$

For every nonempty set $X \subset \mathbb{R}^m$ we denote by $C_S(X)$ the space of mappings

$$w \colon X \ni x \to w(x) \in B(S), \quad \text{where } w(x) \colon S \ni i \to w^i(x) \in \mathbb{R},$$

and the functions w^i are continuous in X. For w we use the notation $w = \{w^i\}_{i \in S}$, as well.

Let $f = \{f^i\}_{i \in S}$, $\varphi = \{\varphi^i\}_{i \in S}$, $g_j = \{g_j^i\}_{i \in S}$ and $h_j = \{h_j^i\}_{i \in S}$, $j = 1, \dots, r$ be given functions such that

$$f^i \colon \overline{\Omega} \times B(S) \times C_S(\overline{\Omega}) \to \mathbb{R}, \quad \varphi^i \colon \mathbb{R}^m \to \mathbb{R},$$
$$g_j^i \colon \mathbb{R}^m \times C_S(Z_j) \to \mathbb{R}, \quad h_j^i \colon \mathbb{R}^m \to \mathbb{R}.$$

Let $u = \{u^i\}_{i \in S}$, where each u^i is an unknown of the variables $(t, x) = (t, x_1, \dots, x_m)$, and set

$$\mathcal{F}^i := \frac{\partial}{\partial t} - \mathcal{A}^i, \quad \mathcal{A}^i := \sum_{j,k=1}^{m} a_{jk}^i(t, x) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{m} b_j^i(t, x) \frac{\partial}{\partial x_j} + c^i(t, x).$$

The purpose of the paper is to find, using the monotone iterative method, a function $u = \{u^i\}_{i \in S}$ such that $u^i \in C^{1,2}(\overline{\Omega})$ and is Hölder continuous with respect to x uniformly in t and satisfies the following infinite system of weakly coupled (1) semilinear parabolic nonlocal initial-value problem

$$\mathcal{F}^i[u^i](t, x) = f^i(t, x, u(t, x), u) \qquad \text{for } (t, x) \in \Omega,\ i \in S, \tag{2.1}$$

$$u(0, x) + \sum_{j=1}^{r} h_j(x) g_j(x, u|_{Z_j}) = \varphi(x) \quad \text{for } x \in \mathbb{R}^m, \tag{2.2}$$

(1) That is every equation contains all unknown functions and derivatives of only one unknown function

where functions $f = \{f^i\}_{i\in S}$, $\varphi = \{\varphi^i\}_{i\in S}$, $h_j = \{h^i_j\}_{i\in S}$ and $g_j = \{g^i_j\}_{i\in S}$, $j = 1,\dots,r$ satisfy some assumptions. The function u (as above) is said to be the *solution* of the system (2.1)–(2.2).

3. Notations, definitions and assumptions

For any $\xi, \widetilde{\xi} \in B(S)$ we write $\xi \le \widetilde{\xi}$ if $\xi^i \le \widetilde{\xi}^i$ for all $i \in S$.

Let $CB_S(X)$ denote the space of functions $w = \{w^i\}_{i\in S}$, such that $w \in C_S(X)$ and each w^i is bounded on X. This space, endowed with the supremum norm

$$\|w\|_0 := \sup\{|w^i(x)| : x \in X,\ i \in S\},$$

is a Banach space.

In space $CB_S(\overline{\Omega})$ we define a functional $\|\cdot\|_{0,t}$ by the formula

$$\|w\|_{0,t} := \sup\{|w^i(\widetilde{t}, x)| : (\widetilde{t}, x) \in \overline{\Omega},\ \widetilde{t} \le t,\ i \in S\}$$

for $w \in \{w^i\}_{i\in S} \in CB_S(\overline{\Omega})$, $t \in [0,T]$.

In the space $C_S(\overline{\Omega})$ the following partial order is introduced: for $z, \widetilde{z} \in C_S(\overline{\Omega})$, and $t \in [0,T]$ the inequality $z \le^t \widetilde{z}$ means that $z(\tau, x) \le \widetilde{z}(\tau, x)$ for all $x \in \mathbb{R}^m$, $\tau \in [0,t]$.

We notice that the notation $f(t, x, u(t,x), u)$ means that the functions f^i are the functional of the function u. We consider the functional dependence of Volterra-type, i.e.

($\mathcal{V}$) for any $(t,x) \in \Omega$, $u, \widetilde{u} \in C_S(\overline{\Omega})$, $i \in S$ the following implication holds

$$u \le^t \widetilde{u} \ \Rightarrow\ f^i(t,x,u(t,x),u) \le f^i(t,x,\widetilde{u}(t,x),\widetilde{u}).$$

This means that the values of the reaction functions $f^i(t,x,u(t,x),u)$, $i \in S$ depend only on the past of history of the process. Therefore, such functionals can describe delays and deviations or be integrals "over past". Many interesting models of this type applied in natural science can be found in [13].

Now, we recall the definitions of subsolutions and supersolutions.

Functions $v = \{v^i\}_{i\in S}$, $w = \{w^i\}_{i\in S}$ such that $v^i, w^i \in C^{1,2}(\overline{\Omega})$ and are Hölder continuous with respect to x uniformly in t for all $i \in S$ and satisfy the system of parabolic semilinear functional-differential inequalities together with nonlocal initial inequalities

$$\begin{aligned}
&\mathcal{F}^i[v^i](t,x) \le f^i(t,x,v(t,x),v), && \text{for } (t,x) \in \Omega,\ i \in S,\\
&v(0,x) + \sum_{j=1}^{r} h_j(x) g_j(x, v|_{Z_j}) \le \varphi(x) && \text{for } x \in \mathbb{R}^m,\\
&\mathcal{F}^i[w^i](t,x) \ge f^i(t,x,w(t,x),w), && \text{for } (t,x) \in \Omega, i \in S,
\end{aligned}$$

$$w(0,x) + \sum_{j=1}^{r} h_j(x) g_j(x, w|_{Z_j}) \geq \varphi(x) \quad \text{for } x \in \mathbb{R}^m$$

are called, respectively, a *subsolution* and a *supersolution* for problem (2.1)–(2.2) in $\overline{\Omega}$.

Throughout the paper we will assume:

($\mathcal{P}$) the operators $\mathcal{F}^i$, $i \in S$ are uniformly parabolic in $\overline{\Omega}$, i.e. there is $\mu > 0$ such that

$$\sum_{j,k=1}^{m} a^i_{jk}(t,x)\xi_j\xi_k \geq \mu \sum_{j=1}^{m} \xi_j^2$$

for all $(t,x) \in \overline{\Omega}$, $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$ and $i \in S$.

($\mathcal{H}_t$) the coefficients $a^i_{jk}(t,x)$, $b^i_j(t,x)$, $c^i(t,x)$, $i \in S$, $j,k = 1,\dots,m$ are bounded, continuous functions in $\overline{\Omega}$ such that $a^i_{jk}(t,x) = a^i_{kj}(t,x)$ and satisfy the following uniform *Hölder conditions* with exponent α $(0 < \alpha \leq 1)$ in $\overline{\Omega}$: there exists $H > 0$ such that

$$\begin{aligned} |a^i_{jk}(t,x) - a^i_{jk}(t',x')| &\leq H(|x-x'|^\alpha + |t-t'|^{\alpha/2}), \\ |b^i_j(t,x) - b^i_j(t,x')| &\leq H|x-x'|^\alpha, \\ |c^i(t,x) - c^i(t,x')| &\leq H|x-x'|^\alpha \end{aligned}$$

for all $(t,x),(t,x') \in \overline{\Omega}$ $j,k = 1,\dots,m$, and $i \in S$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^m$.

Now, let us recall a theorem on the existence of fundamental solution and its estimate, whose proof can be found in [6] or [7].

Lemma 3.1. *If assumptions* ($\mathcal{P}$) *and* ($\mathcal{H}$) *hold then there exist fundamental solutions* $\Gamma^i(t,x;\tau,\xi)$ *of the equations* $\mathcal{F}^i[u^i](t,x) = 0$, $i \in S$, *and the following inequalities*

$$|\Gamma^i(t,x;\tau,\xi)| \leq c(t-\tau)^{-m/2} \exp\left(-\frac{\mu^*|x-\xi|^2}{4(t-\tau)} \right), \quad i \in S$$

hold for some $\mu^* < \mu$ *where* μ^* *depends on* μ *and* H *whereas* c *depends on* μ, α, T *and the character of continuity* $a^i_{jk}(t,x)$ *in* t.

Let us notice that from the proof of this lemma and the above assumptions on the coefficients of the operators $\mathcal{F}^i$, $i \in S$ it follows that the constants c and μ^* are independent of i.

By $\mathcal{S}$ we will denote in Section 4 the set of all bounded classical solutions of problem (2.1)–(2.2) and in Section 5, $\mathcal{S}$ will denote the set of all classical solutions of this problem that satisfy the growth condition of the form

$$|u^i(t,x)| \leq D \exp(\phi(t)|x|^2).$$

For each two functions $u, \widetilde{u} \in \mathcal{S}$ we define an equivalence relation $\sim$ by formula

$$\sum_{j=1}^{r} h_j(x) g_j(x, u|_{Z_j}) = \sum_{j=1}^{r} h_j(x) g_j(x, \widetilde{u}|_{Z_j}) \quad \text{for } x \in \mathbb{R}^m,\ i \in S.$$

By $\mathcal{N}_u(\overline{\Omega})$ we denote an equivalence class of element $u \in \mathcal{S}$.

Before ending this section we introduce the following notation.

For every sufficiently smooth function β, let $\gamma = \mathcal{P}[\beta]$ be the unique solution of the initial value problem

$$\begin{aligned} \mathcal{F}^i[\gamma^i](t,x) &= f^i(t,x,\beta(t,x),\beta), && \text{for } (t,x) \in \Omega,\ i \in S, \\ \gamma(0,x) &= \varphi(x) - \sum_{j=1}^{r} h_j(x) g_j(x, \beta|_{Z_j}) && \text{for } x \in \mathbb{R}^m. \end{aligned} \tag{3.1}$$

4. Bounded solutions

In this section we construct two sequences of successive approximations as solutions of some linear infinite systems of functional-differential equations. These sequences converge to a common limit. We prove that this limit is a desired solution of the equation (2.1) with nonlocal initial condition (2.2).

First, let us formulate the following assumptions.

Assumptions 4.1. *All components $f^i(t,x,s,p)$ of $f = \{f^i\}_{i \in S}$ are*

(C_f) *continuous in $\overline{\Omega} \times B(S) \times CB_S(\overline{\Omega})$;*

(B_f) *uniformly bounded in $\overline{\Omega} \times B(S) \times CB_S(\overline{\Omega})$;*

(I_f) *weakly increasing with respect to s and p;*

(L_f) *Lipchitz continuous with respect to s and p: there exists $L_1 > 0$ and $L_2 > 0$ such that*

$$|f^i(t,x,s,p) - f^i(t,x,\widetilde{s},\widetilde{p})| \le L_1 \|s - \widetilde{s}\|_{B(S)} + L_2 \|p - \widetilde{p}\|_{0,t}$$

for $(t,x) \in \Omega$, $s, \widetilde{s} \in B(S)$, $p, \widetilde{p} \in CB_S(\overline{\Omega})$.

Moreover, for all $u \in CB_S(\overline{\Omega})$ the functions $\widetilde{f}^i(t,x) := f^i(t,x,u(t,x),u)$, where $(t,x) \in \overline{\Omega}$, $i \in S$ are

(H_f) *locally Hölder continuous with respect to x uniformly in t and the Hölder constants are independent of the function u.*

Assumptions 4.2. *All components $h^i_j(x)$ of $h_j = \{h^i_j\}_{i \in S}$, $j = 1, \dots, r$ are*

(C_h) *continuous in $\mathbb{R}^m$;*

(B_h) *uniformly bounded in $\mathbb{R}^m$;*

(0_h) *$h^i_j(x) \le 0$ for $x \in \mathbb{R}^m$.*

And all components $g^i_j(x,p)$ *of* $g_j = \{g^i_j\}_{i\in S}$, $j = 1,\dots,r$ *are*

- (C_g) *continuous in* $\mathbb{R}^m \times CB_S(Z_j)$;
- (B_g) *uniformly bounded in* $\mathbb{R}^m \times CB_S(Z_j)$;
- (L_g) *Lipchitz continuous with respect to* x *and* p: *there exists* $L_g > 0$ *such that*
$$|g^i_j(x,p) - g^i_j(\widetilde{x},\widetilde{p})| \le L_g(|x-\widetilde{x}| + \|p-\widetilde{p}\|_0)$$
for $x,\widetilde{x} \in \Omega$, $p,\widetilde{p} \in CB_S(Z_j)$, $j = 1,\dots,r$, $i \in S$;
- (I_g) *weakly increasing with respect to* p:
$$[p,\widetilde{p} \in CB_S(Z_j),\ p \le \widetilde{p} \text{ in } \overline{\Omega}] \ \Rightarrow\ [g^i_j(x,p) \le g^i_j(x,\widetilde{p}),\ x \in \mathbb{R}^m]$$
$j = 1,\dots,r$, $i \in S$.

Remark 4.3. Let the functions $f^i = f^i(t,x,s,p)$, $i \in S$, satisfy condition (I_f) from Assumptions 4.1, $g^i_j = g^i_j(x,p)$, $i \in S$, $j = 1,\dots,r$, satisfy condition (I_g) and (0_h) from Assumptions 4.2. Then the operator $\mathcal{P}$ is weakly increasing.

Proof. Let $\beta_1,\beta_2 \in CB_S(\overline{\Omega})$ be such that $\beta_1 \le \beta_2$ for $(t,x) \in \Omega$. Let γ_1, γ_2 be the unique solutions of the initial value problem (3.1) with $\beta = \beta_1$ and $\beta = \beta_2$, respectively. Conditions (I_f) from Assumptions 4.2 implies

$$\mathcal{F}^i[\gamma^i_1 - \gamma^i_2](t,x) = f^i(t,x,\beta_1(t,x),\beta_1) - f^i(t,x,\beta_2(t,x),\beta_2) \le 0 \tag{4.1}$$

for $(t,x) \in \Omega$, $i \in S$.

By conditions (I_g) and (0_h) from Assumptions 4.2 the following inequality holds

$$(\gamma_1 - \gamma_2)(0,x) = -\sum_{j=1}^{r} h_j(x)[g_j(x,\beta_1|_{Z_j}) - g_j(x,\beta_2|_{Z_j})] \le 0 \tag{4.2}$$

for $x \in \mathbb{R}^m$.

Consequently, using Collorary 1 from [10], we obtain by (4.1) and (4.2)

$$\gamma_1(t,x) \le \gamma_2(t,x) \quad \text{for } (t,x) \in \Omega. \qquad \square$$

Assumptions 4.4. *There exists at least one pair* $v_0 = v_0(t,x)$, $w_0 = w_0(t,x)$ *in* $CB_S(\overline{\Omega})$ *of a subsolution and a supersolution of problem* (2.1)–(2.2) *in* $\overline{\Omega}$ *which are Hölder continuous in* x *uniformly with respect to* t, $v_0(t,x) \le w_0(t,x)$ *for* $(t,x) \in \overline{\Omega}$ *and* $v_0(0,x) = w_0(0,x)$ *for* $x \in \mathbb{R}^m$.

Now, we state and prove the theorem on the existence of solution of problem (2.1)–(2.2) obtained by a simple iterative method, i.e. starting from the subsolution v_0 and the supersolution w_0 we define $v_n := \mathcal{P}[v_{n-1}]$, $w_n := \mathcal{P}[w_{n-1}]$, $n = 1,2,\dots$. Thus, at each step we have an infinite system of linear functional-differential equations. The sequence of successive approximations converges to the desired solution with *power speed*.

Theorem 4.5. *Let the conditions* $(\mathcal{P})$ *and* $(\mathcal{H}_t)$ *hold. Suppose that every component* φ^i *of the initial data* $\varphi = \{\varphi^i\}_{i\in S}$ *is a bounded continuous function on* $\mathbb{R}^m$. *Moreover, let Assumptions* 4.1, 4.2 *and* 4.4 *hold. Consider the following recursive infinite systems of linear equations:*

$$\mathcal{F}^i[v_n^i](t,x) = f^i(t,x,v_{n-1}(t,x),v_{n-1}), \tag{4.3}$$

$$\mathcal{F}^i[w_n^i](t,x) = f^i(t,x,w_{n-1}(t,x),w_{n-1}), \tag{4.4}$$

for $(t,x)\in\Omega$, $i\in S$, $n=1,2,\dots$ *with the nonlocal initial conditions of the form*

$$v_n(0,x) = -\sum_{j=1}^{r} h_j(x)g_j(x,v_{n-1}|_{Z_j}) + \varphi(x), \tag{4.5}$$

$$w_n(0,x) = -\sum_{j=1}^{r} h_j(x)g_j(x,w_{n-1}|_{Z_j}) + \varphi(x), \tag{4.6}$$

for $x\in\mathbb{R}^m$, $i\in S$, *respectively. Then*

(a) *there exist unique classical bounded solutions* v_n *and* w_n, *in* $\overline{\Omega}$, *for* $n = 1,2,\dots,$ *of systems* (4.3), (4.4) *with the nonlocal initial conditions* (4.5), (4.6), *respectively;*

(b) v_n *and* w_n, $n=0,1,\dots$ *are subsolutions and supersolutions for problem* (2.1)–(2.2) *in* $\overline{\Omega}$*;*

(c) *we have*

$$v_0(t,x)\le\dots\le v_n(t,x)\le v_{n+1}(t,x)\le\dots \\ \le w_{n+1}(t,x)\le w_n(t,x)\le\dots\le w_0(t,x) \tag{4.7}$$

for $(t,x)\in\overline{\Omega}$, $n=1,2,\dots;$

(d) $\lim_{n\to\infty}[w_n^i(t,x)-v_n^i(t,x)]=0$ *uniformly in* $\overline{\Omega}$, $i\in S$*;*

(e) $u(t,x)=\lim_{n\to\infty}v_n(t,x)$, *where the limit is meant in the uniform sense, is a classical bounded solution of problem* (2.1)–(2.2) *in* $\overline{\Omega}$. *Moreover,* $u(t,x)$ *is Hölder continuous with respect to* x *uniformly in* t*;*

(f) *in the class of all functions belonging to* $\mathcal{N}_u(\overline{\Omega})$ *the function* u *is the unique solution of problem* (2.1)–(2.2) *in* Ω.

Before the proof of Theorem 4.5 let us stress that similarly as in papers [3], [4] we obtain only the partial uniqueness result, namely there do not exist two different solutions of problem (2.1)–(2.2) satisfying the same nonlocal part of initial condition, i.e. equivalence classes of quotient space $S/_\sim$ are single-element. Unfortunately, we still have not the uniqueness result in space $S/_\sim$.

Proof. (a) Starting from v_0 and w_0 we define v_1, w_1 as solutions of (4.3), (4.4) with the nonlocal initial conditions (4.5), (4.6), respectively, i.e. $v_1 = \mathcal{P}[v_0]$, $w_1=\mathcal{P}[w_0]$. Observe that the systems in question have the following property: the i-th equation depends on the i-th unknown function only, therefore since v_0, w_0 satisfy Assumption 4.4, the classical theorems on the existence and uniqueness of solution of linear parabolic Cauchy problems (cf. [6] or [7]) assert that there exist unique solutions $v_1, w_1 \in CB_S(\overline{\Omega})$ of the above problems and v_1 and w_1 are Hölder continuous with respect to x uniformly in t (cf. [6]). Next, we define by induction $\{v_n\}$, $\{w_n\}$ as solutions of (4.3), (4.4) with the nonlocal initial conditions (4.5), (4.6), respectively, i.e. $v_n = \mathcal{P}[v_{n-1}]$, $w_n = \mathcal{P}[w_{n-1}]$, The preceding reasoning shows that v_n, w_n exist and are uniquely defined. Moreover, for each $i \in S$, $n = 1, 2, \ldots$, v_n^i, w_n^i are bounded, belong to $C^{1,2}(\overline{\Omega})$ and are Hölder continuous in x uniformly in t.

(b) Using the mathematical induction we show that the functions v_n are subsolutions. v_0 is a subsolution by Assumption 4.4. Suppose for a fixed $n \in \mathbb{N}$ that v_n is a subsolution of (2.1)–(2.2) in $\overline{\Omega}$, i.e. v_n satisfies the following inequalities

$$\begin{aligned} \mathcal{F}^i[v_n^i](t,x) &\le f^i(t,x,v_n(t,x),v_n) && \text{for } (t,x) \in \Omega,\ i \in S, \\ v_n(0,x) &\le \varphi(x) - \sum_{j=1}^{r} h_j(x) g_j(x, v_n|_{Z_j}) && \text{for } x \in \mathbb{R}^m. \end{aligned}$$

From the definition of the operator $\mathcal{P}$ it follows that

$$\begin{aligned} \mathcal{F}^i[v_{n+1}^i](t,x) &= f^i(t,x,v_n(t,x),v_n) && \text{for } (t,x) \in \Omega,\ i \in S, \\ v_{n+1}(0,x) &= \varphi(x) - \sum_{j=1}^{r} h_j(x) g_j(x, v_n|_{Z_j}) && \text{for } x \in \mathbb{R}^m. \end{aligned}$$

The function $v_n^i - v_{n+1}^i$ satisfies the assumptions of Corollary 1 from [10], thus its thesis yields

$$[v_n - v_{n+1}](t,x) \le 0 \quad \text{for } (t,x) \in \overline{\Omega},$$

i.e.

$$v_n(t,x) \le \mathcal{P}[v_n](t,x) \quad \text{for } (t,x) \in \overline{\Omega}.$$

Now, condition (I_f) from Assumptions 4.1 and definition of the operator $\mathcal{P}$ implies

$$\begin{aligned} &\mathcal{F}^i[v_{n+1}^i](t,x) - f^i(t,x,v_{n+1}(t,x),v_{n+1}) \\ &\qquad = f^i(t,x,v_n(t,x),v_n) - f^i(t,x,\mathcal{P}[v_n](t,x),\mathcal{P}[v_n]) \le 0 \end{aligned}$$

for all $i \in S$, $(t,x) \in \overline{\Omega}$. We conclude that v_{n+1} is a subsolution as well. The proof that the w_n are supersolutions is similar.

(c) The monotonicity of the sequences $\{v_n\}$, $\{w_n\}$ is a consequence of the fact that v_n, w_n are subsolutions and supersolutions, respectively and definition of $\mathcal{P}$. To finish the proof of thesis (c) it is enough to show the inequality

$$v_n(t,x) \le w_n(t,x) \quad \text{for } (t,x) \in \overline{\Omega},\ n = 0, 1, \ldots \tag{4.8}$$

By Assumption 4.4 the inequality (4.8) holds for $n = 0$. Now, assume that for a fixed $n \in \mathbb{N}$

$$v_{n-1}(t,x) \le w_{n-1}(t,x) \quad \text{for } (t,x) \in \overline{\Omega}. \tag{4.9}$$

Since v_n and w_n are subsolutions and supersolutions, respectively, the following inequalities hold

$$\mathcal{F}^i[v_n^i](t,x) - f^i(t,x,v_n(t,x),v_n) \le 0 \le \mathcal{F}^i[w_n^i](t,x) - f^i(t,x,w_n(t,x),w_n)$$

for $(t,x) \in \overline{\Omega}$, $i \in S$. Moreover, from (4.5), (4.6), (4.9) and conditions (I_g), (0_h) from Assumptions 4.2 we obtain

$$v_n^i(0,x) - w_n^i(0,x) = -\sum_{j=1}^{r} h_j(x)[g_j(x,v_{n-1}|_{Z_j}) - g_j(x,w_{n-1}|_{Z_j})] \le 0,$$

for $x \in \overline{\Omega}$, $i \in S$. Thus, by Proposition 2 from [10],

$$v_n(t,x) \le w_n(t,x) \quad \text{for } (t,x) \in \overline{\Omega}.$$

Therefore, by the mathematical induction, (4.8) is true.

(d) Let

$$m_n^i(t,x) := w_n^i(t,x) - v_n^i(t,x). \tag{4.10}$$

Using the mathematical induction we show that

$$m_n^i(t,x) \le N_0 \frac{[(L_1+L_2)t]^n}{n!}, \quad n = 0,1,\dots, \text{ for } (t,x) \in \overline{\Omega},\ i \in S, \tag{4.11}$$

where $N_0 = \|w_0 - v_0\|_0$. The inequality for m_0^i is obvious. Suppose that the inequality (4.11) holds for fixed $n \in \mathbb{N} \cup \{0\}$. The condition (L_f), yields

$$\begin{aligned}\mathcal{F}^i[m_{n+1}^i](t,x) &= f^i(t,x,w_n(t,x),w_n) - f^i(t,x,v_n(t,x),v_n)\\ &\le L_1\|m_n(t,x)\|_{B(S)} + L_2\|m_n\|_{0,t}.\end{aligned}$$

By the definitions of $\|\cdot\|_{0,t}$ and $\|\cdot\|_{B(S)}$ and the induction assumption both $\|m_n(t,x)\|_{B(S)}$ and $\|m_n\|_{0,t}$ are estimated by $N_0[(L_1+L_2)t]^n/n!$. Thus, finally,

$$\mathcal{F}^i[m_{n+1}^i](t,x) \le N_0 \frac{(L_1+L_2)^{n+1}t^n}{n!} \quad \text{in } \Omega.$$

Moreover, from (4.10), from (4.7) and from the induction assumption

$$0 \le m_{n+1}^i(0,x) \le m_n^i(0,x) \le 0 \quad \text{for } x \in \mathbb{R}^m,\ i \in S.$$

Therefore,

$$m_{n+1}^i(0,x) = 0 \quad \text{for } x \in \mathbb{R}^m,\ i \in S.$$

In order to apply the theorem on differential inequalities, consider the comparison system

$$\mathcal{F}^i[M^i_{n+1}](t,x) = N_0 \frac{(L_1+L_2)^{n+1}t^n}{n!} \quad \text{for } (t,x)\in\Omega,\ i\in S$$

with the initial condition $M^i_{n+1}(0,x) \geq 0$ for $x \in \mathbb{R}^m,\ i \in S$. The functions $M^i_{n+1}(t,x) = N_0[(L_1+L_2)t]^{n+1}/(n+1)!$ are solutions of the comparison problem. Thus

$$\tag{4.12} \begin{aligned} \mathcal{F}^i[m^i_{n+1}](t,x) &\leq \mathcal{F}^i[M^i_{n+1}](t,x) \quad &&\text{for } (t,x)\in\Omega,\ i\in S,\\ m^i_{n+1}(0,x) &\leq M^i_{n+1}(0,x) &&\text{for } x\in\mathbb{R}^m,\ i\in S. \end{aligned}$$

The inequalities (4.12) imply, by Corollary 1 from [10],

$$m^i_{n+1}(t,x) \leq M^i_{n+1}(t,x) = N_0\frac{[(L_1+L_2)t]^{n+1}}{(n+1)!} \quad \text{for } (t,x)\in\overline{\Omega},\ i\in S,$$

and, consequently, by the mathematical induction, (4.11) holds. As a direct consequence of (4.11) we obtain thesis (d).

(e) First, notice that, from (4.7) and from thesis (d), there exists a continuous function $u = \{u^i\}_{i\in S}$ such that

$$\tag{4.13} \begin{aligned} \lim_{n\to\infty}[w^i_n(t,x) - u^i(t,x)] &= 0 \quad \text{in } \overline{\Omega},\ i\in S;\\ \lim_{n\to\infty}[v^i_n(t,x) - u^i(t,x)] &= 0 \quad \text{in } \overline{\Omega},\ i\in S; \end{aligned}$$

uniformly in $\overline{\Omega}$ for all $i \in S$. Moreover, by thesis (a), by (4.13) and by (L_g) from Assumptions 4.2 the function $u = \{u^i\}_{i\in S}$ satisfies the nonlocal initial condition (2.2). Now, we prove that u satisfies (2.1). It is enough to show that u fulfills (2.1) in any compact set contained in Ω. Consider the cylinder $D_R := \{(t,x) : \sum_{j=1}^m x_j^2 \leq R^2,\ 0\leq t\leq T\}$, where $R > 0$. Let $\Gamma_R := \{(t,x) : \sum_{j=1}^m x_j^2 = R^2,\ 0\leq t\leq T\}$ and S^0_R stands for the base of D_R, i.e. the set $\{(t,x) : \sum_{j=1}^m x_j^2 \leq R^2,\ t=0\}$. Thus, we only need to prove it in D_R for any $R > 0$. Due to assumption (I_f) from Assumptions 4.1 and the inequalities (4.7) it follows that $f^i(t,x,v_{n-1}(t,x),v_{n-1})$ are uniformly bounded in D_R (with respect to n). Therefore the solution $v_n(t,x)$ of the linear system

$$\mathcal{F}^i[v^i_n](t,x) = f^i(t,x,v_{n-1}(t,x),v_{n-1}), \quad i\in S$$

with suitable initial condition is Hölder continuous with exponent α with respect to x uniformly in t, with a constant independent of n (cf. [6]). Consequently, $u(t,x)$ also satisfies the Hölder condition with respect to x uniformly in t. Now, consider the system:

$$\tag{4.14} \mathcal{F}^i[z^i](t,x) = f^i(t,x,u(t,x),u), \quad \text{for } (t,x)\in D_R,\ i\in S$$

with the conditions

$$z(t,x) = u(t,x) \quad \text{on } \Gamma_R, \tag{4.15}$$

$$z(0,x) = \varphi(x) - \sum_{j=1}^{r} h_j(x) g_j(x, u|_{Z_j}) \quad \text{on } S_R^0. \tag{4.16}$$

From conditions (H_f) and (L_f) from Assumptions 4.1 and the fact that $u(t,x)$ is Hölder continuous with respect to x, the right hand sides of this system are continuous in D_R and locally Hölder continuous with respect to x. Thus, on the base of the classical existence and uniqueness theorems for linear parabolic initial-boundary valued problems (cf. [7]) there exists a unique classical solution $z(t,x)$ of the problem (4.14)–(4.16) in $\overline{D}_R$. On the other hand, by condition (L_f) from Assumptions 4.1 and by (4.13)

$$\lim_{n\to\infty} f^i(t,x,v_{n-1}(t,x),v_{n-1}) = f^i(t,x,u(t,x),u) \quad \text{uniformly in } \overline{D}_R.$$

Moreover, the boundary values $v_n(t,x)$ converge uniformly to $u(t,x)$ on Γ_R and due to the condition (L_f) from Assumptions 4.2 the initial values converge uniformly in S_R^0. Consequently, the functional-differential version of the theorem on the continuous dependence of the solution on the right hand sides and initial-boundary values (which is a consequence of Theorem 2.1 from [12] and of a similar argument as in the proof of Theorem 2.1 from [5]) gives

$$\lim_{n\to\infty} v_n^i(t,x) = z^i(t,x) \quad \text{uniformly in } \overline{D}_R. \tag{4.17}$$

Thus, by (4.17) and (4.13),

$$z^i(t,x) = u^i(t,x) \quad \text{in } \overline{D}_R, \text{ for all } i \in S, \text{ for arbitrary } R > 0,$$

which means $z(t,x) = u(t,x)$ for all $(t,x) \in \overline{\Omega}$, i.e. $u(t,x)$ is a classical bounded solution of problem (2.1)–(2.2).

(f) Let u be the solution of problem (2.1)–(2.2) from thesis (e) and let $\widetilde{u}$ be the other continuous bounded solution of problem (2.1)–(2.2) such that $\widetilde{u} \in \mathcal{N}_u(\overline{\Omega})$. Then

$$\mathcal{F}^i[u^i](t,x) - f^i(t,x,u(t,x),u) = \mathcal{F}^i[\widetilde{u}^i](t,x) - f^i(t,x,\widetilde{u}(t,x),\widetilde{u}), \tag{4.18}$$

for $(t,x) \in \overline{\Omega}$, $i \in S$, and

$$u(0,x) = \widetilde{u}(0,x) \quad \text{for } x \in \mathbb{R}^m. \tag{4.19}$$

Applying Proposition 2 from [10] to (4.18) and (4.19) we obtain

$$u(t,x) = \widetilde{u}(t,x) \quad \text{for } (t,x) \in \overline{\Omega}.$$

The proof of Theorem 4.5 is complete. □

Remark 4.6. Thus, the existence of solution of problem (2.1)–(2.2) is now (according to Assumption 4.4) equivalent to existence of suitable subsolution and supersolution. However, let us notice, that in case of existence theorems for wide class of problems proved using monotone methods this assumption in literature is typical (e.g. [2]–[4], [8], [12]). And question of existence subsolution and supersolution is usually solved for particular problem by indicating them (e.g. [9]).

5. Unbounded solutions

We denote by C^+ the space of all positive, real-valued, continuous and non-decreasing functions defined on the set $[0,T]$.

For $w \in C_S(\overline{\Omega})$ we define the following weighted norms depending on $\phi \in C^+$ (see [1]):

$$\|w\|_{2,\phi} := \sup_{i\in S} \sup_{(t,x)\in\overline{\Omega}} \frac{|w^i(t,x)|}{[\phi(t)]^{m/2}\exp(\phi(t)|x|^2)},$$

and

$$\|w\|_{2,\phi,t} := \sup_{i\in S} \sup_{x\in\mathbb{R}^m,\ \overline{t}\le t} \frac{|w^i(\overline{t},x)|}{[\phi(\overline{t})]^{m/2}\exp(\phi(\overline{t})|x|^2)}.$$

Let $E_S^{2,\phi}$ be the space of all functions $w \in C_S(\overline{\Omega})$ such that:

- there exists $D \ge 0$ such that for all $(t,x) \in \overline{\Omega}$ and all $i \in S$

$$|w^i(t,x)| \le D\exp(\phi(t)|x|^2)$$

  for $\phi \in C^+$.

Obviously, the space $E_S^{2,\phi}$ endowed with the norm $\|\cdot\|_{2,\phi}$ is a Banach space.

Now, we state a result similar to Theorem 4.5, but concerning functions which behave like $|u^i(t,x)| \le D\exp(\phi(t)|x|^2)$. But first, let us formulate appropriate assumptions.

Assumptions 5.1. *All components $f^i(t,x,s,p)$ of $f = \{f^i\}_{i\in S}$ are*

(C_f) *continuous in $\overline{\Omega} \times B(S) \times E_S^{2,\phi}$;*

(B_f) *expotentially bounded with respect to t and x, i.e. there exists $M_0 \ge 0$ such that for all $i \in S$ and all $(t,x) \in \overline{\Omega}$*

$$|f^i(t,x,0,0)| \le M_0\exp(\phi(t)|x|^2);$$

(I_f) *weakly increasing with respect to s and p;*

(L_f) *weighted Lipschitz continuous in the following sense: there exists $L_1 > 0$, $L_2 > 0$ such that*

$$|f^i(t,x,s,p) - f^i(t,x,\widetilde{s},\widetilde{p})| \le L_1\|s-\widetilde{s}\|_{B(S)} + L_2\|p-\widetilde{p}\|_{2,\phi,t}[\phi(t)]^{m/2}\exp(\phi(t)|x|^2)$$

for $(t,x) \in \Omega$, $s,\widetilde{s} \in B(S)$, $p,\widetilde{p} \in C_S(\overline{\Omega})$.

Moreover, for all $u \in E_S^{2,\phi}$ *the functions* $\widetilde{f}^i(t,x) := f^i(t,x,u(t,x),u)$, *where* $(t,x) \in \overline{\Omega}$, $i \in S$ *are*

(H_f) *locally Hölder continuous with respect to* x *uniformly in* t *and the Hölder constants are independent of the function* u.

Assumptions 5.2. *All components* $h_j^i(x)$ *of* $h_j = \{h_j^i\}_{i\in S}$, $j = 1,\dots,r$

(C_h) *are continuous in* $\mathbb{R}^m$;

(E_h) *satisfy for some* $K_h > 0$ *the inequality* $|h_j^i(t,x)| \le K_h \exp(\phi(0)|x|^2)$ *for all* $x \in \mathbb{R}^m$;

(0_h) $h_j^i(x) \le 0$ *for* $x \in \mathbb{R}^m$.

And all components $g_j^i(x,p)$ *of* $g_j = \{g_j^i\}_{i\in S}$, $j = 1,\dots,r$

(C_g) *are continuous in* $\mathbb{R}^m \times E_S^{2,\phi}$;

(E_g) *satisfy for some* $K_g > 0$ *the inequality* $|g_j^i(t,p)| \le K_g \exp(\phi(0)|x|^2)$ *for all* $x \in \mathbb{R}^m$ *and* $u \in E_S^{2,\phi}$;

(L_g) *Lipchitz continuous with respect to* x *and* p: *there exists* $L_g > 0$ *such that*

$$|g_j^i(x,p) - g_j^i(\widetilde{x},\widetilde{p})| \le L_g(|x-\widetilde{x}| + \|p-\widetilde{p}\|_{2,\phi})$$

for $x,\widetilde{x} \in \Omega$, $p,\widetilde{p} \in E_S^{2,\phi}$, $j = 1,\dots,r$, $i \in S$;

(I_g) *weakly increasing with respect to* p:

$$[p,\widetilde{p} \in E_S^{2,\phi},\ p \le \widetilde{p} \text{ in } \overline{\Omega}] \Rightarrow [g_j^i(x,p) \le g_j^i(x,\widetilde{p}),\ x \in \mathbb{R}^m]$$

for $j = 1,\dots,r$, $i \in S$.

Remark 5.3. Let the functions $f^i = f^i(t,x,s,p)$, $i \in S$, satisfy condition (I_f) from Assumptions 5.1, $g_j^i = g_j^i(x,p)$, $i \in S$, $j = 1,\dots,r$, satisfy condition (I_g) and (0_h) from Assumptions 5.2. Then the operator $\mathcal{P}$ is weakly increasing.

Proof. Proof of Remark 4.6 is analogous to proof of Remark 4.3. $\square$

Assumptions 5.4. *There exists at least one pair* $v_0 = v_0(t,x)$, $w_0 = w_0(t,x)$ *in* $E_S^{2,\phi}$ *of a subsolution and a supersolution of problem* (2.1)–(2.2) *in* $\overline{\Omega}$ *which are Hölder continuous in* x *uniformly with respect to* t, $v_0(t,x) \le w_0(t,x)$ *for* $(t,x) \in \Omega$ *and* $v_0(0,x) = w_0(0,x)$ *for* $x \in \mathbb{R}^m$.

Theorem 5.5. *Let the assumptions* ($\mathcal{P}$) *and* ($\mathcal{H}_t$) *hold. Let* $\phi \in C^+$ *be a function satisfying the inequality*

$$\frac{\mu^*\phi(\tau)}{\mu^* - 4\phi(\tau)(t-\tau)} \le \phi(t) \quad \text{for } 0 \le \tau \le t \le T,$$

where μ^* *is the constant which appeared in Lemma* 3.1. *Let Assumptions* 5.1, 5.2 *and* 5.4 *hold. Moreover, let all the components of the initial data* $\varphi = \{\varphi^i\}_{i\in S}$

be such that $|\varphi^i(x)| \le \overline{K} \exp(\phi(0)|x|^2)$ *for all* $x \in \mathbb{R}^m$. *Consider the following infinite system of linear equations*

$$\mathcal{F}^i[v_n^i](t,x) = f^i(t,x,v_{n-1}(t,x),v_{n-1}), \tag{5.1}$$

$$\mathcal{F}^i[w_n^i](t,x) = f^i(t,x,w_{n-1}(t,x),w_{n-1}), \tag{5.2}$$

for $(t,x) \in \Omega,\ i \in S,\ n = 1,2,\dots$ *with the nonlocal initial conditions of the form*

$$v_n(0,x) = -\sum_{j=1}^{r} h_j(x) g_j(x, v_{n-1}|_{Z_j}) + \varphi(x), \tag{5.3}$$

$$w_n(0,x) = -\sum_{j=1}^{r} h_j(x) g_j(x, w_{n-1}|_{Z_j}) + \varphi(x), \tag{5.4}$$

for $x \in \mathbb{R}^m,\ i \in S$, *respectively. Then*

(a) *there exist unique classical solutions* $v_n \in E_S^{2,\phi}$ *and* $w_n \in E_S^{2,\phi}$, $n = 1,2,\dots,$ *of systems* (5.1) *and* (5.2) *with the nonlocal initial conditions* (5.3), (5.4), *respectively, in* $\overline{\Omega}$;

(b) v_n *and* w_n, $n = 0,1,\dots,$ *are subsolutions and supersolutions for problem* (2.1)–(2.2) *in* $\overline{\Omega}$, *respectively;*

(c) *we have*

$$\begin{aligned} v_0(t,x) \le \dots \le v_n(t,x) &\le v_{n+1}(t,x) \le \dots \\ &\le w_{n+1}(t,x) \le w_n(t,x) \le \dots \le w_0(t,x) \end{aligned}$$

for $(t,x) \in \overline{\Omega}$, $n = 1,2,\dots;$

(d) $u(t,x) = \lim_{n\to\infty} v_n(t,x)$, *where the limit is meant in the uniform sense, is a classical solution of problem* (2.1)–(2.2) *in* $\overline{\Omega}$ *satisfying the growth condition* $|u^i(t,x)| \le D \exp(\phi(t)|x|^2)$ *for* $(t,x) \in \overline{\Omega}$. *Moreover,* $u(t,x)$ *is Hölder continuous with respect to* x *uniform in* t;

(e) *in the class of all functions belonging to* $\mathcal{N}_u(\overline{\Omega})$ *the function* u *is the unique solution of problem* (2.1)–(2.2) *in* Ω.

Proof. (a) As in the proof of Theorem 4.5, starting from v_0 and w_0, we define by induction the sequences $\{v_n\}$, $\{w_n\}$ as solutions of (5.1), (5.2) with the nonlocal initial condition (5.3), (5.4), respectively, in $\overline{\Omega}$, i.e.

$$v_1 = \mathcal{P}[v_0], \quad v_n = \mathcal{P}[v_{n-1}], \quad w_1 = \mathcal{P}[v_0], \quad w_n = \mathcal{P}[v_{n-1}]$$

for $n = 1,2,\dots$. Here, the i-th equation depends on the i-th unknown function only as well, and Assumption 5.4 holds, therefore the classical theorems on the existence and uniqueness of solution for linear Cauchy problems assert that there exist unique classical solutions v_n, w_n, in $E_S^{2,\phi}$ of problems (5.1), (5.3) and (5.2), (5.4), respectively (cf. [7]).

The proofs of steps (b) and (c) are analogous to those in Theorem 4.5, with Proposition 2 from [10] replaced by Proposition 3 from [10], Corollary 1 from [10] replaced by Corollary 2 from [10] and on noticing that the inequalities (c) guarantee that u satisfies the desired growth condition.

(d) First, we show that $u(t,x) = \lim_{n\to\infty} v_n(t,x)$ is continuous. To this end we show using the mathematical induction that $m_n^i(t,x) := w_n^i(t,x) - v_n^i(t,x) \geq 0$ satisfies

$$m_n^i(t,x) \leq N_0 \frac{[(L_1+L_2)t]^n}{n!} [\phi(t)]^{m/2} \exp(\phi(t)|x|^2), \tag{5.5}$$

$n = 0,1,\ldots$, for $(t,x) \in \overline{\Omega}$, $i \in S$.

The inequality for m_0^i is obvious. Suppose (5.5) holds for fixed $n \in \mathbb{N} \cup \{0\}$. Similarly to the proof of Theorem 4.5, the (L_f) condition yields

$$\begin{aligned} \mathcal{F}^i[m_{n+1}^i](t,x) &= f^i(t,x,w_n(t,x),w_n) - f^i(t,x,v_n(t,x),v_n) \\ &\leq L_1 \|m_n(t,x)\|_{B(S)} + L_2 \|m_n\|_{2,\phi,t} [\phi(t)]^{m/2} \exp(\phi(t)|x|^2) \\ &\leq (L_1+L_2)\|m_n\|_{2,\phi,t} [\phi(t)]^{m/2} \exp(\phi(t)|x|^2). \end{aligned}$$

By the definitions of $\|\cdot\|_{2,\phi,t}$ and the induction assumption

$$\|m_n\|_{2,\phi,t} \leq N_0 \frac{[(L_1+L_2)t]^n}{n!}.$$

Thus, finally,

$$\mathcal{F}^i[m_{n+1}^i](t,x) \leq N_0 \frac{(L_1+L_2)^{n+1} t^n}{n!} [\phi(t)]^{m/2} \exp(\phi(t)|x|^2) \quad \text{in } \Omega.$$

By the same argument as in the proof of Theorem 4.5, $m_{n+1}^i(0,x) = 0$ for $x \in \mathbb{R}^m$, $i \in S$. In order to apply the theorem on differential inequalities, let us consider the comparison system

$$\mathcal{F}^i[M_{n+1}^i](t,x) = N_0 \frac{(L_1+L_2)^{n+1} t^n}{n!} [\phi(t)]^{m/2} \exp(\phi(t)|x|^2)$$

for $(t,x) \in \overline{\Omega}$, $i \in S$, with the initial condition $M_{n+1}^i(0,x) = 0$ for $x \in \mathbb{R}^m$, $i \in S$. The solution of this comparison system can be estimated (comp. [10]) as follows

$$M_{n+1}^i(t,x) \leq N_0 \frac{t^{n+1}}{(n+1)!} (L_1+L_2)^{n+1} [\phi(t)]^{m/2} \exp(\phi(t)|x|^2).$$

Therefore, owing to Proposition 3 from [10] we get

$$m_{n+1}^i(t,x) \leq M_{n+1}^i(t,x) \leq N_0 \frac{[(L_1+L_2)t]^{n+1}}{(n+1)!} [\phi(t)]^{m/2} \exp(\phi(t)|x|^2),$$

for $(t,x) \in \overline{\Omega}$, $i \in S$, so, the induction step is proved. Thus,

$$\|m_n\|_{2,\phi} = \|\widetilde{m}_n\|_0 \xrightarrow{n\to\infty} 0, \quad \text{where } \widetilde{m}_n = \widetilde{w}_n - \widetilde{v}_n,$$
$$\widetilde{v}_n = v_n [\phi(t)]^{-m/2} \exp(-\phi(t)|x|^2), \quad \widetilde{w}_n = w_n [\phi(t)]^{-m/2} \exp(-\phi(t)|x|^2).$$

Therefore, as in the proof of Theorem 4.5 we conclude that $\widetilde{u} := \lim_{n\to\infty} \widetilde{v}_n$ is continuous and consequently so is $u = \widetilde{u}\exp(\phi(t)|x|^2)$.

To end the proof it is enough to repeat proofs of step (e) and (f) of Theorem 4.5. □

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