Rev. Mat. Iberoamericana **22** (2006), no. 3, 867–892

m-Berezin transform and compact operators

Kyesook Nam, Dechao Zheng and Changyong Zhong

Abstract

m-Berezin transforms are introduced for bounded operators on the Bergman space of the unit ball. The norm of the m-Berezin transform as a linear operator from the space of bounded operators to L^∞ is found. We show that the m-Berezin transforms are commuting with each other and Lipschitz with respect to the pseudo-hyperbolic distance on the unit ball. Using the m-Berezin transforms we show that a radial operator in the Toeplitz algebra is compact iff its Berezin transform vanishes on the boundary of the unit ball.

1. Introduction

Let B denote the unit ball in n-dimensional complex space $\mathbb{C}^n$ and dz be normalized Lebesgue volume measure on B. The Bergman space $L_a^2 = L_a^2(B, dz)$ is the space of analytic functions h on B which are square-integrable with respect to Lebesgue volume measure. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, let $\langle z, w\rangle = \sum_{i=1}^n z_i \overline{w}_i$ and $|z|^2 = \langle z, z\rangle$.

For $z \in B$, let P_z be the orthogonal projection of $\mathbb{C}^n$ onto the subspace $[z]$ generated by z and let $Q_z = I - P_z$. Then

$$\phi_z(w) = \frac{z - P_z(w) - (1-|z|^2)^{1/2} Q_z(w)}{1 - \langle w, z\rangle}$$

is the automorphism of B that interchanges 0 and z. The pseudo-hyperbolic metric on B is defined as $\rho(z, w) = |\phi_z(w)|$.

The reproducing kernel in L_a^2 is given by

$$K_z(w) = \frac{1}{(1 - \langle w, z\rangle)^{n+1}},$$

for $z, w \in B$ and the normalized reproducing kernel k_z is $K_z(w)/\|K_z(\cdot)\|_2$.

2000 Mathematics Subject Classification: 47B35, 47B38.
Keywords: m-Berezin transforms, Toeplitz operators.

If $\langle \cdot, \cdot \rangle$ denotes the inner product in L^2, then $\langle h, K_z\rangle = h(z)$, for every $h \in L_a^2$ and $z \in B$. The fundamental property of the reproducing kernel $K_z(w)$ plays an important role in this paper:

$$K_z(w) = \overline{k_\lambda(z)} K_{\phi_\lambda(z)}(\phi_\lambda(w)) k_\lambda(w). \tag{1.1}$$

Given $f \in L^\infty$, the Toeplitz operator T_f is defined on B by $T_f h = P(fh)$ where P denotes the orthogonal projection P of L^2 onto L_a^2.

Let $\mathfrak{L}(L_a^2)$ be the algebra of bounded operators on L_a^2. The Toeplitz algebra $\mathfrak{T}(L^\infty)$ is the closed subalgebra of $\mathfrak{L}(L_a^2)$ generated by $\{T_f : f \in L^\infty\}$.

For $z \in B$, let U_z be the unitary operator given by

$$U_z f = (f \circ \phi_z) \cdot J\phi_z$$

where $J\phi_z = (-1)^n k_z$. For $S \in \mathfrak{L}(L_a^2)$, set

$$S_z = U_z S U_z.$$

Since U_z is a selfadjoint unitary operator on L^2 and L_a^2, $U_z T_f U_z = T_{f\circ\phi_z}$ for every $f \in L^\infty$.

Let $\mathcal{T}$ denote the class of trace operators on L_a^2. For $T \in \mathcal{T}$, we will denote the trace of T by $tr[T]$ and let $\|T\|_{C_1}$ denote the C_1 norm of T given by ([12])

$$\|T\|_{C_1} = tr[\sqrt{T^*T}].$$

Suppose f and g are in L_a^2. Consider the operator $f \otimes g$ on L_a^2 defined by

$$(f \otimes g)h = \langle h, g\rangle f,$$

for $h \in L_a^2$. It is easily proved that $f \otimes g$ is in $\mathcal{T}$ and with norm equal to $\|f \otimes g\|_{C_1} = \|f\|_2 \, \|g\|_2$ and

$$tr[f \otimes g] = \langle f, g\rangle.$$

For a nonnegative integer m, the m-Berezin transform of an operator $S \in \mathfrak{L}(L_a^2)$ is defined by

$$\begin{aligned} B_m S(z) &= C_n^{m+n} tr\left[S_z\left(\sum_{|k|=0}^{m} C_{m,k} \frac{n!k!}{(n+|k|)!} \frac{u^k}{\|u^k\|} \otimes \frac{u^k}{\|u^k\|} \right)\right] \\ &= C_n^{m+n} tr\left[S_z\left(\sum_{|k|=0}^{m} C_{m,k} u^k \otimes u^k \right)\right] \end{aligned} \tag{1.2}$$

where $k = (k_1, \cdots, k_n) \in N^n$, N is the set of nonnegative integers, $|k| = \sum_{i=0}^n k_i$, $u^k = u_1^{k_1} \cdots u_n^{k_n}$, $k! = k_1! \cdots k_n!$,

$$C_n^{m+n} = \binom{m+n}{n} \quad \text{and} \quad C_{m,k} = C_{|k|}^m (-1)^{|k|} \frac{|k|!}{k_1! \cdots k_n!}.$$

Clearly, $B_m : \mathfrak{L}(L^2_a) \to L^\infty$ is a bounded linear operator, the norm of B_m will be given.

Given $f \in L^\infty$, define

$$B_m(f)(z) = B_m(T_f)(z).$$

$B_m(f)(z)$ equals the nice formula in [1]:

$$B_m(f)(z) = \int_B f \circ \phi_z(u) d\nu_m(u),$$

for $z \in B$ where $d\nu_m(u) = C_n^{m+n}(1 - |u|^2)^m du$.

Berezin first introduced the Berezin transform $B_0(S)$ of bounded operators S and the m-Berezin transform of functions in [5]. Because the Berezin transform encodes operator-theoretic information in function-theory in a striking but somewhat impenetrable way, the Berezin transform $B_0(S)$ has found useful applications in studying operators of "function-theoretic significance" on function spaces ([2], [3], [4], [6], [7], [11], and [15]). Suárez [16] introduced m-Berezin transforms of bounded operators on the Bergman space of the unit disk. We will show that our m-Berezin transform coincides with the one defined in [16] on the unit disk D by means of an integral representation of m-Berezin transform. The integral representation shows that many useful properties of the m-Berezin transforms inherit from the identity (1.1) of the reproducing kernel. On the unit ball, some useful properties of the m-Berezin transforms of functions were obtained by Ahern, Flores and Rudin [1]. Recently, Coburn [10] proved that $B_0(S)$ is Lipschitz with respect to the pseudo-hyperbolic distance $\rho(z, w)$. In this paper, we will show that $B_m S(z)$ is Lipschitz with respect to pseudo-hyperbolic distance $\rho(z, w)$. We will show that the m-Berezin transforms B_m are invariant under the Mobious transform,

$$B_m(S_z) = (B_m S) \circ \phi_z, \tag{1.3}$$

and commuting with each other,

$$B_j(B_m S)(z) = B_m(B_j S)(z) \tag{1.4}$$

for any nonnegative integers j and m. Properties (1.3) and (1.4) were obtained for $S = T_f$ in [1] and for operators S on the Bergman space of the unit disk [16].

A common intuition is that for operators on the Bergman space L^2_a "closely associated with function theory", compactness is equivalent to having vanishing Berezin transform on the boundary of the unit ball B. On the unit disk, Axler and Zheng [2] showed that if the operator S equals the finite sum of finite products of Toeplitz operators with bounded symbols then S is compact if and only if $B_0(S)(z) \to 0$ as $z \to \partial D$. Englis extended this result to the unit ball even the bounded symmetric domains [11]. But the problem remains open whether the result is true if S is in the Toeplitz algebra. Recently, Suárez [17] solved the problem for radial operator S on the unit disk via the m-Berezin transform. Using the m-Berezin transform, we will show that for a radial operator S in the Toeplitz algebra on the unit ball, S is compact iff $B_0 S(z) \to 0$ as $|z| \to 1$.

Throughout the paper $C(m,n)$ will denote constant depending only on m and n, which may change at each occurrence.

The authors thank the referees for their suggestions.

2. m-Berezin transform

In this section we will show some useful properties of the m-Berezin transform. First we give an integral representation of the m-Berezin transform $B_m(S)$. For $z \in B$ and a nonnegative integer m, let

$$K_z^m(u) = \frac{1}{(1 - \langle u, z\rangle)^{m+n+1}}, \qquad u \in B.$$

For $u, \lambda \in B$, we can easily see that

$$\sum_{|k|=0}^{m} C_{m,k} u^k \overline{\lambda^k} = (1 - \langle u, \lambda\rangle)^m. \tag{2.1}$$

Proposition 2.1 *Let $S \in \mathfrak{L}(L_a^2)$, $m \geq 0$ and $z \in B$. Then*

$$B_m S(z) = C_n^{m+n}(1 - |z|^2)^{m+n+1} \times \int_B \int_B (1 - \langle u, \lambda\rangle)^m K_z^m(u) \overline{K_z^m(\lambda) S^* K_\lambda(u)} du d\lambda.$$

Proof. For $\lambda \in B$, the definition of B_m implies

$$\begin{aligned} B_m S(z) &= C_n^{m+n} \sum_{|k|=0}^{m} C_{m,k} \left\langle S_z \lambda^k, \lambda^k \right\rangle \\ &= C_n^{m+n} \sum_{|k|=0}^{m} C_{m,k} \int_B S(\phi_z^k k_z)(\lambda) \overline{\phi_z^k(\lambda) k_z(\lambda)} d\lambda \\ &= C_n^{m+n} \sum_{|k|=0}^{m} C_{m,k} \int_B \int_B \phi_z^k(u) k_z(u) \overline{\phi_z^k(\lambda) k_z(\lambda) S^* K_\lambda(u)} du d\lambda \end{aligned} \tag{2.2}$$

where the last equality holds by $S(\phi_z^k k_z)(\lambda) = \left\langle S(\phi_z^k k_z), K_\lambda \right\rangle = \left\langle \phi_z^k k_z, S^* K_\lambda \right\rangle$.

Using (2.1) and (1.1), (2.2) equals

$$C_n^{m+n}\int_B\int_B (1-\langle\phi_z(u),\phi_z(\lambda)\rangle)^m k_z(u)\overline{k_z(\lambda)S^*K_\lambda(u)}dud\lambda$$

$$= C_n^{m+n}\int_B\int_B\left(\frac{k_z(u)\overline{k_z(\lambda)}}{K_\lambda(u)}\right)^{m/(n+1)} k_z(u)\overline{k_z(\lambda)S^*K_\lambda(u)}dud\lambda$$

$$= C_n^{m+n}(1-|z|^2)^{m+n+1}\int_B\int_B (1-\langle u,\lambda\rangle)^m K_z^m(u)\overline{K_z^m(\lambda)S^*K_\lambda(u)}dud\lambda$$

as desired. ■

Proposition 2.2 gives another form of B_m.

Proposition 2.2 *Let $S\in\mathfrak{L}(L_a^2)$, $m\geq 0$ and $z\in B$. Then*

$$B_mS(z) = C_n^{m+n}(1-|z|^2)^{m+n+1}\sum_{|k|=0}^{m} C_{m,k}\left\langle S(u^kK_z^m),u^kK_z^m\right\rangle. \tag{2.3}$$

Proof. Since

$$\int_B\int_B (1-\langle u,\lambda\rangle)^m K_z^m(u)\overline{K_z^m(\lambda)S^*K_\lambda(u)}dud\lambda$$

$$= \sum_{|k|=0}^{m} C_{m,k}\int_B\int_B u^k\overline{\lambda^k}K_z^m(u)\overline{K_z^m(\lambda)S^*K_\lambda(u)}dud\lambda$$

$$= \sum_{|k|=0}^{m} C_{m,k}\int_B S(u^kK_z^m)(\lambda)\overline{\lambda^kK_z^m(\lambda)}d\lambda,$$

Proposition 2.1 implies (2.3). ■

For $n=1$, the right hand side of (2.3) was used by Suárez in [16] to define the m-Berezin transforms on the unit disk.

Recall that given $f\in L^\infty$, define

$$B_m(f)(z) = B_m(T_f)(z).$$

The following proposition gives a nice formula of $B_m(f)(z)$. Let $d\nu_m(u) = C_n^{m+n}(1-|u|^2)^m du$.

Proposition 2.3 *Let $z\in B$ and $f\in L^\infty$. Then*

$$B_m(f)(z) = \int_B f\circ\phi_z(u)d\nu_m(u).$$

Proof. By the change of variables, Theorem 2.2.2 in [14] and (2.3), we have

$$\begin{aligned}
&\int_B f\circ\phi_z(u)d\nu_m(u)\\
&= C_n^{m+n}\int_B f(u)\left(\frac{(1-|z|^2)(1-|u|^2)}{|1-\langle u,z\rangle|^2}\right)^m\left(\frac{(1-|z|^2)}{|1-\langle u,z\rangle|^2}\right)^{n+1}du\\
&= C_n^{m+n}(1-|z|^2)^{m+n+1}\sum_{|k|=0}^m C_{m,k}\int_B f(u)|u^k|^2|K_z^m(u)|^2du\\
&= C_n^{m+n}(1-|z|^2)^{m+n+1}\sum_{|k|=0}^m C_{m,k}\left\langle T_f(u^kK_z^m),u^kK_z^m\right\rangle = B_m(T_f)(z).
\end{aligned}$$

The proof is complete. ∎

The formula in the above proposition was used in [1] to define the m-Berezin transform of functions f.

Clearly, (1.2) gives $\|B_mS\|_\infty \le C(m,n)\|S_z\| = C(m,n)\|S\|$ for $S\in\mathfrak{L}(L_a^2)$. Thus, $B_m:\mathfrak{L}(L_a^2)\to L^\infty$ is a bounded linear operator. The following theorem gives the norm of B_m.

Theorem 2.4 *Let $m\ge 0$. Then*

$$\|B_m\| = C_n^{m+n}\sum_{|k|=0}^m |C_{m,k}|\frac{n!k!}{(n+|k|)!}.$$

Proof. From [8], we have the duality result $\mathfrak{L}(L_a^2)=\mathcal{T}^*$. So, the definition of B_m gives the norm of B_m. In fact,

$$\begin{aligned}
\|B_m\| &= \left\|C_n^{m+n}\sum_{|k|=0}^m C_{m,k}\frac{n!k!}{(n+|k|)!}\frac{u^k}{\|u^k\|}\otimes\frac{u^k}{\|u^k\|}\right\|_{C_1}\\
&= C_n^{m+n}\sum_{|k|=0}^m |C_{m,k}|\frac{n!k!}{(n+|k|)!}
\end{aligned}$$

as desired. ∎

The Mobius map $\phi_z(w)$ has the following property ([14]):

$$\phi_z'(0) = -(1-|z|^2)P_z-(1-|z|^2)^{1/2}Q_z. \tag{2.4}$$

To show that m-Berezin transforms are Lipschitz with respect to the pseudo-hyperbolic distance we need the following lemmas.

For $z,w\in\mathbb{C}^n$, $z\hat{\otimes}w$ on $\mathbb{C}^n$ is defined by $(z\hat{\otimes}w)\lambda=\langle\lambda,w\rangle z$.

Lemma 2.5 *Let $z, w \in B$ and $\lambda = \phi_z(w)$. Then*

$$\phi_z'(w) = (1 - \langle \lambda, z\rangle)(I - \lambda \hat{\otimes} z)[\phi_z'(0)]^{-1}.$$

Proof. Suppose that P_z and Q_z have the matrix representations as $((P_z)_{ij})$ and $((Q_z)_{ij})$ under the standard base of $\mathbb{C}^n$, respectively. In fact,

$$(P_z)_{ij} = \frac{z_i \bar{z}_j}{|z|^2} \qquad \text{if} \quad z \neq 0.$$

Let $(a_{ij}(w)) = \phi_z'(w)$. Write $\phi_z(w) = (f_1(w), \ldots, f_n(w))$. Then

$$a_{ij}(w) = \frac{\partial f_i}{\partial w_j}(w).$$

Noting that

$$f_i(w) = \frac{z_i - (P_z w)_i - (1 - |z|^2)^{1/2}(Q_z w)_i}{1 - \langle w, z\rangle},$$

we have

$$\begin{aligned} a_{ij}(w) &= \frac{(z_i - (P_z w)_i - (1 - |z|^2)^{1/2}(Q_z w)_i)\bar{z}_j}{(1 - \langle w, z\rangle)^2} - \frac{(P_z)_{ij} + (1 - |z|^2)^{1/2}(Q_z)_{ij}}{1 - \langle w, z\rangle} \\ &= \frac{f_i(w)\bar{z}_j}{1 - \langle w, z\rangle} - \frac{(P_z)_{ij} + (1 - |z|^2)^{1/2}(Q_z)_{ij}}{1 - \langle w, z\rangle}. \end{aligned}$$

Let $\lambda = \phi_z(w)$. The above equality becomes

$$a_{ij}(w) = \frac{\lambda_i \bar{z}_j - ((P_z)_{ij} + (1 - |z|^2)^{1/2}(Q_z)_{ij})}{1 - \langle w, z\rangle}$$

Thus

$$\phi_z'(w) = \frac{\lambda \hat{\otimes} z - (P_z + (1 - |z|^2)^{1/2} Q_z)}{1 - \langle w, z\rangle}.$$

From Theorem 2.2.5 in [14], we have

$$\frac{1}{1 - \langle w, z\rangle} = \frac{1 - \langle \lambda, z\rangle}{1 - |z|^2}.$$

Thus (2.4) implies

$$\begin{aligned} \phi_z'(w)\phi_z'(0) &= \frac{-(1 - |z|^2)\lambda \hat{\otimes} z + (1 - |z|^2)P_z + (1 - |z|^2)Q_z}{1 - \langle w, z\rangle} \\ &= \frac{(1 - |z|^2)(-\lambda \hat{\otimes} z + I)}{1 - \langle w, z\rangle} \\ &= (1 - \langle \lambda, z\rangle)(I - \lambda \hat{\otimes} z) \end{aligned}$$

where the first equality follows from $P_z Q_z = Q_z P_z = 0$, $P_z z = z$, and $Q_z z = 0$. The proof is complete. ■

Lemma 2.6 *Suppose* $|z| > 1/2$ *and* $|w| > 1/2$. *If* $|\phi_z(w)| \le \epsilon < 1/2$, *then*

$$\|P_z - P_w\| \le 50\epsilon(1-|z|^2)^{1/2}.$$

Proof. First we will get the estimate of the distance between z and w. Since $|\phi_z(w)| \le \epsilon < 1/2$, w is in the ellipsoid:

$$\phi_z(\epsilon B) = \{w \in B : \frac{|P_z w - c|^2}{\epsilon^2\rho^2} + \frac{|Q_z w|^2}{\epsilon^2\rho} < 1\}$$

with center $c = \frac{(1-\epsilon^2)z}{(1-\epsilon^2|z|^2)}$ and $\rho = \frac{1-|z|^2}{1-\epsilon^2|z|^2}$. Noting that $|z| > 1/2$ and $\epsilon < 1/2$, we have $\rho \le 2(1-|z|^2)$. Thus

$$|Q_z w|^2 \le \epsilon^2\rho \le 2\epsilon^2(1-|z|^2), \qquad |P_z w - c| \le \epsilon\rho \le 2\epsilon(1-|z|^2)$$

and

$$|z - c| \le \frac{\epsilon^2(1-|z|^2)}{(1-\epsilon^2|z|^2)} \le 2\epsilon^2(1-|z|^2).$$

So, we have

$$|P_z w - z| \le |P_z w - c| + |z - c| \le 3\epsilon(1-|z|^2).$$

Because $I = P_z + Q_z$ and $P_z Q_z = 0$, writing

$$(z-w) = P_z(z-w) + Q_z(z-w),$$

we have

$$\begin{aligned} |z-w|^2 &= |P_z(z-w)|^2 + |Q_z(z-w)|^2 \\ &= |P_z w - z|^2 + |Q_z w|^2 \\ &\le 11\epsilon^2(1-|z|^2). \end{aligned} \tag{2.5}$$

Noting that

$$\frac{z}{|z|}\hat{\otimes}\frac{z}{|z|} = \frac{(z-w)}{|z|}\hat{\otimes}\frac{z}{|z|} + \frac{w}{|z|}\hat{\otimes}\frac{(z-w)}{|z|} + \left[\left(\frac{1}{|z|^2} - \frac{1}{|w|^2}\right)w\right]\hat{\otimes}w + \frac{w}{|w|}\hat{\otimes}\frac{w}{|w|},$$

we have

$$P_z - P_w = \frac{(z-w)}{|z|}\hat{\otimes}\frac{z}{|z|} + \frac{w}{|z|}\hat{\otimes}\frac{(z-w)}{|z|} + \left[\left(\frac{1}{|z|^2} - \frac{1}{|w|^2}\right)w\right]\hat{\otimes}w,$$

to obtain

$$\begin{aligned} \|P_z - P_w\| &\le \frac{|z-w|}{|z|} + \frac{2|z-w|}{|z|} + \frac{||z|^2 - |w|^2|}{|z|^2} \\ &\le 2|z-w| + 4|z-w| + 8|z-w| \\ &\le 14\sqrt{11}\epsilon(1-|z|^2)^{1/2} \\ &\le 50\epsilon(1-|z|^2)^{1/2} \end{aligned}$$

where the last inequality holds by (2.5). ■

For given $z, w \in B$, set $A(z,w) = -(1-|z|^2)P_w - (1-|z|^2)^{1/2}Q_w$.

Lemma 2.7 *Suppose* $|z| > 1/2$ *and* $|w| > 1/2$. *If* $|\phi_z(w)| \le \epsilon < 1/2$, *then*

$$\|\phi_z'(0) - A(z,w)\| \le 150\epsilon(1-|z|^2).$$

Proof. Using (2.4), we have

$$\begin{aligned}\|\phi_z'(0) - A(z,w)\| &= \|(1-|z|^2)(P_w - P_z) + (1-|z|^2)^{1/2}(P_z - P_w)\| \\ &\le 3(1-|z|^2)^{1/2}\|P_z - P_w\| \\ &\le 150\epsilon(1-|z|^2)\end{aligned}$$

as desired. The last inequality follows from Lemma 2.6. ■

Let $\mathfrak{U}(n)$ be the group of $n \times n$ complex unitary matrices.

Lemma 2.8 *Let* $z, w \in B$. *Then* $U_z U_w = V_{\mathcal{U}} U_{\phi_w(z)}$ *where*

$$(V_{\mathcal{U}} f)(u) = f(\mathcal{U}u)\det \mathcal{U}$$

for $f \in L_a^2$ *and* $\mathcal{U} = \phi_{\phi_w(z)} \circ \phi_w \circ \phi_z$ *satisfying*

$$\|I + \mathcal{U}\| \le C(n)\rho(z,w).$$

Proof. The map $\phi_{\phi_w(z)} \circ \phi_w \circ \phi_z$ is an automorphism of B that fixes 0, hence it is unitary by the Cartan theorem in [14]. Thus $\phi_w \circ \phi_z = \phi_{\phi_w(z)} \circ \mathcal{U}$ for some $\mathcal{U} \in \mathfrak{U}(n)$. Since ϕ_w is an involution, we have

$$\begin{aligned}&U_z U_w f(u) = (f \circ \phi_w \circ \phi_z)(u) J\phi_w(\phi_z(u)) J\phi_z(u) \\ &= (f \circ \phi_{\phi_w(z)})(\mathcal{U}u) J\phi_w(\phi_w \circ \phi_{\phi_w(z)}(\mathcal{U}u)) J\phi_w(\phi_{\phi_w(z)}(\mathcal{U}u)) J\phi_{\phi_w(z)}(\mathcal{U}u)\det \mathcal{U} \\ &= (f \circ \phi_{\phi_w(z)})(\mathcal{U}u) J\phi_{\phi_w(z)}(\mathcal{U}u)\det \mathcal{U} \\ &= V_{\mathcal{U}} U_{\phi_w(z)} f(u)\end{aligned}$$

as desired.

Now we will show that

$$\|I + \mathcal{U}\| \le C(n)\rho(z,w).$$

Noting that $\mathcal{U}$ is continuous for $|z| \le 1/2$ and $|w| \le 1/2$, we need only to prove

$$\|I + \mathcal{U}\| \le 20000\rho(z,w),$$

for $|z| > 1/2$, $|w| > 1/2$ and $|\phi_w(z)| < 1/2$. Let $\lambda = \phi_w(z)$. Then $|\lambda| = \rho(z,w)$ and $z = \phi_w(\lambda)$. Since

$$\phi_w \circ \phi_z(u) = \phi_\lambda(\mathcal{U}u),$$

taking derivatives both sides of the above equations and using the chain rule give

$$\phi_w'(\phi_z(u))\,\phi_z'(u) = \phi_\lambda'(\mathcal{U}u)\mathcal{U}.$$

Letting $u = 0$, the above equality gives

$$\mathcal{U} = [\phi'_\lambda(0)]^{-1}\phi'_w(z)\phi'_z(0).$$

By Lemma 2.5, write

$$\begin{aligned}\mathcal{U} + I &= [\phi'_\lambda(0)]^{-1}(1-\langle\lambda,w\rangle)(I-\lambda\hat{\otimes}w)[\phi'_w(0)]^{-1}\phi'_z(0) + I\\ &= [\phi'_\lambda(0)]^{-1}(1-\langle\lambda,w\rangle)(I-\lambda\hat{\otimes}w)[\phi'_w(0)]^{-1}[\phi'_z(0) - A(z,w)]\\ &\qquad + \big([\phi'_\lambda(0)]^{-1}(1-\langle\lambda,w\rangle)(I-\lambda\hat{\otimes}w)[\phi'_w(0)]^{-1}A(z,w) + I\big)\\ &:= I_1 + I_2.\end{aligned}$$

By Lemma 2.7, we have

$$\begin{aligned}\|I_1\| &\le \|[\phi'_\lambda(0)]^{-1}\||1-\langle\lambda,w\rangle|\|I-\lambda\hat{\otimes}w\|\|[\phi'_w(0)]^{-1}\|\|\phi'_z(0) - A(z,w)\|\\ &\le 4\times 2\times 2\times \frac{3}{(1-|w|^2)}\left[150|\lambda|(1-|z|^2)\right].\end{aligned}$$

Theorem 2.2.2 in [14] leads to

$$\frac{1-|z|^2}{1-|w|^2} = \frac{1-|\lambda|^2}{|1-\langle\lambda,w\rangle|^2}.$$

Thus

$$\|I_1\| \le 4\times 2\times 2\times 3\times 2\times 150|\lambda| = 14400|\lambda|.$$

Also, we have

$$\left|1 - \frac{(1-|z|^2)^{1/2}}{(1-|w|^2)^{1/2}}\right| \le \left|1 - \frac{1-|z|^2}{1-|w|^2}\right| \le 32|\lambda|.$$

Hence, we get

$$\left\|I - \frac{1-|z|^2}{1-|w|^2}P_w - \frac{(1-|z|^2)^{1/2}}{(1-|w|^2)^{1/2}}Q_w\right\| \le 32|\lambda|.$$

On the other hand, clearly,

$$\|[\phi'_\lambda(0)]^{-1} + I\| \le 4|\lambda|, \quad |(1-\langle\lambda,w\rangle) - 1| \le |\lambda|$$

and

$$\|(I-\lambda\hat{\otimes}w) - I\| \le |\lambda|.$$

These give

$$\|I + [\phi'_\lambda(0)]^{-1}(1-\langle\lambda,w\rangle)(I-\lambda\hat{\otimes}w)\| \le 16|\lambda|.$$

Hence, we have

$$\begin{aligned}\|I_2\| &\le \|[\phi'_\lambda(0)]^{-1}(1-\langle\lambda,w\rangle)(I-\lambda\hat{\otimes}w)[\phi'_w(0)]^{-1}A(z,w)\\ &\quad -[\phi'_\lambda(0)]^{-1}(1-\langle\lambda,w\rangle)(I-\lambda\hat{\otimes}w)\|\\ &\quad +\|[\phi'_\lambda(0)]^{-1}(1-\langle\lambda,w\rangle)(I-\lambda\hat{\otimes}w)+I\|\\ &\le \left\|[\phi'_\lambda(0)]^{-1}(1-\langle\lambda,w\rangle)(I-\lambda\hat{\otimes}w)\right\|\left\|I-\frac{1-|z|^2}{1-|w|^2}P_w-\frac{(1-|z|^2)^{1/2}}{(1-|w|^2)^{1/2}}Q_w\right\|\\ &\quad +16|\lambda|\\ &\le 4\times 2\times 2\times 32|\lambda|+16|\lambda|<600|\lambda|.\end{aligned}$$

Combining the above estimates we conclude that

$$\|\mathcal{U}+I\|\le 14400|\lambda|+600|\lambda|<20000|\lambda|. \qquad \blacksquare$$

Theorem 2.9 *Let $S\in\mathfrak{L}(L_a^2)$, $m\ge 0$ and $z\in B$. Then $B_mS_z=(B_mS)\circ\phi_z$.*

Proof. Proposition 2.2 and (1.2) give

$$B_mS_z(0)=C_n^{m+n}\sum_{|k|=0}^{m}C_{m,k}\left\langle S_zu^k,u^k\right\rangle=B_mS(z)=(B_mS)\circ\phi_z(0).$$

For any $w\in B$, Proposition 2.1 and Lemma 2.8 imply

$$\begin{aligned}(B_mS_z)\circ\phi_w(0)&=B_m((S_z)_w)(0)\\ &=C_n^{m+n}\int_B\int_B(1-\langle u,\lambda\rangle)^m\overline{U_wU_zS^*U_zU_wK_\lambda(u)}dud\lambda\\ &=C_n^{m+n}\int_B\int_B(1-\langle u,\lambda\rangle)^m\overline{V_{\mathcal{U}}U_{\phi_z(w)}S^*U_{\phi_z(w)}V_{\mathcal{U}}^*K_\lambda(u)}dud\lambda\\ &=B_mS_{\phi_z(w)}(0)\end{aligned}$$

where $V_{\mathcal{U}}$ is in Lemma 2.8. Thus, $B_mS_z(w)=(B_mS)\circ\phi_z(w)$. $\blacksquare$

Lemma 2.10 *Let $S\in\mathfrak{L}(L_a^2)$, $m\ge 1$ and $z\in B$. Then*

$$B_mS(z)=\frac{m+n}{m}B_{m-1}\left(S-\sum_{i=1}^{n}T_{\overline{(\phi_z)_i}}ST_{(\phi_z)_i}\right)(z)$$

where $(\phi_z)_i$ is i-th variable of ϕ_z.

Proof. By Theorem 2.9, we just need to show that

$$B_mS(0) = \frac{m+n}{m} B_{m-1}\left(S - \sum_{i=1}^{n} T_{\overline{u_i}} S T_{u_i} \right)(0).$$

Using Proposition 2.1 and (2.1), we get

$$\begin{aligned} B_mS(0) &= C_n^{m+n} \int_B \int_B (1 - \langle u, \lambda \rangle)^m \overline{S^* K_\lambda(u)} du d\lambda \\ &= \frac{m+n}{m} B_{m-1}S(0) - C_n^{m+n} \sum_{i=1}^{n} \sum_{|k|=0}^{m-1} C_{m-1,k} \int_B \int_B u_i \overline{\lambda_i} u^k \overline{\lambda^k S^* K_\lambda(u)} du d\lambda \\ &= \frac{m+n}{m} B_{m-1}S(0) - C_n^{m+n} \sum_{i=1}^{n} \sum_{|k|=0}^{m-1} C_{m-1,k} \int_B S(u^k u_i)(\lambda) \overline{\lambda^k \lambda_i} d\lambda \\ &= \frac{m+n}{m} B_{m-1}S(0) - C_n^{m+n} \sum_{i=1}^{n} \sum_{|k|=0}^{m-1} C_{m-1,k} \left\langle ST_{u_i}(u^k), T_{u_i}(u^k) \right\rangle \end{aligned}$$

as desired. ■

For $m = 0$, the following result was obtained in [10].

Theorem 2.11 *Let $S \in \mathfrak{L}(L_a^2)$ and $m \geq 0$. Then there exists a constant $C(m, n) > 0$ such that*

$$|B_mS(z) - B_mS(w)| < C(m, n)\|S\|\rho(z, w).$$

Proof. We will prove this theorem by induction on m. If $m = 0$, (1.2) gives

$$\begin{aligned} |B_0S(z) - B_0S(w)| &= |tr[S_z(1 \otimes 1)] - tr[S_w(1 \otimes 1)]| \\ &= |tr[S_z(1 \otimes 1) - SU_w(1 \otimes 1)U_w]| \\ &= |tr[S_z(1 \otimes 1) - SU_z(U_zU_w1 \otimes U_zU_w1)U_z]| \end{aligned}$$

From Lemma 2.8, the last term equals

$$\begin{aligned} |tr[S_z(1 \otimes 1 - U_{\phi_w(z)}1 \otimes U_{\phi_w(z)}1)]| &\leq \|S_z\| \|1 \otimes 1 - U_{\phi_w(z)}1 \otimes U_{\phi_w(z)}1\|_{C^1} \\ &\leq \sqrt{2}\|S_z\|(2 - 2|\langle 1, k_{\phi_w(z)} \rangle|^2)^{1/2} \\ &= 2\|S\|[1 - (1 - |\phi_w(z)|^2)^{n+1}]^{1/2} \\ &\leq C(n)\|S\||\phi_w(z)| \end{aligned}$$

where the second equality holds by $\|T\|_{C^1} \leq \sqrt{l}(tr[T^*T])^{1/2}$ where l is the rank of T.

Suppose $|B_{m-1}S(z)-B_{m-1}S(w)| < C(m,n)\|S\|\rho(z,w)$. By Lemma 2.10, we have

$$\begin{aligned}
&|B_mS(z)-B_mS(w)|\\
&\quad\le \frac{m+n}{m}|B_{m-1}S(z)-B_{m-1}S(w)|\\
&\quad\quad+\frac{m+n}{m}\sum_{i=1}^{n}\left|B_{m-1}\left(T_{\overline{(\phi_z)_i}}ST_{(\phi_z)_i}\right)(z)-B_{m-1}\left(T_{\overline{(\phi_w)_i}}ST_{(\phi_w)_i}\right)(w)\right|.
\end{aligned}$$

Since the term in the summation is less than or equals

$$\begin{aligned}
&\left|B_{m-1}\left(T_{\overline{(\phi_z)_i}}ST_{(\phi_z)_i}\right)(z)-B_{m-1}\left(T_{\overline{(\phi_w)_i}}ST_{(\phi_z)_i}\right)(z)\right|\\
&\quad+\left|B_{m-1}\left(T_{\overline{(\phi_w)_i}}ST_{(\phi_z)_i}\right)(z)-B_{m-1}\left(T_{\overline{(\phi_w)_i}}ST_{(\phi_w)_i}\right)(z)\right|\\
&\quad+\left|B_{m-1}\left(T_{\overline{(\phi_w)_i}}ST_{(\phi_w)_i}\right)(z)-B_{m-1}\left(T_{\overline{(\phi_w)_i}}ST_{(\phi_w)_i}\right)(w)\right|,
\end{aligned}$$

it is sufficient to show that

$$\left|B_{m-1}\left(T_{\overline{(\phi_z)_i}}ST_{(\phi_z)_i}\right)(z)-B_{m-1}\left(T_{\overline{(\phi_w)_i}}ST_{(\phi_z)_i}\right)(z)\right|<C(m,n)\|S\|\rho(z,w).$$

Lemma 2.8 gives

$$\begin{aligned}
&\left|B_{m-1}\left(T_{\overline{(\phi_z)_i}-\overline{(\phi_w)_i}}ST_{(\phi_z)_i}\right)(z)\right|\\
&=C_n^{m+n-1}\left|tr\left[\left(T_{\overline{(\phi_z)_i}-\overline{(\phi_w)_i}}ST_{(\phi_z)_i}\right)_z\sum_{|k|=0}^{m-1}C_{m-1,k}\frac{n!k!}{(n+|k|)!}\frac{u^k}{\|u^k\|}\otimes\frac{u^k}{\|u^k\|}\right]\right|\\
&\le C_n^{m+n-1}\sum_{|k|=0}^{m-1}|C_{m-1,k}|\frac{n!k!}{(n+|k|)!}\left|\left\langle S_zT_{(\phi_z)_i\circ\phi_z}\frac{u^k}{\|u^k\|},T_{((\phi_z)_i-(\phi_w)_i)\circ\phi_z}\frac{u^k}{\|u^k\|}\right\rangle\right|
\end{aligned}$$

$$\le C(m,n)\|S_z\|\left\|T_{((\phi_z)_i-(\phi_w)_i)\circ\phi_z}\frac{u^k}{\|u^k\|}\right\|_2. \tag{2.6}$$

Let $\lambda=\phi_w(z)$. Then

$$\begin{aligned}
\left\|T_{((\phi_z)_i-(\phi_w)_i)\circ\phi_z}\frac{u^k}{\|u^k\|}\right\|_2^2&\le\int_B|(\phi_z\circ\phi_z)_i(u)-(\phi_w\circ\phi_z)_i(u)|^2du\\
&=\int_B|(\mathcal{U}u)_i-(\phi_\lambda(u))_i|^2du\\
&\le 2\int_B|(\mathcal{U}u)_i+u_i|^2+|u_i+(\phi_\lambda(u))_i|^2du
\end{aligned}$$

where $\phi_w\circ\phi_z=\phi_\lambda\circ\mathcal{U}$ for some $\mathcal{U}\in\mathfrak{U}(n)$.

Noting that

$$\phi_\lambda(u) + u = \frac{\lambda - \langle u, \lambda\rangle u + [1 - (1 - |\lambda|^2)^{1/2}]Q_\lambda(u)}{1 - \langle u, \lambda\rangle},$$

we have that for $|\lambda| \le 1/2$,

$$|\phi_\lambda(u) + u| \le 2(|\lambda| + |\lambda| + |\lambda|^2) \le 6|\lambda|.$$

By Lemma 2.8 we also have

$$\int_B |(\mathcal{U}u)_i + u_i|^2 du = \int_B |((\mathcal{U} + I)u)_i|^2 du \le C\|\mathcal{U} + I\|^2 \le C|\lambda|^2.$$

Thus (2.6) is less than or equal to

$$C(m,n)\|S_z\|[36|\lambda|^2 + C|\lambda|^2]^{1/2} \le C(m,n)\|S\||\lambda|.$$

The proof is complete. ∎

Lemma 2.12 *Let $S \in \mathfrak{L}(L^2_a)$ and $m, j \ge 0$. If $|S^*K_\lambda(z)| \le C$ for any $z \in B$ then $(B_mB_j)(S) = (B_jB_m)(S)$.*

Proof. By Theorem 2.9, it is enough to show that $(B_mB_j)S(0) = (B_jB_m)S(0)$. From Proposition 2.3, Proposition 2.1 and Fubini's Theorem, we have

$$\begin{aligned}
B_m(B_jS)(0) &= B_m(T_{B_jS})(0) \\
&= C_n^{m+n}\int_B B_jS(z)(1-|z|^2)^m dz \\
&= C_n^{m+n}C_n^{j+n}\int_B\int_B\int_B (1-|z|^2)^{m+j+n+1}(1-\langle u,\lambda\rangle)^j \times \\
&\qquad K_z^j(u)\overline{K_z^j(\lambda)}\,\overline{S^*K_\lambda(u)}du d\lambda dz \\
&= C_n^{m+n}C_n^{j+n}\int_B\int_B (1-\langle u,\lambda\rangle)^j \int_B (1-|z|^2)^{m+j+n+1} \times \\
&\qquad K_z^j(u)\overline{K_z^j(\lambda)}dz\overline{S^*K_\lambda(u)}du d\lambda.
\end{aligned}$$

Let

$$F_{m,j}(u,\lambda) = (1-\langle u,\lambda\rangle)^j\int_B (1-|z|^2)^{m+j+n+1}K_z^j(u)\overline{K_z^j(\lambda)}dz.$$

Then $F_{m,j}(u,\lambda) = \sum_{i=1}^l H_i(u)\overline{G_i}(\lambda)$ where H_i and G_i are holomorphic functions and for some $l \ge 0$. Thus, from Lemma 9 in [9], we just need to show $F_{m,j}(\lambda,\lambda) = F_{j,m}(\lambda,\lambda)$ for $\lambda \in B$.

The change of variables implies

$$\begin{aligned} F_{m,j}(\lambda,\lambda) &= (1-|\lambda|^2)^j \int_B (1-|z|^2)^{m+j+n+1}|K^j_\lambda(z)|^2 dz \\ &= (1-|\lambda|^2)^j \int_B (1-|\phi_\lambda(z)|^2)^{m+j+n+1}|K^j_\lambda(\phi_\lambda(z))|^2 |k_\lambda(z)|^2 dz \\ &= (1-|\lambda|^2)^m \int_B (1-|z|^2)^{m+j+n+1}|K^m_\lambda(z)|^2 dz \\ &= F_{j,m}(\lambda,\lambda) \end{aligned}$$

as desired. ■

Lemma 2.13 *For any $S \in \mathfrak{L}(L^2_a)$, there exists sequences $\{S_\alpha\}$ satisfying*

$$|S^*_\alpha K_\lambda(u)| \le C(\alpha)$$

such that $B_m(S_\alpha)$ converges to $B_m(S)$ pointwise.

Proof. Since H^∞ is dense in L^2_a and the set of finite rank operators is dense in the ideal $\mathcal{K}$ of compact operators on L^2, the set $\{\sum_{i=1}^l f_i \otimes g_i : f_i,\ g_i \in H^\infty\}$ is dense in the ideal $\mathcal{K}$ in the norm topology. Since $\mathcal{K}$ is dense in the space of bounded operators on L^2_a in strong operator topology, (2.3) gives that for any $S \in \mathfrak{L}(L^2_a)$, there exists a finite rank operator sequences $S_\alpha = \sum_{i=1}^l f_i \otimes g_i$ such that $B_m(S_\alpha)$ converges to $B_m(S)$ pointwise for some $f_i,\ g_i$ in H^∞. Also, for $l \ge 0$, for such $S_\alpha = \sum_{i=1}^l f_i \otimes g_i$, we have

$$\begin{aligned} |S^*_\alpha K_\lambda(u)| &= \left|\sum_{i=1}^l (g_i \otimes f_i) K_\lambda(u)\right| = \left|\sum_{i=1}^l \langle K_\lambda(u), f_i(u)\rangle\, g_i(u)\right| \\ &\le \sum_{i=1}^l |f_i(\lambda)||g_i(u)| \le \sum_{i=1}^l \|f_i\|_\infty \|g_i\|_\infty < C. \end{aligned}$$

The proof is complete. ■

Proposition 2.14 *Let $S \in \mathfrak{L}(L^2_a)$ and $m, j \ge 0$. Then*

$$(B_m B_j)(S) = (B_j B_m)(S).$$

Proof. Let $S \in \mathfrak{L}(L^2_a)$. Then Lemma 2.13 implies that there exists a sequence $\{S_\alpha\}$ satisfying $|S^*_\alpha K_\lambda(u)| \le C(\alpha)$, hence $B_m(B_j S_\alpha)(z) = B_j(B_m S_\alpha)(z)$ by Lemma 2.12. From Proposition 2.3, we know

$$B_m(B_j S_\alpha)(z) = \int_B (B_j S_\alpha) \circ \phi_z(u) d\nu_m(u)$$

and $\|(B_j S_\alpha) \circ \phi_z\|_\infty \le C(j,n)\|S\|$. Also, $(B_j S_\alpha) \circ \phi_z(u)$ converges to $(B_j S) \circ \phi_z(u)$. Therefore $B_m(B_j S_\alpha)(z)$ converges to $B_m(B_j S)(z)$. By the uniqueness of the limit, we have $(B_m B_j)(S) = (B_j B_m)(S)$. ■

Proposition 2.15 *Let $S \in \mathfrak{L}(L^2_a)$ and $m \geq 0$. If $B_0S(z) \to 0$ as $z \to \partial B$ then $B_mS(z) \to 0$ as $z \to \partial B$.*

Proof. Suppose $B_0S(z) \to 0$ as $z \to \partial B$. Then we will prove that $S_z \to 0$ in the $\mathcal{T}^*$-norm as $z \to \partial B$. Suppose it is not true. Then for some net $\{w_\alpha\} \in B$ and an operator $V \neq 0$ in $\mathfrak{L}(L^2_a)$, there exists a sequence $\{S_{w_\alpha}\}$ such that $S_{w_\alpha} \to V$ in the $\mathcal{T}^*$-norm as $w_\alpha \to \partial B$, hence $tr[S_{w_\alpha}T] \to tr[VT]$ for any $T \in \mathcal{T}$. Let $T = k_\lambda \otimes k_\lambda$ for fixed $\lambda \in B$. Then Theorem 2.9 implies

$$\begin{aligned} tr[S_{w_\alpha}T] &= tr[S_{w_\alpha}(k_\lambda \otimes k_\lambda)] = \langle S_{w_\alpha}k_\lambda, k_\lambda\rangle \\ &= B_0S_{w_\alpha}(\lambda) = (B_0S)\circ\phi_{w_\alpha}(\lambda) \to 0 \end{aligned}$$

as $w_\alpha \to \partial B$. Since $tr[VT] = B_0V(\lambda)$ and B_0 is one-to-one mapping, $V = 0$. This is the contradiction. Thus $S_z \to 0$ as $z \to \partial B$ in the $\mathcal{T}^*$-norm. (1.2) finishes the proof of this proposition. ■

3. Operators S approximated by Toeplitz operators $T_{B_m(S)}$

In this section we will give a criterion for operators approximated by Toeplitz operators with symbol equal to their m-Berezin transforms. The main result in this section is Theorem 3.7. It extends and improves Theorem 2.4 in [17]. Even on the unit disk, we will show an example that the result in the theorem is sharp on the unit disk.

From Proposition 1.4.10 in [14], we have the following lemma

Lemma 3.1 *Suppose $a < 1$ and $a + b < n + 1$. Then*

$$\sup_{z\in B}\int_B \frac{d\lambda}{(1-|\lambda|^2)^a|1-\langle\lambda, z\rangle|^b} < \infty.$$

This lemma gives the following lemma which extends Lemma 4.2 in [13].

Let $1 < q < \infty$ and p be the conjugate exponent of q. If we take $p > n+2$, then $q < (n+2)/(n+1)$.

Lemma 3.2 *Let $S \in \mathfrak{L}(L^2_a)$ and $p > n + 2$. Then there exists $C(n,p) > 0$ such that $h(z) = (1-|z|^2)^{-a}$ where $a = (n+1)/(n+2)$ satisfies*

$$\int_B |(SK_z)(w)|h(w)dw \leq C(n,p)\|S_z1\|_p h(z) \tag{3.1}$$

for all $z \in B$ and

$$\int_B |(SK_z)(w)|h(z)dz \leq C(n,p)\|S^*_w1\|_p h(w) \tag{3.2}$$

for all $w \in B$.

Proof. Fix $z \in B$. Since

$$U_z 1 = (-1)^n (1 - |z|^2)^{(n+1)/2} K_z,$$

we have

$$\begin{aligned} SK_z &= (-1)^n (1 - |z|^2)^{-(n+1)/2} SU_z 1 \\ &= (-1)^n (1 - |z|^2)^{-(n+1)/2} U_z S_z 1 \\ &= (1 - |z|^2)^{-(n+1)/2} (S_z 1 \circ \phi_z) k_z. \end{aligned}$$

Thus, letting $\lambda = \phi_z(w)$, the change of variables implies

$$\begin{aligned} \int_B \frac{|(SK_z)(w)|}{(1 - |w|^2)^a} dw &= \frac{1}{(1 - |z|^2)^{(n+1)/2}} \int_B \frac{|(S_z 1 \circ \phi_z)(w)||k_z(w)|}{(1 - |w|^2)^a} dw \\ &= \frac{1}{(1 - |z|^2)^a} \int_B \frac{|S_z 1(\lambda)|}{(1 - |\lambda|^2)^a |1 - \langle \lambda, z \rangle|^{n+1-2a}} d\lambda \\ &\le \frac{\|S_z 1\|_p}{(1 - |z|^2)^a} \left(\int_B \frac{1}{(1 - |\lambda|^2)^{aq} |1 - \langle \lambda, z \rangle|^{(n+1-2a)q}} d\lambda \right)^{\frac{1}{q}}. \end{aligned}$$

The last inequality comes from Holder's inequality. Since $aq < 1$ and $aq + (n + 1 - 2a)q < n + 1$, Lemma 3.1 implies (3.1).

To prove (3.2), replace S by S^* in (3.1), interchange w and z in (3.1) and then use the equation

$$(S^* K_w)(z) = \langle S^* K_w, K_z \rangle = \langle K_w, SK_z \rangle = \overline{SK_z}(w) \tag{3.3}$$

to obtain the desired result. ■

Lemma 3.3 *Let $S \in \mathfrak{L}(L_a^2)$ and $p > n + 2$. Then*

$$\|S\| \le C(n, p) \left(\sup_{z \in B} \|S_z 1\|_p \right)^{1/2} \left(\sup_{z \in B} \|S_z^* 1\|_p \right)^{1/2}$$

where $C(n, p)$ is the constant of Lemma 3.2.

Proof. (3.3) implies

$$(Sf)(w) = \langle Sf, K_w \rangle = \int_B f(z) \overline{(S^* K_w)}(z) dz = \int_B f(z) (SK_z)(w) dz$$

for $f \in L_a^2$ and $w \in B$. Thus, Lemma 3.2 and the classical Schur's theorem finish the proof. ■

Lemma 3.4 *Let S_m be a bounded sequence in $\mathfrak{L}(L_a^2)$ such that*

$$\|B_0 S_m\|_\infty \to 0 \quad \text{as } m \to \infty.$$

Then

$$\sup_{z\in B} |\langle (S_m)_z 1, f\rangle| \to 0 \tag{3.4}$$

as $m \to \infty$ for any $f \in L_a^2$ and

$$\sup_{z\in B} |(S_m)_z 1| \to 0 \tag{3.5}$$

uniformly on compact subsets of B as $m \to \infty$.

Proof. To prove (3.4), we only need to have

$$\sup_{z\in B} \left|\left\langle (S_m)_z 1, w^k \right\rangle\right| \to 0 \tag{3.6}$$

as $m \to \infty$ for any multi-index k.

Since

$$K_z(w) = \sum_{|\alpha|=0}^{\infty} \frac{(n+|\alpha|)!}{n!\alpha!} \overline{z}^\alpha w^\alpha, \tag{3.7}$$

we have

$$\begin{aligned} &B_0 S_m(\phi_z(\lambda)) = B_0(S_m)_z(\lambda) \\ &= (1-|\lambda|^2)^{n+1} \sum_{|\alpha|=0}^{\infty} \sum_{|\beta|=0}^{\infty} \frac{(n+|\alpha|)!}{n!\alpha!} \frac{(n+|\beta|)!}{n!\beta!} \left\langle (S_m)_z w^\alpha, w^\beta \right\rangle \overline{\lambda}^\alpha \lambda^\beta \end{aligned}$$

where α, β are multi-indices.

Then for any fixed k and $0 < r < 1$,

$$\begin{aligned} &\int_{rB} \frac{B_0 S_m(\phi_z(\lambda)) \overline{\lambda}^k}{(1-|\lambda|^2)^{n+1}} d\lambda \\ &= \sum_{|\alpha|=0}^{\infty} \sum_{|\beta|=0}^{\infty} \frac{(n+|\alpha|)!}{n!\alpha!} \frac{(n+|\beta|)!}{n!\beta!} \left\langle (S_m)_z w^\alpha, w^\beta \right\rangle \int_{rB} \overline{\lambda}^{\alpha+k} \lambda^\beta d\lambda \\ &= r^{2n+2|k|} \left(\left\langle (S_m)_z 1, w^k \right\rangle + \sum_{|\alpha|=1}^{\infty} \frac{(n+|\alpha|)!}{n!\alpha!} \left\langle (S_m)_z w^\alpha, w^{\alpha+k} \right\rangle r^{2|\alpha|} \right). \end{aligned}$$

Since S_m is bounded sequence, we have

$$\begin{aligned}\left|\left\langle (S_m)_z 1, w^k\right\rangle\right| \leq r^{-2n-2|k|}&\left|\int_{rB} \frac{B_0 S_m(\phi_z(\lambda))\overline{\lambda}^k}{(1-|\lambda|^2)^{n+1}} d\lambda\right| + \\ &\sum_{|\alpha|=1}^{\infty} \frac{(n+|\alpha|)!}{n!\alpha!} \|(S_m)_z\| \|w^\alpha\| \|w^{\alpha+k}\| r^{2|\alpha|} \\ &\leq r^{-2n-2|k|} \|B_0 S_m\|_\infty \int_{rB} \frac{|\lambda^k|}{(1-|\lambda|^2)^{n+1}} d\lambda + C \sum_{|\alpha|=1}^{\infty} r^{2|\alpha|},\end{aligned}$$

hence, by assumption

$$\limsup_{m\to\infty} \sup_{z\in B} \left| \left\langle (S_m)_z 1, w^k \right\rangle \right| \leq C \sum_{|\alpha|=1}^{\infty} r^{2|\alpha|}.$$

Letting $r \to 0$, we have (3.6).

Now we prove (3.5). From (3.7), we get

$$\begin{aligned}|(S_m)_z 1(\lambda)| &= |\langle (S_m)_z 1, K_\lambda\rangle| \\ &\leq \sum_{|\alpha|=0}^{\infty} \frac{(n+|\alpha|)!}{n!\alpha!} |\langle (S_m)_z 1, w^\alpha\rangle| \, |\lambda^\alpha| \\ &\leq \sum_{|\alpha|=0}^{l-1} \frac{(n+|\alpha|)!}{n!\alpha!} |\langle (S_m)_z 1, w^\alpha\rangle| + \sum_{|\alpha|=l}^{\infty} \frac{(n+|\alpha|)!}{n!\alpha!} \|S_m\| \|w^\alpha\| |\lambda^\alpha|\end{aligned}$$

for $z \in B$, $\lambda \in rB$ and $l \geq 1$. Since the second summation is less than or equals to

$$\begin{aligned}\sum_{j=l}^{\infty} \left(\frac{(n+j)!}{n!j!}\right)^{1/2} \sum_{|\alpha|=j} \left(\frac{j!}{\alpha!}\right)^{1/2} |\lambda^\alpha| &\leq \sum_{j=l}^{\infty} \frac{(n+j)!}{n!j!} \left[\sum_{|\alpha|=j} \frac{j!}{\alpha!} |\lambda^\alpha|^2\right]^{1/2} \\ &\leq \sum_{j=l}^{\infty} \frac{(n+j)!}{n!j!} r^j,\end{aligned}$$

for any $\epsilon > 0$, we can find sufficiently large l such that the second summation is less than ϵ. Thus, (3.6) imply $\sup_{z\in B} |(S_m)_z 1| \to 0$ uniformly on compact subsets of B as $m \to \infty$. ■

Lemma 3.5 *Let $\{S_m\}$ be a sequence in $\mathfrak{L}(L_a^2)$ such that for some $p > n+2$, $\|B_0 S_m\|_\infty \to 0$ as $m \to \infty$,*

$$\sup_{z\in B} \|(S_m)_z 1\|_p \leq C \qquad \text{and} \qquad \sup_{z\in B} \|(S_m^*)_z 1\|_p \leq C$$

where $C > 0$ is independent of m, then $S_m \to 0$ as $m \to \infty$ in $\mathfrak{L}(L_a^2)$-norm.

Proof. Lemma 3.3 implies

$$\|S_m\| \le C(n,p)\left(\sup_{z\in B}\|(S_m)_z1\|_p\right)^{1/2}\left(\sup_{z\in B}\|(S_m^*)_z1\|_p\right)^{1/2} \le C(n,p),$$

hence, Lemma 3.4 gives

$$\sup_{z\in B}|(S_m)_z1| \to 0 \tag{3.8}$$

uniformly on compact subsets of B as $m \to \infty$.

Here, for $n+2 < s < p$, Holder's inequality gives

$$\begin{aligned}\sup_{z\in B}\|(S_m)_z1\|_s^s &\le \sup_{z\in B}\int_{B\setminus r\overline{B}}|(S_m)_z1(w)|^s dw + \sup_{z\in B}\int_{r\overline{B}}|(S_m)_z1(w)|^s dw \\ &\le C\sup_{z\in B}\|(S_m)_z1\|_p^s(1-r)^{1-s/p} + \sup_{z\in B}\int_{r\overline{B}}|(S_m)_z1(w)|^s dw\end{aligned}$$

and (3.8) implies the second term tends to 0 as $m \to \infty$. Also, the first term is less than or equals to $C^s(1-r)^{1-s/p}$ which can be small by taking r close to 1. Consequently, Lemma 3.3 gives

$$\begin{aligned}\|S_m\| &\le C(n,s)\left(\sup_{z\in B}\|(S_m)_z1\|_s\right)^{1/2}\left(\sup_{z\in B}\|(S_m^*)_z1\|_s\right)^{1/2}. \\ &\le C(n,s)\left(\sup_{z\in B}\|(S_m)_z1\|_s\right)^{1/2} \to 0\end{aligned}$$

■

Corollary 3.6 *Let $S \in \mathfrak{L}(L_a^2)$ such that for some $p > n+2$,*

$$\sup_{z\in B}\|S_z1 - (T_{B_mS})_z1\|_p \le C \quad \text{and} \quad \sup_{z\in B}\|S_z^*1 - (T_{B_m(S^*)})_z1\|_p \le C, \tag{3.9}$$

where $C > 0$ is independent of m. Then $T_{B_mS} \to S$ as $m \to \infty$ in $\mathfrak{L}(L_a^2)$-norm.

Proof. Let $S_m = S - T_{B_mS}$. Then Proposition 2.14 and Theorem 2.11 imply

$$B_0(S_m) = B_0S - B_0(T_{B_mS}) = B_0S - B_0(B_mS) = B_0S - B_m(B_0S)$$

which tends uniformly to 0 as $m \to \infty$, hence $\|B_0(S_m)\|_\infty \to 0$. Consequently, by Lemma 3.5 we complete the proof. ■

Theorem 3.7 *Let $S \in \mathfrak{L}(L_a^2)$. If there is $p > n+2$ such that*

$$\sup_{z\in B} \|T_{(B_m S)\circ\phi_z} 1\|_p < C \quad \textit{and} \quad \sup_{z\in B} \|T^*_{(B_m S)\circ\phi_z} 1\|_p < C \tag{3.10}$$

where $C > 0$ is independent of m, then $T_{B_m S} \to S$ as $m \to \infty$ in $\mathfrak{L}(L_a^2)$-norm.

Proof. By Corollary 3.6, we only need to show that (3.10) implies (3.9). Since $T_{(B_m S)\circ\phi_z} = (T_{B_m S})_z$ and

$$T^*_{(B_m S)\circ\phi_z} = T_{\overline{B_m S_z}} = T_{B_m(S_z^*)} = T_{(B_m(S^*))\circ\phi_z},$$

it is sufficient to show that

$$\sup_{z\in B} \|S_z 1\|_p < \infty.$$

By Lemma 3.3, we get

$$\|T_{B_m S}\| \le C(n,p) \left(\sup_{z\in B} \|T_{B_m S\circ\phi_z} 1\|_p\right)^{1/2} \left(\sup_{z\in B} \|T^*_{B_m S\circ\phi_z} 1\|_p\right)^{1/2} < C$$

where C is independent of m, hence writing $S_m = S - T_{B_m S}$, we have $\|S_m\| \le C$ where C is independent of m. Also, the proof of Corollary 3.6 implies

$$\|B_0 S_m\|_\infty \to 0$$

as $m \to \infty$.

Let f be a polynomial with $\|f\|_q = 1$. Then Lemma 3.4 implies

$$\sup_{z\in B} |\langle (S_m)_z 1, f\rangle| \to 0$$

as $m \to \infty$. Thus, for any $\epsilon > 0$ and $z_0 \in B$, we have

$$|\,\langle S_{z_0} 1, f\rangle\,| \le \sup_{z\in B} |\,\langle (S_m)_z 1, f\rangle\,| + |\,\langle (T_{B_m S})_{z_0} 1, f\rangle\,| \le \epsilon + C$$

for sufficiently large m, where C is independent of m. Since ϵ is arbitrary, we get

$$\sup_{z\in B} \|S_z 1\|_p < \infty$$

as desired. ■

4. Compact Radial operator

Given $\mathcal{U} \in \mathfrak{U}(n)$, define $V_{\mathcal{U}} f(w) = f(\mathcal{U}w) \det \mathcal{U}$ for $f \in L^2_a$. Then $V_{\mathcal{U}}$ is a unitary operator on L^2_a. We say that $S \in \mathfrak{L}(L^2_a)$ is a radial operator if $SV_{\mathcal{U}} = V_{\mathcal{U}} S$ for any $\mathcal{U} \in \mathfrak{U}(n)$.

If $S \in \mathfrak{L}(L^2_a)$, the radialization of S is defined by

$$S^\sharp = \int_{\mathfrak{U}} V_{\mathcal{U}}{}^* S V_{\mathcal{U}} \, d\mathcal{U}$$

where $d\mathcal{U}$ is the Haar measure on the compact group $\mathfrak{U}(n)$ and the integral is taken in the weak sense. Then $S^\sharp = S$ if S is radial and $\mathfrak{U}$-invariance of $d\mathcal{U}$ shows that $S^\sharp$ is indeed a radial operator.

If $f \in L^\infty$ and $g, h \in L^2_a$ then

$$\langle V_{\mathcal{U}}{}^* T_f V_{\mathcal{U}} g, h\rangle = \int_B f(w) V_{\mathcal{U}} g(w) \overline{V_{\mathcal{U}} h(w)} dw = \int_B f(\mathcal{U}^* w) g(w) \overline{h(w)} dw.$$

Thus $V_{\mathcal{U}}{}^* T_f V_{\mathcal{U}} = T_{f \circ \mathcal{U}^*}$ and

$$V_{\mathcal{U}}{}^* T_{f_1} \cdots T_{f_l} V_{\mathcal{U}} = T_{f_1 \circ \mathcal{U}^*} \cdots T_{f_l \circ \mathcal{U}^*}$$

for $f_1, \ldots, f_l \in L^\infty$, $l \geq 0$.

Lemma 4.1 *Let $S \in \mathfrak{L}(L^2_a)$ be a radial operator. Then $T_{B_m(S)} = \int_B S_w d\nu_m(w)$.*

Proof. Let $z \in B$. By (2.3) and Lemma 2.8, we obtain

$$\begin{aligned} B_0 \left(\int_B S_w d\nu_m(w) \right)(z) &= \left\langle \left(\int_B S_w d\nu_m(w) \right)_z 1, 1 \right\rangle \\ &= \int_B \langle U_z U_w S U_w U_z 1, 1\rangle \, d\nu_m(w) \\ &= \int_B \left\langle U_{\phi_z(w)} V_{\mathcal{U}}^* S V_{\mathcal{U}} U_{\phi_z(w)} 1, 1 \right\rangle d\nu_m(w) \end{aligned}$$

where $V_{\mathcal{U}}$ is in Lemma 2.8. Since S is a radial operator, Theorem 2.9, Proposition 2.3 and Proposition 2.14 imply that the last integral equals

$$\begin{aligned} \int_B \left\langle U_{\phi_z(w)} S U_{\phi_z(w)} 1, 1 \right\rangle d\nu_m(w) &= \int_B B_0 S \circ \phi_z(w) d\nu_m(w) \\ &= B_m B_0 S(z) \\ &= B_0 B_m S(z) \\ &= B_0(T_{B_m(S)})(z). \end{aligned}$$

Since B_0 is one-to-one mapping, the proof is complete. ■

Theorem 4.2 *Let $S \in \mathfrak{T}(L^\infty)$ be a radial operator. Then S is compact if and only if $B_0 S \equiv 0$ on ∂B.*

Proof. Suppose $B_0 S \equiv 0$ on ∂B. Then $B_m S \equiv 0$ on ∂B by Proposition 2.15, hence $T_{B_m S}$ is compact for all $m \geq 0$.

Let

$$Q = \int_{\mathfrak{U}} T_{f_1 \circ \mathcal{U}^*} \cdots T_{f_l \circ \mathcal{U}^*} \, d\mathcal{U}$$

with $f_1, \ldots, f_l \in L^\infty$ for some $l \geq 0$. Then $Q \in \mathfrak{L}(L_a^2)$. By Lemma 4.1, for any $z \in B$, we have

$$\begin{aligned} T_{(B_m(Q)) \circ \phi_z} &= \int_B ((Q)_z)_w d\nu_m(w) \\ &= \int_B \int_{\mathfrak{U}} T_{f_1 \circ \mathcal{U}^* \circ \phi_z \circ \phi_w} \cdots T_{f_l \circ \mathcal{U}^* \circ \phi_z \circ \phi_w} \, d\mathcal{U} \, d\nu_m(w). \end{aligned}$$

Consequently,

$$\begin{aligned} \|T_{(B_m(Q)) \circ \phi_z}\| &\leq C(l) \|f_1 \circ \mathcal{U}^* \circ \phi_z \circ \phi_w\|_\infty \cdots \|f_l \circ \mathcal{U}^* \circ \phi_z \circ \phi_w\|_\infty \\ &= C(l) \|f_1\|_\infty \cdots \|f_l\|_\infty. \end{aligned}$$

Similarly, we have

$$\|T^*_{(B_m(Q)) \circ \phi_z}\| \leq C(l) \|f_1\|_\infty \cdots \|f_l\|_\infty.$$

Thus, Theorem 3.7 gives that

$$T_{B_m(Q)} \to Q \tag{4.1}$$

in $\mathfrak{L}(L_a^2)$-norm.

Since $S \in \mathfrak{T}(L^\infty)$, there exists a sequence $\{S_k\}$ such that $S_k \to S$ in $\mathfrak{L}(L_a^2)$-norm where each S_k is a finite sum of finite products of Toeplitz operators. Since the radialization is continuous and S is radial, $S_k^\sharp \to S^\sharp = S$. From Lemma 4.1, we have

$$\|T_{B_m S}\| = \left\| \int_B S_w d\nu_m(w) \right\| \leq \int_B \|S_w\| d\nu_m(w) = \|S\|.$$

Thus

$$\begin{aligned} \|S - T_{B_m S}\| &\leq \|S - S_k^\sharp\| + \|S_k^\sharp - T_{B_m(S_k^\sharp)}\| + \|T_{B_m(S_k^\sharp)} - T_{B_m S}\| \\ &\leq 2\|S - S_k^\sharp\| + \|S_k^\sharp - T_{B_m(S_k^\sharp)}\| \end{aligned}$$

and (4.1) imply $T_{B_m(S)} \to S$ as $m \to \infty$ in $\mathfrak{L}(L_a^2)$-norm, hence S is compact.

The other direction is trivial. ■

Example. This example shows that for $n = 1$, the number $n + 2 = 3$ in Theorem 3.7 is sharp. We show that there is a bounded operator S on L_a^2 such that

$$\sup_{z\in D} \max\{\|T_{(B_m S)\circ\phi_z} 1\|_3, \|T^*_{(B_m S)\circ\phi_z} 1\|_3\} < \infty,$$

and for each $m \geq 0$, $B_m(S)(z) \to 0$ as $z \to \partial D$, but S is not compact on L_a^2.

The following operator S was constructed in [3] to show that $B_0(S)(z) \to 0$ as $z \to \partial D$, but S is not compact on L_a^2. Let S be defined on L_a^2 by

$$S\left(\sum_{l=0}^{\infty} a_l w^l\right) = \sum_{l=0}^{\infty} a_{2^l} w^{2^l}.$$

It is clear that S is a self-adjoint projection with infinite-dimensional range. Thus S is not compact on L_a^2. From

$$B_0(S)(z) = \langle Sk_z, k_z\rangle = \|Sk_z\|_2^2 = (1 - |z|^2)^2 \sum_{l=0}^{\infty} (2^l + 1)(|z|^2)^{2^l},$$

it is easy to see that

$$B_0(S)(z) \to 0 \quad \text{as } z \to \partial D.$$

By Proposition 2.15, we see that

$$B_m(S)(z) \to 0 \quad \text{as } z \to \partial D.$$

This gives that $T_{B_m(S)}$ is compact. Hence $T_{B_m(S)}$ does not converge to S in the norm topology.

By means of the Zygmund theorem on gap series [18], it was proved in [13] that

$$C = \sup_{z\in D} \max\{\|S_z 1\|_3, \|S_z^* 1\|_3\} < \infty.$$

Clearly, S is a radial operator. By Lemma 4.1, we have

$$T_{(B_m S)\circ\phi_z} 1 = \int_D (S_w)_z 1 d\nu_m(w) = \int_D S_{\phi_z(w)} 1 d\nu_m(w) = \int_D S_\lambda 1 d\nu_m \circ \phi_z(\lambda).$$

Noting that for each $z \in D$, $d\nu_m \circ \phi_z$ is a probability measure on D, we have

$$\|T_{(B_m S)\circ\phi_z} 1\|_3 \leq \int_D \|S_\lambda 1\|_3 d\nu_m \circ \phi_z(\lambda) \leq C.$$

Similarly, we also have

$$\|T^*_{(B_m S)\circ\phi_z} 1\|_3 \leq C.$$

References

[1] Ahern, P., Flores, M. and Rudin, W.: An invariant volume-mean-value property. *J. Funct. Anal.* **111** (1993), 380–397.

[2] Axler, S. and Zheng, D.: The Berezin transform on the Toeplitz algebra. *Studia Math.* **127** (1998), 113–136.

[3] Axler, S. and Zheng, D.: Compact operators via the Berezin transform. *Indiana Univ. Math. J.* **47** (1998), 387–399.

[4] Békollé, D., Berger, C. A., Coburn, L. A. and Zhu, K. H.: BMO in the Bergman metric on bounded symmetric domains. *J. Funct. Anal.* **93** (1990), 310–350.

[5] Berezin, F. A.: Covariant and contravariant symbols of operators. *Math. USSR-Izv.* **6** (1972), 1117–1151.

[6] Berger, C. A. and Coburn, L. A.: Toeplitz operators on the Segal-Bargmann space. *Trans. Amer. Math. Soc.* **301** (1987), 813–829.

[7] Berger, C. A. and Coburn, L. A.: Heat flow and Berezin-Toeplitz estimates. *Amer. J. Math.* **116** (1994), 563–590.

[8] Bratteli, O. and Robinson, D. W.: *Operator algebras and quantum statistical mechanics. I, II.* Texts and Monographs in Physics. Springer Verlag, New York 1979, 1981.

[9] Choe, B. R. and Lee, Y. J.: Pluriharmonic symbols of commuting Toeplitz operators. *Illinois J. Math.* **37** (1993), 424–436.

[10] Coburn, L. A.: A Lipschitz estimate for Berezin's operator calculus. *Proc. Amer. Math. Soc.* **133** (2005), 127–131.

[11] Engliš, M.: Compact Toeplitz operators via the Berezin transform on bounded symmetric domains. *Integral Equations Operator Theory* **33** (1999), 426–455.

[12] Gohberg, I. C. and Kreĭn, M. G.: *Introduction to the theory of linear nonselfadjoint operators.* Translations of Mathematical Monographs **18**. American Mathematical Society, Providence, R.I., 1969.

[13] Miao, J. and Zheng, D.: Compact operators on Bergman spaces. *Integral Equations Operator Theory* **48** (2004), 61–79.

[14] Rudin, W.: *Function Theory in the unit ball of* $\mathbb{C}^n$. Fundamental Principles of Mathematical Science **241**. Springer-Verlag, New York-Berlin, 1980.

[15] Stroethoff, K.: The Berezin transform and operators on spaces of analytic functions. In *Linear operators (Warsaw, 1994)*, 361–380. Banach Center Publ. **38**. Polish Acad. Sci., Warsaw, 1997.

[16] Suárez, D.: Approximation and symbolic calculus for Toeplitz algebras on the Bergman space. *Rev. Mat. Iberoamericana* **20** (2004), 563–610.

[17] Suárez, D.: Approximation and the n-Berezin transform of operators on the Bergman space. *J. Reine Angew. Math.* **581** (2005), 175–192.

[18] Zygmund, A.: *Trigonometric series*. Cambridge University Press, New York, 1959.

Recibido: 13 de septiembre de 2004
Revisado: 5 de abril de 2005

Kyesook Nam
Department of Mathematics
Hanshin University
Gyeonggi 447-791, KOREA
ksnam@hanshin.ac.kr

Dechao Zheng
Department of Mathematics
Vanderbilt University
Nashville, Tennessee 37240, USA
zheng@math.vanderbilt.edu

Changyong Zhong
Department of Mathematics
Vanderbilt University
Nashville, Tennessee 37240, USA
zhong@math.vanderbilt.edu

The first author was partially supported by KOSEF(R01-2003-000-10243-0). The second author was partially supported by the National Science Foundation.