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AN ALTERNATE SOLUTION TO SCOTTISH BOOK 157

Abstract

In 1971, D. Ornstein proved a theorem that completely solved *Problem 157* of the *Scottish Book.* The purpose of this paper is to give an independent proof.

1 Introduction

In 1971, D. Ornstein, [4] proved a theorem that directly solves *Problem 157* of the *Scottish Book*, see [5]. In this issue of the *Exchange* there are two related Inroads papers, [1] and [2]. In [1] the history of *Problem 157* is described and a solution is given using O'Malley's Theorem for the existence of approximate extrema of approximately continuous functions. In [2] a separate proof of O'Malley's Theorem is presented. The purpose of this paper is to present an independent proof of the original *Scottish Book Problem 157*.

We adopt the notation introduced in [2] repeating several of the definitions for completeness. All sets and functions considered here will be assumed to be measurable with respect to λ, Lebesgue measure on $\mathbb{R}$. Suppose $E \subset \mathbb{R}$ and J is a given interval with length $|J|$. Then the density (or relative measure) of E in J is $\Delta(E, J) = \lambda(E \cap J)/|J|$. The upper density of E at a point $x \in \mathbb{R}$ is defined as $\limsup_{r\to 0^+} \Delta(E, (x - r, x + r))$ and is denoted by $\overline{\delta}(E, x)$. The

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lower density at x, $\underline{\delta}(E,x)$ is defined similarly where lim inf replaces lim sup. If these two are equal at x, their common value is called the density of E at x and is denoted $\delta(E,x)$.

A function $f:\mathbb{R}\to\mathbb{R}$ is approximately continuous at x_0 if there is a set E with density 1 at x_0, so that

$$\lim_{x\in E,\ x\to x_0} f(x) = f(x_0).$$

If $y\in\mathbb{R}$, the function f determines two associated sets that we'll make use of; $f^{-1}((-\infty,y))$ and $f^{-1}([y,\infty))$. These are denoted as L_y and U_y respectively when the function f is established.

Ornstein's Theorem is the following, see [4].

Ornstein's Theorem. *Let $f(x)$ be a real–valued function of a real variable satisfying the following:*

(A) *$f(x)$ is approximately continuous,*

(B) *For each x_0, $\limsup_{h\to 0^+}\Delta(U_{f(x_0)},(x_0,x_0+h))\neq 0$.*

Then, f is monotone increasing and continuous.

2 Proof of Ornstein's Theorem

First note that if a function is both monotone and approximately continuous then it's continuous, so monotonicity is the only issue. So suppose $f:[a,b]\to\mathbb{R}$ satisfies conditions (A) and (B) of Ornstein's hypothesis above. We must show $f(a)\le f(b)$. To do this we have a closer look at the conditions (A) and (B). First, (A) implies both of the following, considerably weaker, one-sided conditions.

(A1) $$\forall x\in[a,b)\ \forall\epsilon>0\ \exists\delta>0 \text{ such that } \forall\ z\in(x,x+\delta),$$
$$\Delta(U_{f(x)-\epsilon},(x,z))>\frac{1}{2}.$$

(A2) $$\forall x\in(a,b]\ \forall\epsilon>0\ \exists\delta>0 \text{ such that } \forall\ z\in(x-\delta,x),$$
$$\Delta(L_{f(x)+\epsilon},(z,x))>1-\epsilon.$$

Condition (B) above can be restated in a similar fashion as:

(B) $$\forall x \in [a,b)\ \exists \epsilon > 0 \text{ such that } \forall \delta > 0\ \exists\ z \in (x, x+\delta),$$
$$\text{with } f(z) \geq f(x) \text{ and } \Delta(U_{f(x)}, (x,z)) > \epsilon.$$

Conditions (A1) and (B) have a somewhat complementary structure and together these conditions imply a useful global density condition, (C) below.

(C) $$\forall x \in [a,b)\ \forall \delta > 0\ \exists z \in (x, x+\delta) \text{ such that } f(z) \geq f(x)$$
$$\text{and } \forall\ \epsilon \in (0, \frac{1}{2}),\ \Delta(U_{f(x)-\epsilon}, (x,z)) > \epsilon.$$

Lemma 1. *If $f : [a,b] \to \mathbb{R}$ satisfies* (A1) *and* (B) *then f also satisfied* (C).

Proof. Let $x \in [a,b)$ be fixed. Since f satisfies (B), there is an $\epsilon_o > 0$ so that

$$\forall \delta > 0\ \exists z \in (x, x+\delta) \text{ with } f(z) \geq f(x) \text{ and } \Delta(U_{f(x)}, (x,z)) > \epsilon_o. \tag{1}$$

Applying condition (A1) for this ϵ_o yields a $\delta' > 0$ so that for every $z \in (x, x+\delta')$,

$$\Delta(U_{f(x)-\epsilon_o}, (x,z)) > \frac{1}{2}. \tag{2}$$

Now fix $\delta > 0$ and let $\delta_o = \min(\delta, \delta^{'})$. Then by (1) there is a $z_o \in (x, x+\delta_o)$ with

$$f(z_o) \geq f(x) \text{ and } \Delta(U_{f(x)}, (x, z_o)) > \epsilon_o \tag{3}$$

And since $\delta_o \leq \delta^{'}$ we also have that

$$\Delta(U_{f(x)-\epsilon_o}, (x, z_o)) > \frac{1}{2}. \tag{4}$$

Finally, let $\epsilon \in (0, \frac{1}{2})$.

Case 1 $\epsilon \in (0, \epsilon_o]$

In this case, $U_{f(x)} \subset U_{f(x)-\epsilon}$ so that by (3)

$$\Delta(U_{f(x)-\epsilon}, (x, z_o)) > \epsilon_o \geq \epsilon.$$

Case 2 $\epsilon \in (\epsilon_o, \frac{1}{2})$

Here, $U_{f(x)-\epsilon_o} \subset U_{f(x)-\epsilon}$ so that by (2),

$$\Delta(U_{f(x)-\epsilon}, (x, z_o)) \geq \Delta(U_{f(x)-\epsilon_o}, (x, z_o)) > \frac{1}{2} \geq \epsilon.$$

This completes the proof of Lemma 1. □

Remark 2. *If* $f : [a, b] \to \mathbb{R}$ *satisfies* (C) *then for every* $x \in [a, b)$ *there is a* $z = z(x) \in (x, b]$ *such that*

i. $f(x) \leq f(z)$, *and*

ii. If $\epsilon \in (0, \frac{1}{2})$, *then* $\Delta(U_{f(x)-\epsilon}, (x, z)) > \epsilon$.

We're now prepared to prove the following.

Theorem 3. *Let* $f : [a, b] \to \mathbb{R}$. *Then* f *is monotone increasing if and only if* f *satisfies conditions* (A2) *and* (C).

Proof. If f is increasing, then it immediately follows from the definitions that f satisfies both conditions.

So suppose that f satisfies both (A2) and (C). We begin by using Remark 2 to define a (possibly transfinite) sequence, $\{x_\alpha\}$ as follows. Let $x_0 = a$ and suppose that x_β has been defined for all $\beta < \alpha$. If $\alpha = \alpha_o + 1$ is a successor ordinal, then define $x_\alpha = z(x_{\alpha_o})$ as per the remark above. If α is a limit ordinal, simply let $x_\alpha = \sup\{x_\beta : \beta < \alpha\}$.

Then this process terminates after countably many, say γ steps, and necessarily $x_\gamma = b$. It suffices to show that $\{f(x_\alpha) : \alpha \leq \gamma\}$ is a monotone increasing sequence. At non-limit ordinals, monotonicity is simply a consequence of Remark 2*i.* However, at limit ordinals there's some work to be done. To this end, suppose $\lambda \leq \gamma$ is a limit ordinal and assume $\{f(x_\alpha) : \alpha < \lambda\}$ is monotone increasing. We must show that $f(x_\lambda) \geq \lim_{\alpha<\lambda} f(x_\alpha)$.

Let $\epsilon \in (0, \frac{1}{2})$. Using the fact that $\{[x_\beta, x_{\beta+1}) : \alpha \leq \beta < \lambda\}$ partitions the interval $[x_\alpha, x_\lambda)$ and Remark 2*ii,* we have that for all $\alpha < \lambda$,

$$\Delta(U_{f(x_\alpha)-\epsilon}, (x_\alpha, x_\lambda)) > \epsilon. \tag{5}$$

Since λ is a limit ordinal, $\lim_{\alpha<\lambda} x_\alpha = x_\lambda$ and so by (A2), x_α can be chosen sufficiently close to x_λ that

$$\Delta(L_{f(x_\lambda)+\epsilon}, (x_\alpha, x_\lambda)) > 1 - \epsilon. \tag{6}$$

However, (5) and (6) entailes that $f(x_\lambda) + \epsilon \geq f(x_\alpha) - \epsilon$. Since $\epsilon > 0$ is arbitrary this implies $f(x_\lambda) \geq \lim_{\alpha<\lambda} f(x_\alpha)$ as claimed. □

Remark 4. *Ornstein's Theorem follows directly from Theorem 3 since monotone functions that are approximately continuous are indeed continuous.*

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