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INTERVALS CONTAINING ALL THE PERIODIC POINTS

Abstract

For any map f from $\mathbb{R}$ to $\mathbb{R}$, if an interval J contains all periodic points of period 1 and 2, then $f(f(J))$ contains all periodic points (and therefore contains the centre of f).

1 Introduction

This paper is in continuation of the investigation on dynamics on the real line made in [6], [7], [8], [1], and [5]. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. For every positive integer n, $f^1 = f$ and $f^n = f \circ f^{n-1}$. An element $x \in \mathbb{R}$ is said to be a periodic point of period n if $f^n(x) = x$ and $f^i(x) \neq x$ for $1 \leq i \leq n-1$. Let $P(f)$ denote the set of all periodic points of f and $\text{Fix}(f)$ denote the set of all fixed points of f. A point $x \in I$ is a recurrent point if $x \in \omega(x)$, where $\omega(x) = \bigcap_{m \geq 0} \overline{\bigcup_{n \geq m} f^n(x)}$. The set of recurrent points is denoted by $R(f)$ and the *centre* of f equals the closure of the set of all recurrent points. By the convex hull of A we mean the smallest interval containing A.

The intermediate value theorem guarantees that if $P(f)$ is non-empty, then $\text{Fix}(f^2)$ is nonempty and the convex hull of every periodic orbit contains at least one point of $\text{Fix}(f^2)$ (in fact, $\text{Fix}(f)$). Dually we ask: will the convex hull of $\text{Fix}(f)$ or $\text{Fix}(f^2)$ meet every cycle? For $\text{Fix}(f)$ it need not be true; that it is true for $\text{Fix}(f^2)$ has been proved in this paper.

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Already there are known results [2] describing the location of periodic points forced by a given cycle. Working in the reverse direction: when $\mathrm{Fix}(f^2)$ is known, we have results about the location of $P(f)$. Here we have succeeded in proving that $\mathrm{Fix}(f^2)$ has to be well-spread in two different senses: (1) For every periodic point p, there exists a point x between two elements of $\mathrm{Fix}(f^2)$ such that $f^2(x) = p$; (2) Every periodic orbit contains a point that lies between two points of $\mathrm{Fix}(f^2)$. The analogue of this main theorem is not true in the plane $\mathbb{R}^2$ or in the circle. Counterexamples can be easily constructed.

If $f : \mathbb{R} \to \mathbb{R}$ is the Tent map given by $f(x) = 1 - |1 - 2x|$, then we can easily calculate the following: $\mathrm{Fix}(f^2) = \{0, \frac{2}{5}, \frac{2}{3}, \frac{4}{5}\}$. The convex hull of $\mathrm{Fix}(f^2)$ is $[0, \frac{4}{5}]$. The image of the convex hull of $\mathrm{Fix}(f^2))$ is $[0,1]$. $P(f) = \{\frac{2m}{2n+1} \mid m \leq n \text{ in } \mathbb{N}\}$. We find $P(f) \subset f(\text{convex hull of } \mathrm{Fix}(f^2))$. We are interested in proving some general theorems that assert that for all continuous maps, inclusions similar to the above hold. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. If there are closed subintervals $I_0, I_1, ..., I_l$ of $\mathbb{R}$ with $I_l = I_0$ such that $f(I_i) \supset I_{i+1}$ for $i = 0, 1, ..., l-1$, then $I_0 I_1 I_2 ... I_l$ is called a cycle of length l. We write $I_i \to I_j$, if $f(I_i) \supset I_j$.

Lemma 1. *[4] If $I_0 I_1 I_2 ... I_l$ is a cycle of length l, then there exists a periodic point x of f such that $f^i(x) \in I_i$ for $i = 0, 1, ..., l-1$ and $f^l(x) = x$.*

2 Main results

Theorem 2. *For every real map f, $P(f) \subset f(f(\text{convex hull of } \mathrm{Fix}(f^2)))$.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Let $a = \inf\{x \in \mathbb{R} : f^2(x) = x\}$, $b = \sup\{x \in \mathbb{R} : f^2(x) = x\}$, where $-\infty \leq a < b \leq \infty$. Claim: If there is any periodic point of f to the right of b, then $f - \mathrm{id}$ and $f^2 - \mathrm{id}$ are both negative on (b, ∞). Let y be the rightmost point of a periodic orbit of period k intersecting (b, ∞). By definition, $k > 2$; thus $f(y)$ and $f^2(y)$, both points of this periodic orbit, must lie to the left of y. But the sign of $f - \mathrm{id}$ and $f^2 - \mathrm{id}$ is constant on any component of the complement of the convex hull of $\mathrm{Fix}(f^2)$. A similar argument (or looking at a conjugate of f via an orientation-reversing homeomorphism) shows that if there is any periodic point to the left of a, then both $f - \mathrm{id}$ and $f^2 - \mathrm{id}$ are positive on $(-\infty, a)$.

Now let us consider the case that one of a and b is finite; let $a = -\infty$ and some periodic orbit of period k intersects (b, ∞). Let y be the rightmost point in its orbit. Since $f - \mathrm{id}$ is negative on (b, ∞), $f^{k-1}(y) \notin (b, \infty)$, so $f^{k-1}(y) \in (-\infty, b]$. And therefore y belongs to $f(\text{convex hull of } \mathrm{Fix}(f^2)$, so orbit of y. Similarly if this convex hull is $[a, \infty)$.

If none of a and b is finite then the proof is trivial.

Assume now that both a and b are finite, let $[c,d] = f([a,b])$ and $[e,q] = f([c,d])$. Note that $[e,q] \supseteq [c,d] \supseteq [a,b]$. Suppose some periodic orbit of period k intersects (q,∞) and let y be its rightmost point. Then $k > 2$, and consider $f^{k-1}(y)$; since y is the highest point in its orbit, $f^{k-1}(y) < y$, but since $f(f^{k-1}(y)) = y$ and $f - \mathrm{id} < 0$ on (b,∞), we must have $f^{k-1}(y) < b$. It cannot belong to $[c,d]$ since $y \notin [e,q]$. So we have $f^{k-1}(y) < c$. Now consider $f^{k-2}(y)$: since $f \circ f(f^{k-2}(y)) = y$, if $f^{k-2}(y) > b$ we have a contradiction to $f^2 - \mathrm{id} < 0$ on (b,∞). Also, since y is not in $[e,q]$ we cannot have $f^{k-2}(y)$ in $[a,b]$. But then $f^{k-2}(y) < a$ and hence $f^{k-2}(y) < f^{k-1}(y)$ since $f - \mathrm{id}$ is positive on $(-\infty,a)$. Now let us suppose for some m, $0 < m < k$, we have $f^m(y) \in [b,y)$; note that $f^{k-1}(y), f^{k-2}(y)$ must lie below c, so $m < k-2$. Assume without loss of generality that m is the maximum value (among $0 < m < k-2$) with $f^m(y) \in [b,y)$. Then for $j = m+1,\ldots,k-1, f^j(y) < a$ and hence $f^j(y) < f^{j+1}(y)$ (since $f - -\mathrm{id}$ is positive on (∞,a). This means we have $f^{m+1}(y) < f^{m+2}(y) < \ldots < f^{k-1}(y) < c < b < f^m(y) < y$ and in particular, $f[f^{k-1}(y), f^m(y)] \supset [f^{m+1}(y), y] \supset [f^{k-2}(y), f^{k-1}(y)]$ while $f[f^{k-2}(y), f^{k-1}(y)] \supset [f^{k-1}(y), y] \supset [f^{k-1}(y), f^m(y)]$. Thus, $f^2[f^{k-2}(y), f^{k-1}(y)] \supset [f^{k-2}(y), f^{k-1}(y)]$ and this is an interval disjoint from $[a,b]$ intersecting $\mathrm{Fix}(f^2)$, a contradiction. □

Theorem 3. *Let f be a continuous function on the real line.* $\mathrm{Fix}(f^2)$ *is bounded if and only if $P(f)$ is bounded.*

Proof. This follows from Theorem 2. □

Corollary 4. *For a real map f, if* $\mathrm{Fix}(f^2)$ *is bounded above but not below, then f(convex hull of* $\mathrm{Fix}(f^2)$*)* $\supset P(f)$.

Proof. This follows from the first part of the proof of Theorem 2. □

Remark 1. *The analogue of the above corollary is true if f is bounded below but not above. The proof is similar to the above.*

Theorem 5.

(1) *For every real map f, if* $\mathrm{Fix}(f^2)$ *is unbounded, then*

$$P(f) \subset f(\textit{convex hull of } \mathrm{Fix}(f^2)).$$

(2) *There exists a real map f such that $P(f) \not\subseteq f$(convex hull of* $\mathrm{Fix}(f^2)$*).*

Proof.

(1) This follows from the first part of the proof of Theorem 2.

(2) Define $f : [0,6] \to [0,6]$ as $f(x) = \begin{cases} -x+6 & x \in [0,3] \\ -3x+12 & x \in [3,4] \\ x-4 & x \in [4,6] \end{cases}$.

Then f is a piecewise linear map (see Figure 1 below) such that $f^2([0,1)) = (1,2]$ and $f^2((5,6]) = [4,5)$. Thus there is no periodic point of period 2 in $[0,1) \cup (5,6]$. On the other hand, since 1 and 5 are periodic with period 2, the convex hull of $\mathrm{Fix}(f^2)$ is $[1,5]$. Now, $f([1,5]) = [0,5]$, which does not contain the periodic point 6 of period 4. Hence $P(f) \not\subset f(\text{convex hull of } \mathrm{Fix}(f^2))$.

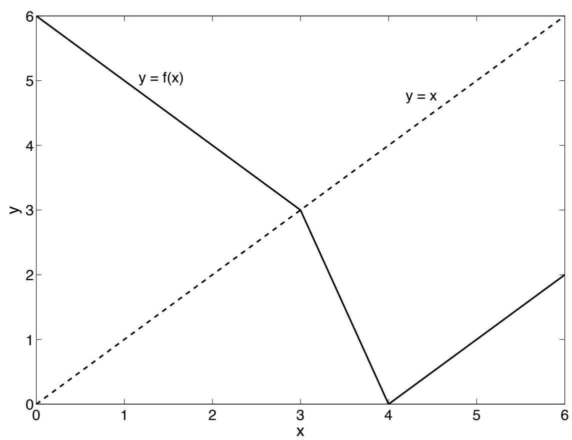

Figure 1: The function, f

□

Remark 2. *There is a real map f such that $\mathrm{Fix}(f^2)$ is bounded above, but $P(f)$ is not bounded above.*

Define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \begin{cases} -x & \text{if } x > 0 \\ 0 & \text{if } x \in [-1,0] \\ (4n+3)x + 8n^2 + 12n + 3 & \text{if } x \in [-2n-2, -2n-1] \text{ and } n \geq 0 \\ (-4n-1)x - 8n^2 - 4n - 1 & \text{if } x \in [-2n-1, -2n] \text{ and } n > 0 \end{cases}.$$

This f is nothing but the "linear extension" of the integer function

$$\begin{cases} f(n) = -n & \text{if } n \geq 0 \\ f(2n) = 2n-1 & \text{if } n < 0 \\ f(2n-1) = -2n & \text{if } n \leq 0 \end{cases}.$$

This f has no positive fixed point; no positive point of period 2. But every even positive integer is of period 3. Therefore $\mathrm{Fix}(f^2)$ is bounded above, but $P(f)$ is not bounded above.

Theorem 6. *For a real map f, for all $p \in P(f)$, there exists $n \in \mathbb{N}$ and x, $y \in \mathrm{Fix}(f^2)$ such that $x \leq f^n(p) \leq y$.*

Proof. First, let $\mathrm{Fix}(f^2)$ be bounded and $[a,b]$ be its convex hull. From the previous theorem, $P(f) \subset f^2(\text{convex hull of } \mathrm{Fix}(f^2))$. Let there be a periodic point of period $k(>2)$, whose orbit is in the complement of the convex hull of $\mathrm{Fix}(f^2)$, and let p be the rightmost point in its orbit. By an argument similar to the one in the proof of Theorem 2, the signs of $f - \mathrm{id}$, $f^2 - \mathrm{id}$ are positive on $(-\infty, a)$ and negative on (b, ∞). Then $p > b$ and $f^{k-2}(p) < f^{k-1}(p) < a$. Let us choose m as in the proof of Theorem 2, and proceeding in the same way, we get that there is a fixed point for f^2 in $[f^{k-2}(p), f^{k-1}(p)]$, which is a contradiction. So the orbit of p meets the convex hull of $\mathrm{Fix}(f^2)$.

If $\mathrm{Fix}(f^2)$ is unbounded then the proof is trivial. □

Remark 3. *For a continuous map f on I, let J be the smallest interval containing $\mathrm{Fix}(f^2)$. The following question is natural to ask: how is the convex hull of $P(f)$, denoted by $[P(f)]$, situated with respect to J, $f(J)$ and $f^2(J)$? We answer this question through examples.*

It is clear that $J \subset f(J) \subset f^2(J)$ and $[P(f)] \subset f^2(J)$.

1. *$J = [P(f)] = f(J) = f^2(J)$ for $f(x) = 1 - x$ on $[0,1]$.*

2. *$J \subset [P(f)] = f(J) \subset f^2(J)$ for the tent map*

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}.$$

3. *$J \subset [P(f)] = f(J) \subset f^2(J)$ for the following function: Define $f : [0,8] \to [0,8]$ linearly on every interval $[n, n+1]$, $0 \leq n \leq 7$, after defining, $f(0) = 5$, $f(1) = 6$, $f(2) = 7$, $f(3) = 8$, $f(4) = 4$, $f(5) = 0$, $f(6) = 3$, $f(7) = 2$ and $f(8) = 1$.*

4. *$J \subset f(J) \subset [P(f)] \subset f^2(J)$ for the following function: Define f : $[0,8] \to [0,8]$ linearly on every interval $[n, n+1]$, $0 \le n \le 7$, after defining, $f(0) = 7$, $f(1) = 8$, $f(2) = 6$, $f(3) = 5$, $f(4) = 4$, $f(5) = 0$, $f(6) = 2$, $f(7) = 3 = f(8)$.*

5. *$J \subset f(J) \subset [P(f)] = f^2(J)$ for the function given in (2) of Theorem 5.*

We note that these are the only possibilities.

Remark 4. *In fact $f^2(J)$ contains the centre of f. This will follow from:*

- $\mathrm{Fix}(f^2)$ *is closed.*
- $\overline{P(f)} = \overline{R(f)}$ *[3].*

3 Some final remarks

In this paper we have proved that (1) $f^2($ convex hull of $\mathrm{Fix}(f^2)) \supset P(f)$, and (2) every member of $P(f)$, at some time or the other, should come between two elements of $\mathrm{Fix}(f^2)$. In other words the smaller set $\mathrm{Fix}(f^2)$ is in some sense spread fairly enough in the bigger set $P(f)$.

We conclude the paper with the following open question that looks simple:

- Whenever m forces n (in the Sarkovski-sense and m, n being integers ≥ 2), should every m-cycle of f be contained in the f^2-image of the convex hull of the union of all n-cycles of f?

  If this is proved, the main theorem of this paper is the particular case when n = 2. (Some partial affirmative results can be proved.)

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