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LONG TIME EXISTENCE OF SOLUTIONS TO 2D NAVIER-STOKES EQUATIONS WITH INFLOW-OUTFLOW AND HEAT CONVECTION

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Abstract. Global existence of regular solutions to the Navier–Stokes equations for velocity and pressure coupled with the heat convection equation for temperature in cylindrical pipe with inflow and outflow in the two-dimensional case is shown. We assume the slip boundary conditions for velocity and the Neumann condition for temperature. First an appropriate estimate is shown and next the existence of solutions is proved by the Leray–Schauder fixed point theorem.

1. Introduction

We consider the problem

$$
\begin{aligned}
&v_{,t} + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) = \alpha(\theta)g && \text{in } \Omega^T = \Omega\times(0,T),\\
&\operatorname{div} v = 0 && \text{in } \Omega^T,\\
&\theta_{,t} + v\cdot\nabla\theta - \chi\Delta\theta = 0 && \text{in } \Omega^T,\\
&\overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau} = 0 && \text{on } S^T = S\times(0,T),\\
&v\cdot\overline{n} = 0,\quad \overline{n}\cdot\nabla\theta = 0 && \text{on } S_1^T,\\
&v\cdot\overline{n} = d,\quad \overline{n}\cdot\nabla\theta = \varphi > 0 && \text{on } S_2^T,\\
&v|_{t=0} = v_0,\quad \theta|_{t=0} = \theta_0 && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

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where $\alpha > 0$, $\Omega \subset \mathbb{R}^2$ is a bounded domain, Ω^T satisfies the weak horn condition (see [2, Section 8]) and is not axially symmetric, $S = \partial\Omega$, $v = (v_1(x,t), v_2(x,t)) \in \mathbb{R}^2$ is the velocity of the fluid, $\theta = \theta(x,t) \in \mathbb{R}$ the temperature, $p = p(x,t) \in \mathbb{R}$ the pressure, $g = (g_1(x,t), g_2(x,t)) \in \mathbb{R}^2$ the external force, $\nu > 0$ the constant viscosity coeficient, $\chi > 0$ the constant heat coefficient. We assume that $S = S_1 \cup S_2$, where S_1 is the part of the boundary which is parallel to the axis x_2 and S_2 is perpendicular to x_2. Hence

$$\begin{aligned} S_1 &= \{x \in \mathbb{R}^2 : x_1 = b_1 < 0 \vee \ x_1 = b_2 > 0,\ -a \le x_2 \le a\}, \\ S_2 &= \{x \in \mathbb{R}^2 : b_1 \le x_1 \le b_2,\ x_2 = -a \vee x_2 = a\} \end{aligned}$$

and $\overline{n}$ the unit outward vector normal to S, $\overline{\tau}$ is the tangent wector to S.

By $\mathbb{T}(v,p)$ we denote the stress tensor

$$\mathbb{T}(v,p) = \nu\mathbb{D}(v) - pI,$$

where I is the unit matrix and $\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2}$ is the dilatation tensor.

To describe inflow and outflow we define

$$d_1 = -v \cdot \overline{n}|_{S_2(-a)}, \qquad d_2 = v \cdot \overline{n}|_{S_2(a)},$$

so $d_i \ge 0$, $i = 1,2$, and by $(1.1)_{2,6}$ we have the compatibility condition

$$\int_{S_2(-a)} d_1\, dS_2 = \int_{S_2(a)} d_2\, dS_2.$$

This paper extends the result from [5] to the inflow-outflow case.

Now we formulate the main result

Theorem 1.1. *Assume that $\alpha \in C^1(\mathbb{R})$, $v_0, \theta_0 \in W_s^{2-2/s}(\Omega)$, $v_{,t}(0), \theta_{,t}(0) \in L_2(\Omega)$, $d \in W_s^{2-1/s,1-1/2s}(S_2^T)$, $\varphi \in W_s^{1-1/s,1/2-1/2s}(S_2^T)$, $g, g_{,t} \in L_\infty(\Omega^T)$, $2 < s < 6$. Then there exists a solution (v,p,θ) of problem (1.1) such $v, \theta \in W_s^{2,1}(\Omega^T)$, $\nabla p \in L_s(\Omega^T)$ and constants $c > 0$, $c_1 > 0$, $c_2 > 0$ are such that*

$$\|v\|_{W_s^{2,1}(\Omega^T)} + \|\nabla p\|_{L_s(\Omega^T)} + \|\theta\|_{W_s^{2,1}(\Omega^T)} \le c$$

and $c_1 \le \theta \le c_2$.

The above result is an extension of the result from [5], where the long time existence of solutions is proved in the case without inflow and outflow. The considered in this paper problem has nonhomogeneous boundary conditions for velocity and temperature (see $(1.1)_6$) which needs many additional considerations comparing with [5] (see proofs of Lemmas 3.2–3.4). Since the basic estimates in this paper are obtained by the energy method the nonhomogeneous Dirichlet boundary condition for velocity must be made homogeneous by an appropriate extension (see (3.7) and the transformation before (3.7)). The inflow-outflow problem for the Navier–Stokes equations was considered in [8], where the Besov spaces are used so the proof becomes more complicated.

2. Notation

Let us consider the Stokes problem

$$\begin{aligned} &v_{,t} - \operatorname{div}\mathbb{T}(v,p) = f && \text{in } \Omega^T,\\ &\operatorname{div} v = 0 && \text{in } \Omega^T,\\ &\overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau} = g && \text{on } S^T,\\ &v\cdot\overline{n} = d && \text{on } S^T,\\ &v|_{t=0} = v_0 && \text{in } \Omega. \end{aligned} \tag{2.1}$$

Theorem 2.1. *Let $f\in L_q(\Omega^T)$, $v_0\in W_q^{2-2/q}(\Omega)$, $d\in W_q^{2-1/q,1-1/2q}(S^T)$, $\int_S d\,dS = 0$, $g\in W_q^{1-1/q,1/2-1/2q}(S^T)$, $q\in(1,\infty)$. Then there exists a unique solution to problem* (2.1) *such that $v\in W_q^{2,1}(\Omega^T)$, $\nabla p\in L_q(\Omega^T)$ and*

$$\begin{aligned} \|v\|_{W_q^{2,1}(\Omega^T)} + \|\nabla p\|_{L_q(\Omega^T)} \le c(&\|f\|_{L_q(\Omega^T)} + \|v_0\|_{W_q^{2-2/q}(\Omega)}\\ &+ \|d\|_{W_q^{2-1/q,1-1/2q}(S^T)} + \|g\|_{W_q^{1-1/q,1/2-1/2q}(S^T)}). \end{aligned}$$

Next we consider the following problem

$$\begin{aligned} &\theta_{,t} - \Delta\theta = f && \text{in } \Omega^T,\\ &\overline{n}\cdot\nabla\theta = d && \text{on } S^T,\\ &\theta|_{t=0} = \theta_0 && \text{in } \Omega. \end{aligned} \tag{2.2}$$

Theorem 2.2. *Let $f\in L_q(\Omega^T)$, $v_0\in W_q^{2-2/q}(\Omega)$, $d\in W_q^{1-1/q,1/2-1/2q}(S^T)$, $q\in(1,+\infty)$. Then there exists a unique solution to problem (2.2) such that $\theta\in W_q^{2,1}(\Omega^T)$ and*

$$\|\theta\|_{W_q^{2,1}(\Omega)} \le c(\|f\|_{L_q(\Omega^T)} + \|\theta_0\|_{W_q^{2-2/q}(\Omega)} + \|d\|_{W_q^{1-1/q,1/2-1/2q}(S^T)}). \tag{2.3}$$

Theorem 2.3 (see [2, Chapter 3, Section 10]). *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain such that Ω^T satisfies the weak horn condition and let $u\in W_s^{2,1}(\Omega^T)\cap L_2(\Omega^T)$. Then the following interpolation inequality holds*

$$\|\nabla u\|_{L_q(\Omega^T)} \le \varepsilon\|u\|_{W_s^{2,1}(\Omega^T)} + c/\varepsilon\|\nabla u\|_{L_2(\Omega^T)}, \tag{2.4}$$

for $s,q\in(1,\infty)$ satisfying $1/s-1/q<1/(n+2)$, $q>2$.

Theorem 2.4 (Korn inequality, see [7], [8]). *Assume that $\Omega\subset\mathbb{R}^n$ is not invariant with respect to any rotation. Assume that*

$$\|\mathbb{D}(u)\|_{L_2(\Omega)} < \infty,\quad u\cdot\overline{n}|_S = 0,\quad \operatorname{div} u = 0. \tag{2.5}$$

Then $\|u\|_{H^1(\Omega)} \le c\|\mathbb{D}(u)\|_{L_2(\Omega)}$.

3. Estimates

We show estimates for the temperature

Lemma 3.1. *Assume* $\theta(0)\geq c_1>0$. *Assume that* $\varphi\geq 0$, $d\in L_2(0,T;L_\infty(S_2))$. *Then, for* θ *sufficiently regular, we have*

$$\theta(t)\geq c_1,\quad t\geq 0. \tag{3.1}$$

Proof. Let $(\theta-c_1)_-=\min\{0,\theta-c_1\}$. Multiplying $(1.1)_3$ by $(\theta-c_1)_-$ integrating over Ω we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega(\theta-c_1)_-^2\,dx&+\chi\int_\Omega|\nabla(\theta-c_1)_-|^2\,dx\\&=-\frac{1}{2}\int_S\overline{n}\cdot v(\theta-c_1)_-^2\,dS+\chi\int_S\overline{n}\cdot\nabla(\theta-c_1)_-(\theta-c_1)_-\,dS\\&=-\frac{1}{2}\int_{S_2}d(\theta-c_1)_-^2\,dS_2+\chi\int_{S_2}\varphi(\theta-c_1)_-\,dS_2.\end{aligned}$$

Applying some interpolation inequality and inequality $\varphi>0$, we obtain

$$\frac{d}{dt}\|(\theta-c_1)_-\|^2_{L_2(\Omega)}+\|\nabla(\theta-c_1)_-\|^2_{L_2(\Omega)}\leq c\|d\|^2_{L_\infty(S_2)}\|(\theta-c_1)_-\|^2_{L_2(\Omega)}.$$

Finally, using the Gronwall inequality, we have

$$\begin{aligned}\|(\theta-c_1)_-\|^2_{L_\infty(0,T;L_2(\Omega))}&+\|\nabla(\theta-c_1)_-\|^2_{L_2(\Omega^T)}\\&\leq\exp(c\|d\|^2_{L_2(0,T;L_\infty(S_2))})\|(\theta-c_1)_-(0)\|_{L_2(\Omega)}.\end{aligned}$$

Since $(\theta-c_1)_-(0)=0$ we conclude the proof. □

Lemma 3.2. *Assume* $d\in L_2(0,T;L_\infty(S_2))$, $\varphi\in L_2(S_2^T)$, $\theta(0)\in L_2(\Omega)$. *Then*

$$\begin{aligned}&\|\theta\|^2_{L_\infty(0,T;L_2(\Omega))}+\|\nabla\theta\|^2_{L_2(\Omega^T)}\\&\qquad\leq c\exp(c\|d\|^2_{L_2(0,T;L_\infty(S_2))})\cdot(\|\varphi\|^2_{L_2(S_2^T)}+\|\theta(0)\|^2_{L_2(\Omega)})\equiv A_1.\end{aligned}\tag{3.2}$$

Proof. Multiplying $(1.1)_3$ by θ and integrating over Ω we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega\theta^2\,dx+\chi\int_\Omega|\nabla\theta|^2\,dx&=-\frac{1}{2}\int_S\overline{n}\cdot v\theta^2\,dS+\chi\int_S\overline{n}\cdot\nabla\theta\theta\,dS\\&=-\frac{1}{2}\int_{S_2}d\theta^2\,dS_2+\chi\int_{S_2}\varphi\theta\,dS_2.\end{aligned}$$

Using some interpolation and the Young inequality we obtain

$$\frac{d}{dt}\|\theta\|^2_{L_2(\Omega)}+\|\nabla\theta\|^2_{L_2(\Omega)}\leq c(\|d\|^2_{L_\infty(S_2)}\|\theta\|^2_{L_2(\Omega)}+\|\varphi\|^2_{L_2(S_2)}+\|\theta\|^2_{L_2(\Omega)}).$$

Finally, by the Gronwall inequality, we obtain (3.2). □

Lemma 3.3. *Assume that θ is sufficiently regular. Then there exists a constant $c_2 > 0$ such that*

$$\|\theta\|_{L_\infty(\Omega^T)} \le c_2. \tag{3.3}$$

Proof. Multiplying $(1.1)_3$ by θ^{p-1} integrating over Ω and using the boundary conditions we obtain

$$\frac{1}{p}\frac{d}{dt}\|\theta\|^p_{L_p(\Omega)} + \frac{1}{p}\int_{S_2} d\theta^p\,dS_2 + \chi\frac{4(p-1)}{p^2}\int_\Omega |\nabla\theta^{p/2}|^2\,dx - \chi\int_{S_2}\varphi\theta^{p-1}\,dS_2 = 0.$$

Applying the Hölder and an interpolation inequality

$$\|\theta\|^p_{L_p(S_2)} \le \varepsilon\|\nabla\theta^{p/2}\|^2_{L_2(\Omega)} + c\left(\frac{1}{\varepsilon}\right)\|\theta\|^p_{L_p(\Omega)}$$

we have

$$\frac{d}{dt}\|\theta\|^p_{L_p(\Omega)} + \|\nabla\theta^{p/2}\|^2_{L_2(\Omega)} \le c(\|d\|^2_{L_\infty(S_2)}\|\theta\|^p_{L_p(\Omega)} + \|\varphi\|_{L_\infty(S_2)}\|\theta\|^{p-1}_{L_p(S_2)}). \tag{3.4}$$

Now we estimate the last term from the r.h.s. of (3.4)

$$\begin{aligned}\|\varphi\|_{L_\infty(S_2)}\|\theta\|^{p-1}_{L_p(S_2)} &\le \|\varphi\|_{L_\infty(S_2)}\|\nabla\theta^{p/2}\|^{(p-1)/p}_{L_2(\Omega)}\|\theta^{p/2}\|^{(p-1)/p}_{L_2(\Omega)}\\ &\le \|\varphi\|_{L_\infty(S_2)}(\varepsilon\|\nabla\theta^{p/2}\|^2_{L_2(\Omega)} + c(1/\varepsilon)\|\theta^{p/2}\|^{2(p-1)/(p+1)}_{L_2(\Omega)}) \equiv I.\end{aligned} \tag{3.5}$$

From (3.1) $\int_\Omega \theta^p\,dx \ge \int_\Omega c_1\,dx = c_2$, therefore

$$\frac{\|\theta\|^{2p/(p+1)}_{L_p(\Omega)}}{c_2^{2/(p+1)}} \ge 1.$$

Using the last inequality in (3.5) we obtain

$$I \le \|\varphi\|_{L_\infty(S_2)}(\varepsilon\|\nabla\theta^{p/2}\|^2_{L_2(\Omega)} + c(1/\varepsilon)c_2^{2/(p+1)}\|\theta\|^p_{L_p(\Omega)}). \tag{3.6}$$

Then (3.4), (3.5) and (3.6) imply

$$\|\theta\|^{p-1}_{L_p(\Omega)}\frac{d}{dt}\|\theta\|_{L_p(\Omega)} \le c\|\theta\|^p_{L_p(\Omega)}.$$

Finally, using the Gronwall inequality and passing with p to infinity, we obtain inequality

$$\|\theta\|_{L_\infty(\Omega^T)} \le ce^{cT}\|\theta(0)\|_{L_\infty(\Omega)}.$$ □

To obtain the energy type estimate for solutions to problem (1.1) we introduce the function

$$\beta|_{S_2(-a)} = d_1, \qquad \beta|_{S_2(a)} = d_2.$$

Then we define u by

$$\operatorname{div} u = -\operatorname{div} b, \qquad u\cdot\overline{n}|_S = 0,$$

where $b = (0,\beta)$.

Next we introduce the function φ by

$$\Delta\varphi = -\operatorname{div} b \quad \text{in } \Omega, \qquad \overline{n}\cdot\nabla\varphi = 0 \quad \text{on } S, \qquad \int_\Omega \varphi\, dx = 0.$$

Finally, the function

$$w = v - (b + \nabla\varphi) \equiv v - \delta_1$$

and p is a solution to the problem

$$\begin{aligned}
&w_{,t} - \operatorname{div}\mathbb{T}(w,p) = -\delta_{1,t} + \nu \operatorname{div}\mathbb{D}(\delta_1) && \\
&\quad + (w+\delta_1)\cdot\nabla(w+\delta_1) + \alpha(\theta)g && \text{in } \Omega^T,\\
&\operatorname{div} w = 0 && \text{in } \Omega^T,\\
&\overline{n}\cdot w = 0 && \text{on } S^T,\\
&\nu\overline{n}\cdot\mathbb{D}(w+\delta_1)\cdot\overline{\tau} = 0 && \text{on } S^T,\\
&w|_{t=0} = v(0) && \text{in } \Omega.
\end{aligned} \tag{3.7}$$

Multiplying (3.7) by w and integrating over Ω yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w\|^2_{L_2(\Omega)} - \int_\Omega \operatorname{div}\mathbb{T}(w,p)\cdot w\, dx = \int_\Omega (-\delta_{1,t} + \nu\operatorname{div}\mathbb{D}(\delta_1))\cdot w\, dx \\
+ \int_\Omega (w+\delta_1)\cdot\nabla(w+\delta_1)\cdot w\, dx + \int_\Omega \alpha(\theta) g\cdot w\, dx \equiv \sum_{i=1}^3 I_i.
\end{aligned} \tag{3.8}$$

Since $\alpha\in C^1(\mathbb{R})$ Lemmas 3.1 and 3.3 imply that

$$\|\alpha(\theta)\|^2_{L_\infty(\Omega^T)} \le c_3.$$

The second term on the l.h.s. equals

$$-\int_S \overline{n}\cdot\mathbb{T}(w,p)\cdot w\, dS + \nu\int_\Omega |\mathbb{D}(w)|^2\, dx,$$

where the boundary term assumes the form

$$\int_S \nu\overline{n}\cdot\mathbb{D}(\delta_1)\cdot\overline{\tau} w\cdot\overline{\tau}\, dS.$$

Hence

$$-\int_S \overline{n}\cdot\mathbb{T}(w,p)\cdot w\, dS \le \varepsilon\|w\|^2_{H^1(\Omega)} + c/\varepsilon\|\delta_1\|^2_{H^2(\Omega)}.$$

Now we estimate the terms from the r.h.s. of (3.8)

$$\begin{aligned}
|I_1| &\le \varepsilon\|w\|^2_{H^1(\Omega)} + c/\varepsilon(\|\delta_{1,t}\|^2_{L_2(\Omega)} + \|\delta_1\|^2_{H^2(\Omega)}),\\
|I_2| &\le \varepsilon\|w\|^2_{H^1(\Omega)} + c/\varepsilon(\|\delta_1\|^2_{L_\infty(\Omega)}\|w\|^2_{L_2(\Omega)} + \|\delta_1\|^4_{H^1(\Omega)}),\\
|I_3| &\le \varepsilon\|w\|^2_{H^1(\Omega)} + c/\varepsilon c_3\|g\|^2_{L_2(\Omega)}.
\end{aligned}$$

Using the Korn inequality and assuming that ε is sufficiently small we obtain the inequality

$$
\begin{aligned}
(3.8') \quad \frac{d}{dt}\|w\|^2_{L_2(\Omega)} + \|w\|^2_{H^1(\Omega)} \le{}& c(\|\delta_1\|^2_{L_\infty(\Omega)}\|w\|^2_{L_2(\Omega)} + \|\delta_1\|^2_{H^2(\Omega)} \\
&+ \|\delta_{1,t}\|^2_{L_2(\Omega)} + \|\delta_1\|^4_{H^1(\Omega)} + c_3\|g\|^2_{L_2(\Omega)}).
\end{aligned}
$$

Continuing, we get

$$
\begin{aligned}
(3.9) \quad & \|w\|^2_{L_\infty(0,T;L_2(\Omega))} + \|w\|^2_{L_2(0,T;H^1(\Omega))} \\
&\le c\exp(\|\delta_1\|^2_{L_2(0,T;L_\infty(\Omega))})(\|\delta_1\|^2_{L_2(0,t;H^2(\Omega))} + \|\delta_{1,t}\|^2_{L_2(\Omega^T)} \\
&\quad + \|\delta_1\|^4_{L_4(0,T;H^1(\Omega))} + c_3\|g\|^2_{L_2(\Omega^T)} + \|w(0)\|^2_{L_2(\Omega)}).
\end{aligned}
$$

Differentiating $(3.7)_1$ with respect to t, multiplying by $w_{,t}$, integrating over Ω we obtain

$$
\begin{aligned}
(3.10) \quad \frac{1}{2}\frac{d}{dt}\|w_{,t}\|^2_{L_2(\Omega)} &- \int_\Omega \operatorname{div}\mathbb{T}(w_{,t},p_{,t})\cdot w_{,t}\,dx \\
&= \int_\Omega(-\delta_{1,tt} + \nu\operatorname{div}\mathbb{D}(\delta_{1,t}))\cdot w_t\,dx \\
&\quad + \int_\Omega (w_{,t}+\delta_{1,t})\cdot\nabla(w+\delta_1)\cdot w_{,t}\,dx \\
&\quad + \int_\Omega (w+\delta_1)\cdot\nabla(w_{,t}+\delta_{1,t})\cdot w_{,t}\,dx \\
&\quad + \int_\Omega \alpha_{,\theta}\theta_{,t}gw_{,t}dx + \int_\Omega \alpha(\theta)g_{,t}\cdot w_{,t}\,dx \equiv \sum_{i=1}^5 I_i.
\end{aligned}
$$

The second term on the l.h.s. equals

$$
-\int_S \overline{n}\cdot\mathbb{T}(w_{,t},p_{,t})\cdot w_{,t}\,dS + \nu\int_\Omega |\mathbb{D}(w_{,t})|^2\,dx,
$$

where the boundary term assumes the form

$$
\int_S \nu\overline{n}\cdot\mathbb{D}(\delta_{1,t})\cdot\overline{\tau}w_{,t}\cdot\overline{\tau}\,dS
$$

Hence

$$
\int_S \overline{n}\cdot\mathbb{T}(w_{,t},p_{,t})\cdot w_{,t}\,dS \le \varepsilon\|w_{,t}\|^2_{H^1(\Omega)} + c/\varepsilon\|\delta_{1,t}\|^2_{H^2(\Omega)}.
$$

Now we estimate the terms from the r.h.s. of (3.10)

$$
\begin{aligned}
I_1 &\le \varepsilon\|w_{,t}\|^2_{H^1(\Omega)} + c/\varepsilon(\|\delta_{1,tt}\|^2_{L_2(\Omega)} + \|\delta_{1,t}\|^2_{H^2(\Omega)}), \\
I_2 &\le \|w_{,t}\|^2_{L_4(\Omega)}(\|\nabla w\|_{L_2(\Omega)} + \|\nabla\delta_1\|_{L_2(\Omega)}) \\
&\quad + \|w_{,t}\|_{L_4(\Omega)}\|\delta_{1,t}\|_{L_4(\Omega)}(\|\nabla w\|_{L_2(\Omega)} + \|\nabla\delta_1\|_{L_2(\Omega)}) \\
&\le \varepsilon\|w_{,t}\|^2_{H^1(\Omega)} + c/\varepsilon(\|\nabla w\|^2_{L_2(\Omega)} + \|\nabla\delta_1\|^2_{L_2(\Omega)})\|w_{,t}\|^2_{L_2(\Omega)} \\
&\quad + c/\varepsilon\|\delta_{1,t}\|^2_{L_4(\Omega)}(\|\nabla w\|^2_{L_2(\Omega)} + \|\nabla\delta_1\|^2_{L_2(\Omega)}),
\end{aligned}
$$

$$
\begin{aligned}
I_3 \le{}& \varepsilon\|w_{,t}\|^2_{H^1(\Omega)} + c/\varepsilon(\|\delta_1\|^2_{L_\infty(\Omega)}\|w_{,t}\|^2_{L_2(\Omega)} + \|\delta_1\|^2_{L_4(\Omega)}\|\nabla\delta_{1,t}\|^2_{L_2(\Omega)}\cdot\\
&+ \|w\|^2_{L_4(\Omega)}\|\nabla\delta_{1,t}\|^2_{L_2(\Omega)}),\\
I_4 \le{}& \varepsilon\|w_{,t}\|^2_{H^1(\Omega)} + c/\varepsilon\|g\|^2_{L_\infty(\Omega)}\|\theta_{,t}\|^2_{L_2(\Omega)}c_4,\\
I_5 \le{}& \varepsilon\|w_{,t}\|^2_{H^1(\Omega)} + c/\varepsilon\|g_{,t}\|^2_{L_2(\Omega)}c_3,
\end{aligned}
$$

where we assumed that $\|\alpha(\theta)\|^2_{L_\infty(\Omega^T)} \le c_3$ and $\|\alpha_{,\theta}(\theta)\|^2_{L_\infty(\Omega^T)} \le c_4$.

Using the Korn inequality and assuming that ε is sufficietnly small we obtain the inequality

$$
\begin{aligned}
(3.11)\quad \frac{d}{dt}\|w_{,t}\|^2_{L_2(\Omega)} + \|w_{,t}\|^2_{H^1(\Omega)} \le{}& c(\|\delta_{1,tt}\|^2_{L_2(\Omega)} + \|\delta_{1,t}\|^2_{H^2(\Omega)}\\
&+ (\|\nabla w\|^2_{L_2(\Omega)} + \|\delta_1\|^2_{L_\infty(\Omega)} + \|\nabla\delta_1\|^2_{L_2(\Omega)})\|w_{,t}\|^2_{L_2(\Omega)}\\
&+ \|\delta_{1,t}\|^2_{L_4(\Omega)}(\|\nabla w\|^2_{L_2(\Omega)} + \|\nabla\delta_1\|^2_{L_2(\Omega)}) + \|g_{,t}\|^2_{L_2(\Omega)}c_3\\
&+ (\|\delta_1\|^2_{L_4(\Omega)} + \|w\|^2_{L_4(\Omega)})\|\nabla\delta_{1,t}\|^2_{L_4(\Omega)} + \|g\|^2_{L_\infty(\Omega)}\|\theta_{,t}\|^2_{L_2(\Omega)}c_4).
\end{aligned}
$$

Differentiating $(1.1)_3$ with respect to t, multiplying by $\theta_{,t}$, integrating over Ω we have

$$
\begin{aligned}
(3.12)\quad \frac{1}{2}\frac{d}{dt}\|\theta_{,t}\|^2_{L_2(\Omega)} - \chi\int_\Omega \Delta\theta_{,t}\theta_{,t}\,dx&\\
= -\int_\Omega v_{,t}\cdot\nabla\theta\theta_{,t}\,dx - \int_\Omega v\cdot\nabla\theta_{,t}\theta_{,t}\,dx &= \sum_{i=1}^2 -I_i.
\end{aligned}
$$

The second term on the l.h.s. equals

$$
-\chi\int_{S_2}\varphi_{,t}\theta_{,t}\,dS_2 + \chi\|\nabla\theta_{,t}\|^2_{L_2(\Omega)},
$$

where the boundary term we estimate by

$$
\chi\int_{S_2}\varphi_{,t}\theta_{,t}\,dS_2 \le \chi(\varepsilon\|\theta_{,t}\|^2_{H^1(\Omega)} + c(1/\varepsilon)\|\varphi_{,t}\|^2_{L_2(S_2)}).
$$

Now we estimate the terms from the r.h.s. of (3.12)

$$
\begin{aligned}
I_1 \le{}& \|v_{,t}\|_{L_4(\Omega)}\|\nabla\theta\|_{L_2(\Omega)}\|\theta_{,t}\|_{L_4(\Omega)}\\
\le{}& c(\|\nabla v_{,t}\|^{1/2}_{L_2(\Omega)}\|v_{,t}\|^{1/2}_{L_2(\Omega)} + \|v_{,t}\|_{L_2(\Omega)})(\|\nabla\theta_{,t}\|^{1/2}_{L_2(\Omega)}\|\theta_{,t}\|^{1/2}_{L_2(\Omega)}\\
&+ \|\theta_{,t}\|_{L_2(\Omega)})\|\nabla\theta\|_{L_2(\Omega)}\\
\le{}& c((\|\nabla v_{,t}\|_{L_2(\Omega)}\|v_{,t}\|_{L_2(\Omega)} + \|\nabla\theta_{,t}\|_{L_2(\Omega)}\|\theta_{,t}\|_{L_2(\Omega)})\|\nabla\theta\|_{L_2(\Omega)}\\
&+ (\|v_{,t}\|^2_{L_2(\Omega)} + \|\theta_{,t}\|^2_{L_2(\Omega)})\|\nabla\theta\|^2_{L_2(\Omega)} + \varepsilon\|\theta_{,t}\|^2_{L_2(\Omega)})\\
\le{}& c(\varepsilon(\|\nabla v_{,t}\|^2_{L_2(\Omega)} + \|\nabla\theta_{,t}\|^2_{L_2(\Omega)})\\
&+ (\|v_{,t}\|^2_{L_2(\Omega)} + \|\theta_{,t}\|^2_{L_2(\Omega)})\|\nabla\theta\|^2_{L_2(\Omega)} + \varepsilon\|\theta_{,t}\|^2_{L_2(\Omega)}),\\
I_2 \le{}& c/\varepsilon\|d\|^2_{L_2(S_2)} + \varepsilon\|\theta_{,t}\|^2_{H^1(\Omega)}.
\end{aligned}
$$

Assuming that ε is sufficiently small we obtain the inequality

$$\begin{aligned}(3.13)\quad &\frac{d}{dt}\|\theta_{,t}\|^2_{L_2(\Omega)} + \|\theta_{,t}\|^2_{H^1(\Omega)} \le c(\varepsilon\|\nabla v_{,t}\|^2_{L_2(\Omega)} \\ &\quad + \|\nabla\theta\|^2_{L_2(\Omega)}(\|v_{,t}\|^2_{L_2(\Omega)} + \|\theta_{,t}\|^2_{L_2(\Omega)}) + \|d\|^2_{L_2(S_2)} + \|\varphi_{,t}\|^2_{L_2(S_2)}).\end{aligned}$$

Then adding (3.8′), (3.11), (3.13), we obtain the result

Lemma 3.4. *Assume that* $\alpha \in C^1(\mathbb{R})$, $\delta_1 \in L_\infty(\Omega^T) \cap L_4(0,T;H^2(\Omega))$, $\varphi_{,t} \in L_2(S_2^T)$, $\delta_{1,t} \in L_\infty(\Omega^T) \cap L_2(0,T;H^2(\Omega))$, $\delta_{1,tt} \in L_2(\Omega^T)$, $g \in L_\infty(\Omega^T)$, $g_{,t} \in L_2(\Omega^T)$, $d \in L_2(S_2^T)$, $v_0, \theta_0, v_{,t}(0), \theta_{,t}(0) \in L_2(\Omega^T)$. *Then*

$$\begin{aligned}(3.14)\quad &\|w\|_{L_\infty(0,T;L_2(\Omega))} + \|w\|^2_{L_2(0,T;H^1(\Omega))} \\ &\le c\exp(\|\delta_1\|^2_{L_2(0,T;L_\infty(\Omega))})(\|\delta_1\|^2_{L_2(0,T;H^2(\Omega))} + \|\delta_{1,t}\|^2_{L_2(\Omega^T)} \\ &\quad + \|\delta_1\|^4_{L_4(0,T;H^1(\Omega))} + c_3\|g\|^2_{L_2(\Omega^T)} + \|w(0)\|^2_{L_2(\Omega)}) \equiv A_2\end{aligned}$$

and

$$\begin{aligned}(3.15)\quad &\|w_{,t}\|^2_{L_\infty(0,T;L_2(\Omega))} + \|\theta_{,t}\|^2_{L_\infty(0,T;L_2(\Omega))} \\ &\quad + \|w_{,t}\|^2_{L_2(0,T;H^1(\Omega))} + \|\theta_{,t}\|^2_{L_2(0,T;H^1(\Omega))} \\ &\le c\exp(\|\nabla w\|^2_{L_2(\Omega^T)} + \|\delta_1\|^2_{L_2(0,T;L_\infty(\Omega))} + \|\nabla\delta_1\|^2_{L_2(\Omega^T)} \\ &\quad + \|g\|^2_{L_\infty(\Omega^T)} + \|\nabla\theta\|^2_{L_2(\Omega^T)})(\|\delta_{1,tt}\|^2_{L_2(\Omega^T)} + \|\delta_{1,t}\|^2_{L_2(0,T;H^2(\Omega))} \\ &\quad + \|\delta_{1,t}\|^2_{L_\infty(\Omega^T)}(\|w\|^2_{L_2(0,T;H^1(\Omega))} + \|\nabla\delta_1\|^2_{L_2(\Omega^T)}) \\ &\quad + (\|\delta_1\|^2_{L_\infty(\Omega^T)} + \|w\|^2_{L_2(0,T;H^1(\Omega))})\|\nabla\delta_{1,t}\|^2_{L_\infty(0,T;L_4(\Omega))} \\ &\quad + \|g_{,t}\|^2_{L_2(\Omega^T)} + \|\varphi_{,t}\|^2_{L_2(S_2^T)} \\ &\quad + \|d\|^2_{L_2(S_2^T)} + \|v_{,t}(0)\|^2_{L_2(\Omega)} + \|\theta_{,t}(0)\|^2_{L_2(\Omega^T)}) \equiv A_3.\end{aligned}$$

Lemma 3.5. *Let the assumptions of Lemma 3.4 be satisfied. Moreover, assume that* $v_0, \theta_0 \in W_s^{2-2/s}(\Omega)$, $d \in W_s^{2-1/s,1-1/2s}(S_2^T)$, *where* $s \in (1,6)$. *Then*

$$\begin{aligned}(3.16)\quad &\|v\|_{W_s^{2,1}(\Omega^T)} + \|\nabla p\|_{L_s(\Omega^T)} + \|\theta\|_{W_s^{2,1}(\Omega)} \le c(\|g\|_{L_s(\Omega^T)} \\ &\quad + \|v_0\|_{W_s^{2-2/s}(\Omega)} + \|\theta_0\|_{W_s^{2-2/s}(\Omega)} + \|d\|_{W_s^{2-1/s,1-1/2s}(S_2^T)} \\ &\quad + \|\varphi\|_{W_s^{1-1/s,1/2-1/2s}(S_2^T)} + A_1 + A_2 + A_3),\end{aligned}$$

where A_1, A_2, A_3 *are given by* (3.2), (3.14) *and* (3.15).

Proof. From (3.2), (3.14), (3.15) we have

$$\|v_{,t}\|_{L_\infty(0,t,L_2(\Omega))} + \|v\|_{L_2(0,t;H^1(\Omega))} + \|v\|_{L_\infty(0,t;L_2(\Omega))} \le c, \quad t \le T.$$

Hence $v \in H^1(\Omega^t)$ and then

$$(3.17)\qquad v \in L_6(\Omega^T).$$

Now we want to increase regularity described by (3.16). For this purpose we consider the problem

$$
\text{(3.18)}\qquad
\begin{aligned}
& v_{,t} + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) = \alpha(\theta)g && \text{in } \Omega^T,\\
& \operatorname{div} v = 0 && \text{in } \Omega^T,\\
& \overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau} = 0 && \text{on } S^T,\\
& v\cdot\overline{n} = 0 && \text{on } S_1^T,\\
& v\cdot\overline{n} = d && \text{on } S_2^T,\\
& v|_{t=0} = v_0 && \text{in } \Omega.
\end{aligned}
$$

To apply Theorem 2.1 we examine

$$
\|v\cdot\nabla v\|_{L_s(\Omega^t)} \le \|v\|_{L_{s\lambda_1}(\Omega^t)}\|\nabla v\|_{L_{s\lambda_2}(\Omega^t)} = I,
$$

where $1/(s\lambda_1)+1/(s\lambda_2)=1/s$. Assuming $s\lambda_1=6$ we obtain $s\lambda_2=6s/(6-s)$, then $4/s-4/(s\lambda_2)<1$, where $1<s<6$.

Hence in view of (2.4) and (3.17) we have

$$
I \le c\|\nabla v\|_{L_{s\lambda_2}(\Omega^T)} \le \varepsilon\|v\|_{W_s^{2,1}(\Omega^t)} + c/\varepsilon\|\nabla v\|_{L_2(\Omega^t)},
$$

where $1<s<6$.

Assuming that $g\in L_\infty(\Omega^T)$ we apply Theorem 2.1. Then we have

$$
\begin{aligned}
\text{(3.19)}\qquad \|v\|_{W_s^{2,1}(\Omega^t)} + \|\nabla p\|_{L_s(\Omega^t)} \le{}& c(\|\nabla v\|_{L_2(\Omega^t)}\|\theta\|_{W_s^{2,1}(\Omega^T)}\|g\|_{L_\infty(\Omega^T)}\\
&+ \|v_0\|_{W_s^{2-2/s}(\Omega)} + \|d\|_{W_s^{2-1/s,1-1/2s}(S^t)}),
\end{aligned}
$$

where $s\in(1,6)$. From (3.2), (3.14), (3.15) we have

$$
\text{(3.20)}\qquad \|\theta_{,t}\|_{L_\infty(0,t;L_2(\Omega))} + \|\theta\|_{L_2(0,t;H^1(\Omega))} + \|\theta\|_{L_\infty(0,t;L_2(\Omega))} \le c, \quad t\le T.
$$

Hence $\theta\in H^1(\Omega^t)$ and then

$$
\text{(3.21)}\qquad \theta\in L_6(\Omega^T).
$$

Now we want to increase regularity described by (3.16). For this purpose we consider the problem

$$
\text{(3.22)}\qquad
\begin{aligned}
& \theta_{,t} + v\cdot\nabla\theta - \chi\Delta\theta = 0 && \text{in } \Omega^T,\\
& \overline{n}\cdot\nabla\theta = 0 && \text{on } S_1^T,\\
& \overline{n}\cdot\nabla\theta = \varphi && \text{on } S_2^T,\\
& \theta|_{t=0} = \theta_0 && \text{in } \Omega.
\end{aligned}
$$

To apply Theorem 2.2 we examine

$$
\|v\cdot\nabla\theta\|_{L_s(\Omega^t)} \le \|v\|_{L_{s\lambda_1}(\Omega^t)}\|\nabla\theta\|_{L_{s\lambda_2}(\Omega^t)} = I_1,
$$

where $1/(s\lambda_1)+1/(s\lambda_1)=1$. Assuming $s\lambda_1=6$ we obtain $s\lambda_2=6s/(6-s)$, then $4/s-4/(s\lambda_2)<1$, where $1<s<6$.

Hence in view of (2.4) and (3.21) we have

$$I_1 \le c\|\nabla\theta\|_{L_{s\lambda_2}(\Omega^t)} \le \varepsilon\|\theta\|_{W_s^{2,1}(\Omega^t)} + c/\varepsilon\|\nabla\theta\|_{L_2(\Omega^t)},$$

where $1 < s < 6$. Then, for $s \in (1,6)$, we have

$$\|\theta\|_{W_s^{2,1}(\Omega^t)} \le c(\|\nabla\theta\|_{L_2(\Omega^t)} + \|\theta_0\|_{W_s^{2-2/s}(\Omega)} + \|\varphi\|_{W_s^{1-1/s,1/2-1/2s}(S^t)}). \qquad \square$$

4. Existence

For $2 < \eta < 6$ define $\mathcal{M}(\Omega^T) = \{(v,\theta) \in (L_\infty(0,T;W_\eta^1(\Omega))^2\}$. Let us consider the problems

$$\tag{4.1} \begin{aligned} & v_{,t} - \operatorname{div}\mathbb{T}(v,p) = -\lambda(\widetilde{v}\cdot\nabla\widetilde{v} + \alpha(\widetilde{\theta})g) && \text{in } \Omega^T,\\ & \operatorname{div} v = 0 && \text{in } \Omega^T,\\ & \overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau} = 0 && \text{on } S^T,\\ & v\cdot\overline{n} = 0 && \text{on } S_1^T,\\ & v\cdot\overline{n} = d && \text{on } S_2^T,\\ & v_{,t=0} = v_0 && \text{in } \Omega \end{aligned}$$

and

$$\tag{4.2} \begin{aligned} & \theta_{,t} - \chi\Delta\theta = -\lambda\widetilde{v}\cdot\nabla\widetilde{\theta} && \text{in } \Omega^T,\\ & \overline{n}\cdot\nabla\theta = 0 && \text{on } S_1^T,\\ & \overline{n}\cdot\nabla\theta = \varphi && \text{on } S_2^T,\\ & \theta|_{t=0} = \theta_0 && \text{in } \Omega, \end{aligned}$$

where $\lambda \in [0,1]$ and $\widetilde{v}$, $\widetilde{\theta}$ are treated as given functions.

LEMMA 4.1. *Let* $(\widetilde{v},\widetilde{\theta}) \in \mathcal{M}(\Omega^T)$, $g \in L_s(\Omega^T)$, $v_0,\theta_0 \in W_s^{2-2/s}(\Omega)$, $\alpha \in C^1(\mathbb{R})$, $d \in W_s^{2-1/s,1-1/2s}(S_2^T)$, $\varphi \in W_s^{1-1/s,1/2-1/2s}(S_2^T)$, *where* $2 < s < \eta < 6$, $4/s - 2/\eta < 1$. *Then there exists a unique solution to the problem* (4.1), (4.2) *such that* $v,\theta \in W_s^{2,1}(\Omega^T) \subset L_\infty(0,T;W_\eta^1(\Omega))$, *where the imbedding is compact and*

$$\tag{4.3} \begin{aligned} \|v\|_{L_\infty(0,T;W_\eta^1(\Omega))} &+ \|\theta\|_{W_\infty(0,T;W_\eta^1(\Omega))} \le \|v\|_{W_s^{2,1}(\Omega^T)} + \|\theta\|_{W_s^{2,1}(\Omega)}\\ &\le c(\lambda\|\widetilde{v}\|_{L_\infty(0,T;W_\eta^1(\Omega)}(\|\widetilde{\theta}\|_{L_\infty(0,T;W_\eta^1(\Omega))}\\ &\quad + \|\widetilde{v}\|_{L_\infty(0,T;W_\eta^1(\Omega))}) + c_3^{1/2}\|g\|_{L_s(\Omega^T)})\\ &\quad + \|v_0\|_{W_s^{2-2/s}(\Omega)} + \|\theta_0\|_{W_s^{2-2/s}(\Omega)}\\ &\quad + \|d\|_{W_s^{2-1/s,1-1/2s}(S_2^T)} + \|\varphi\|_{W_s^{1-1/s,1/2-1/2s}(S_2^T)}). \end{aligned}$$

PROOF. We have

$$\|\widetilde{v}\cdot\nabla\widetilde{v}\|_{L_s(\Omega^t)} \le c\|\widetilde{v}\|_{L_\infty(\Omega^T)}\|\nabla\widetilde{v}\|_{L_s(\Omega^T)} \le c\|\widetilde{v}\|^2_{L_\infty(0,T;W_\eta^1(\Omega))},$$

$$\|\widetilde{\theta}^\alpha g\|_{L_s(\Omega^T)} \le \|\alpha(\widetilde{\theta})\|_{L_\infty(\Omega^T)}\|g\|_{L_s(\Omega^T)} \le \alpha(\|\widetilde{\theta}\|_{L_\infty(0,T;W^1_\eta(\Omega))})\|g\|_{L_s(\Omega^T)},$$
$$\|\widetilde{v}\cdot\nabla\widetilde{\theta}\|_{L_s(\Omega^T)} \le c\|\widetilde{v}\|_{L_\infty(\Omega^T)}\|\nabla\widetilde{\theta}\|_{L_s(\Omega^T)} \le c\|\widetilde{v}\|_{L_\infty(0,T;W^1_\eta(\Omega))}\|\widetilde{\theta}\|_{L_\infty(0,T;W^1_\eta(\Omega))}.$$

By Theorems 2.1 and 2.2 the proof is complete. □

To prove the existence of solutions to problem (1.1) we apply the Leray-Schauder fixed point theorem (see [4]). Therefore we introduce the mapping $\phi\colon [0,1]\times\mathcal{M}(\Omega^T)\to (W^{2,1}_s(\Omega^T))^2$, $(\lambda,\widetilde{v},\widetilde{\theta})\to\phi(\lambda,\widetilde{v},\widetilde{\theta})=(v,\theta)$, where (v,θ) is a solution to problems (4.1), (4.2). For $\lambda=0$ we have the existence of a unique solutions. For $\lambda=1$ every fixed point is a solution to problem (1.1).

Lemma 4.2. *Let the assumptions of Lemma* 4.1 *be satisfied. Then mapping* $\phi(\lambda,\,\cdot\,)\colon\mathcal{M}(\Omega^T)\to\mathcal{M}(\Omega^T)$, $\lambda\in[0,1]$ *is completly continuous.*

Proof. By Lemma 4.1 the mapping $\phi(\lambda,\,\cdot\,)$, $\lambda\in[0,1]$, is compact. From this it follows that bounded set in $\eta(\Omega^T)$ are transformed into bounded sets in $M(\Omega^T)$. Let $(\widetilde{v}_i,\widetilde{\theta}_i)\in M(\Omega^T)$, $i=1,2$ be two given elements. Then (v_i,θ_i), $i=1,2$, are solutions to the problems

$$\tag{4.4}\begin{aligned}
&v_{i,t}-\operatorname{div}\mathbb{T}(v_i,p_i)=-\lambda(\widetilde{v}_i\cdot\nabla\widetilde{v}_i+\alpha(\widetilde{\theta}_i)g) &&\text{in } \Omega^T,\\
&\operatorname{div} v_i=0 &&\text{in } \Omega^T,\\
&\overline{n}\cdot\mathbb{D}(v_i)\cdot\overline{\tau}=0 &&\text{on } S^T,\\
&v_i\cdot\overline{n}=0 &&\text{on } S_1^T,\\
&v_i\cdot\overline{n}=d &&\text{on } S_2^T,\\
&v_i|_{t=0}=v_0 &&\text{in } \Omega
\end{aligned}$$

$$\tag{4.5}\begin{aligned}
&\theta_{i,t}-\chi\Delta\theta_i=-\lambda\widetilde{v}_i\cdot\nabla\widetilde{\theta}_i &&\text{in } \Omega^T,\\
&\overline{n}\cdot\nabla\theta_i=0 &&\text{on } S_1^T,\\
&\overline{n}\cdot\nabla\theta_i=\varphi &&\text{on } S_2^T,\\
&\theta_i|_{t=0}=\theta_0 &&\text{in } \Omega.
\end{aligned}$$

To show continuity we introduce the differences

$$V=v_1-v_2,\quad P=p_1-p_2,\quad \mathcal{T}=\theta_1-\theta_2,$$

which are solutions to the problems

$$\tag{4.6}\begin{aligned}
&V_{,t}-\operatorname{div}\mathbb{T}(V,P)=-\lambda(\widetilde{V}\cdot\nabla\widetilde{v}_1+\widetilde{v}_2\cdot\nabla V+(\alpha(\widetilde{\theta}_1)-\alpha(\widetilde{\theta}_2))g) &&\text{in } \Omega^T,\\
&\operatorname{div} V=0 &&\text{in } \Omega^T,\\
&\overline{n}\cdot\mathbb{D}\,(V)\cdot\overline{\tau}=0 &&\text{on } S^T,\\
&V\cdot\overline{n}=0 &&\text{on } S^T,\\
&V|_{t=0}=0 &&\text{in } \Omega
\end{aligned}$$

and

$$
\begin{aligned}
&\mathcal{T}_{,t} - \chi\Delta\mathcal{T} = -\lambda(\widetilde{V}\cdot\nabla\widetilde{\theta}_1 + \widetilde{v}_2\cdot\nabla\widetilde{\mathcal{T}}) && \text{in } \Omega^T,\\
&\overline{n}\cdot\nabla\mathcal{T} = 0 && \text{on } S^T,\\
&\mathcal{T}|_{t=0} = 0 && \text{in } \Omega,
\end{aligned}
\tag{4.7}
$$

where $\widetilde{V} = \widetilde{v}_1 - \widetilde{v}_2$, $\widetilde{\mathcal{T}} = \widetilde{\theta}_1 - \widetilde{\theta}_2$.

In view of Therorems 2.1, 2.2 we have

$$
\begin{aligned}
\|V\|_{W_s^{2,1}(\Omega^T)} &+ \|\mathcal{T}\|_{W_s^{2,1}(\Omega^T)}\\
&\le c(\|\widetilde{V}\|_{L_\infty(\Omega^T)}\|\nabla\widetilde{v}_1\|_{L_s(\Omega^T)} + \|\widetilde{v}_2\|_{L_\infty(\Omega^T)}\|\nabla\widetilde{V}\|_{L_s(\Omega^T)}\\
&\quad + c\|\mathcal{T}\|_{L_\infty(\Omega^T)}\|g\|_{L_\infty(\Omega^T)} + \|\widetilde{V}\|_{L_\infty(\Omega^T)}\|\nabla\widetilde{\theta}_1\|_{L_s(\Omega^T)})\\
&\le c(\|\widetilde{V}\|_{L_\infty(0,T;W_\eta^1(\Omega))} + \|\widetilde{\mathcal{T}}\|_{L_\infty(0,T;W_\eta^1(\Omega))}),
\end{aligned}
$$

so continuity of ϕ follows. □

References

[1] W. Alame, *On existence of solutions for the nonstationary Stokes system with slip boundary conditions*, Appl. Math., Warsaw **32** (2005), 195–223.

[2] O. V. Besov, V. P. Il'in and S. M. Nikol'skiĭ, *Integral Representations of Functions and Imbedding Therorems*, "Nauka", Moscow, 1975. (Russian)

[3] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, "Nauka", Moscow, 1967. (Russian)

[4] I. Pawłow and W. M. Zajączkowski, *Global existence to a three-dimensional nonlinear thermoelasticity system arising in shape memory materials*, Math. Methods Appl. Sci. **28** (2005), 407–442.

[5] J. Socała, W. M. Zajączkowski, *Long time existence of solutions to* 2D *Navier–Stokes equations with heat convection*, Appl. Math., Warsaw **36** (2009), 453–463.

[6] V. A. Solonnikov, *A priori estimates for second order parabolic equations*, Trudy Mat. Inst. Steklov. **70** (1964), 133–212. (Russian)

[7] W. M. Zajączkowski, *Global existence of axially symmetric solutions to Navier–Stokes equations with large angular component of velocity*, Colloq. Math. **100** (2004), 243–263.

[8] ———, *Global special regular solutions to the Navier-Stokes equations in a cylindrical domain under boundary slip conditions*, Gakuto Series in Math. **21** (2004), 188.

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