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ON ω-LIMIT SETS OF TRIANGULAR INDUCED MAPS

Abstract

Let X be a compact topological space and $T : X \to X$ a continuous map. Y.N. Dowker, F.G. Friedlander and A.N. Sharkovsky, independently, introduced and studied the notion of *T-connectedness*. In particular, they showed that any ω-limit set of a dynamical system (X, T) is T-connected. Let $\big(C(X), F\big)$ be the induced dynamical system of a given system (X, T), where $C(X)$ is the hyperspace of all compact connected subsets of X and $F : C(X) \to C(X)$ is the induced map of T. In this paper we give a characterization of the induced-map-connected subsets of $C(I)$, where I is a compact interval. The characterization is given via the structure of the ω-limit sets (located on a fiber) of continuous *triangular induced maps* on $I \times C(I)$.

1 Introduction

Let $f : I \to I$ be a continuous map from a compact interval I into itself. The structure of an ω-limit set of a continuous self-map of a closed interval has been studied in detail (see [16], [17]). As established in [1] (see also [4]), a nonvoid closed subset M of I is an ω-limit set for a point $x \in I$ if and only if M is either a nowhere-dense subset of I or a union of finitely many nondegenerate closed intervals in I. The structure of ω-limit sets for some other classes of (non-continuous) self-maps of a closed interval has been studied in [3]. Up to now, the problem of characterizing which closed sets can be described as an

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ω-limit set for a continuous map in $\mathbb{R}^k$, for $k \geq 2$, is a difficult open problem. So, a natural first step is to study the ω-limit sets of a particular class of continuous maps with dimension two. The triangular maps of the square I^2 into itself, which we shall shortly define, happen to be good examples to begin with.

A map $F : I^2 \to I^2$ is called *triangular* if $F(x, y) = \big(f(x), g(x, y)\big)$, for any $(x, y) \in I^2$. Such a map is continuous if and only if $f : I \to I$ and $g : I^2 \to I$ are continuous. For these maps we can also write $F(x, y) = \big(f(x), g_x(y)\big)$, where $g_x : I \to I$ is a family of continuous maps continuously depending on x. The set of all continuous triangular maps from I^2 into I^2 will be denoted by $S_\Delta(I^2)$. Since the triangular map F splits the square I^2 into fibres ($\{x\} \times I$, where $x \in I$) such that each fibre is mapped by F into a fibre, one may expect that the triangular dynamical system (I^2, F) is close, in its dynamical properties, to one-dimensional dynamical systems. In some respects, this is true; for example, the continuous triangular maps of the square are known to obey the Sharkovsky cycle coexistence ordering [9]. Nevertheless, they prove to have some essential differences when compared with continuous one-dimensional maps (see [11], [10], [18]).

Now let us narrow the focus of our investigation, to concentrate on the ω-limit sets of triangular maps. As mentioned above, it is important to study both the structure and also the general properties of these ω-limit sets, since they can open the door to the world of ω-limit sets of continuous maps defined on Euclidean spaces of dimension greater than one. Bearing in mind that a triangular map sends each fibre into a fibre, it seems sensible to begin by examining those ω-limit sets which lie on a fibre. It turns out that these sorts of ω-limit sets are rather free in terms of the forms they may take. As a matter of fact, it was proved that any closed set located on any fixed fibre can be an ω-limit set for some triangular map from I^2 into itself, with one specific exception. This prohibited form was given in the following theorem from [12], which provided a full characterization of those ω-limit sets of triangular maps which lie on a fibre.

Theorem (A)**.** *Let $a \in I$ be a single point and let $M \subset I$. Then, the following two statements are equivalent:*

(i) There exist $F \in S_\Delta(I^2)$ and $(x, y) \in I^2$ such that $\omega_F(x, y) = \{a\} \times M$.

(ii) M is a nonempty closed subset of I, which cannot be written in the form $M = J_1 \cup J_2 \cup ... \cup J_n \cup K$, where n is a natural number, each J_i, for $1 \leq i \leq n$, is a closed, nondegenerate interval, K is a nonempty countable set, the collection $\{J_1, \ldots, J_n, K\}$ is pairwise disjoint and, finally, for at least one $i \in \{1, ..., n\}$ we have that $\operatorname{dist}_H(K, J_i) > 0$.

Also in the paper [12], a wider range of ω-limit sets are considered and, in some sense, a more thorough description of them is constructed. More precisely, the assumption that the ω-limit set must lie on one fibre is dropped, thus achieving a partial description of other types of ω-limit set. The question is then whether any arbitrary closed subset of a fibre is the intersection of that particular fibre with the ω-limit set of some triangular map. The theorem below (from [12]) gives the answer.

Theorem (B)**.** *Let $a \in I$ be a single point and M be any closed subset of I. Then, there exists a map $F \in S_\Delta(I^2)$ and a point $(x,y) \in I^2$ such that $\omega_F(x,y) \cap I_a = \{a\} \times M$, where $I_a = \{a\} \times I$.*

For further results concerning the ω-limit sets of triangular maps, see also [8] and [15].

In the present work we introduce and study a particular class of *induced* continuous maps. Before moving forward we would like to define the type of induced maps which will be considered. Let $f : X \to X$ be a continuous map from a continuum (compact connected metric space) X into itself. Denote by $C(X)$ the hyperspace of all compact connected subsets of X endowed with the Hausdorff metric. It is known that if X is compact, so is the hyperspace $C(X)$ (see [13], pages 52-63). We define the induced continuous map $\mathcal{F} : C(X) \to C(X)$ by setting $\mathcal{F}(x_c) := \{f(x) : x \in x_c\}$ for each $x_c \in C(X)$. It is well known that continuity of f implies continuity of $\mathcal{F}$. It is natural to study the relations between the dynamical properties of f and its induced map $\mathcal{F}$.

Dynamical properties of induced maps on an interval have been studied before [6], [7]. It turns out that in the one-dimensional situation there is a close connection between the dynamics of the induced map and those of the original inducing map. If consider the induced map of the interval I, then the ω-limit set of a point of $C(I)$ is a union of singletons or a finite subset of $C(I)$.

Suppose that X is a compact metric space. By $C(X)^2$ we denote the Cartesian product $C(X) \times C(X)$ endowed with the Hausdorff metric. We will consider a skew product of two induced maps on $C(X)^2$. More precisely, we consider a map $\tilde{F} : C(X)^2 \to C(X)^2$ such that $\tilde{F}(x_c, y_c) = \big(\mathcal{F}(x_c), \mathcal{G}(x_c, y_c)\big)$, where $\mathcal{F} : C(X) \to C(X)$ is a continuous induced map. In other words, $\mathcal{F}(x_c) := \{f(x) : x \in x_c\}$, and $\mathcal{G}_{x_c} : C(X)^2 \to C(X)$ is a family of continuous maps continuously depending on x_c. The map $\tilde{F}$ will be referred to as a *triangular induced map* and the set of all continuous triangular induced maps from the hyperspace $C(X)^2$ into itself will be denoted by $S_\Delta\big(C(X)^2\big)$.

Giving a complete characterization of the possible ω-limit sets for triangular induced maps on $C(X)^2$ in general is presently too complicated. However, if (X^2, F) is an invariant subsystem of $\big(C(X)^2, \tilde{F}\big)$, it can be easier to begin by studying the ω-limit sets of the smaller system, then finding connections from there to the larger system (if any exist), and then trying to generalize the previously-obtained results. By a subsystem (Y, g) of a dynamical system (X, f), we mean that Y is a closed f-invariant subset of X, that is, $f(Y) \subseteq Y$, and g is the restriction of f to Y. One thing is obvious: the dynamics of the system (X, f) cannot be simpler than that of the system (Y, g), because the system (Y, g) is present in the system (X, f). Here, instead of considering the general case where X could be any arbitrary continuum, we will consider the case where X is the compact interval $I = [0, 1]$. Again, since it is not easy to characterize the ω-limit sets of the dynamical system $\big(C(I)^2, \tilde{F}\big)$ in general, we start from the simpler system $\big(I \times C(I), \tilde{F}\big)$. This is because, on the one hand, $\big(I \times C(I), \tilde{F}\big)$ is an invariant subsystem of $\big(C(I)^2, \tilde{F}\big)$ and, on the other hand, it contains the system (I^2, F), for which a full characterization of the ω-limit sets which lie on a fixed fibre is already known. Therefore, in the current paper we compare some of the facts and properties established for the ω-limit sets of the system (I^2, F) with those of the, roughly speaking, wider system $\big(I \times C(I), \tilde{F}\big)$. Once more, we emphasize that the main goal here is to characterize those ω-limit sets of the system $\big(I \times C(I), \tilde{F}\big)$ which are located on a fixed fibre.

2 Notations and definitions

We defined $C(I)$ as the space of all compact connected subsets of the compact interval I, which means that each element of $C(I)$ is in fact a closed subinterval of I. On the other hand, despite the set-nature of each of these elements, we treat them as a single whole object. That is, if roughly speaking, any space consists of points, then our space $C(I)$ consists of points, which are as a matter of fact, closed intervals in I. For this reason and in order to avoid any ambiguity, we use the word *pointerval* to mean the described elements of the phase space $C(I)$. Moreover, since any singleton $\{x\} \subset I$ is also a compact connected set, i.e. a pointerval in $C(I)$, we prefer the phrase *degenerate pointerval* when we talk about such a set as an element of $C(I)$. Consequently, the phrase *nondegenerate pointerval* is applied otherwise. Furthermore, it is necessary to introduce appropriate notation in order to remove any confusion. That is, if for instance, two points a and b are arbitrary points in I such that $a \leq b$, then the notation $J = [a, b]$ can confuse one. The problem is that it is not clear if we regard $J = [a, b]$ as an ordinary interval $[a, b] = \{x \in I | a \leq x \leq b\} \subseteq I$ or we mean the pointerval $[a, b]$ as a single element of $C(I)$. To solve this, we use $[a, b]_c$ (or equivalently J_c) for a pointerval in $C(I)$ and $[a, b]$ (equivalently J) to mean the set $\{x \in I | a \leq x \leq b\} \subseteq I$. In addition, and for ease, when we talk about a degenerate pointerval, say $\{a\}_c \in C(I)$, we may only write a_c, when it does not cause any confusion. Finally, we would like you to remember that the phrase "single point" is used, while we merely mean a single element of I.

In addition, we have to mention that, in spite of the fact that using phrases like "convex set", "connected set" or "dense set" do not raise any ambiguity when we are talking about subsets of, e.g. $\mathbb{R}$, $\mathbb{R}^2$, it is not clear yet what a "convex" or "connected" set of pointervals is. Here, we introduce some of these conceptions regarding sets of pointervals. To this end, first, we give a specific interpretation of a pointerval at the beginning of the next paragraph. In fact, we will build a one-to-one correspondence between the elements of $C(I)$ and the points of two dimensional set $I^2_{y\geq x} = \{(x,y) \in I^2 \mid y \geq x\}$ (i.e. points of the square I^2 located on and above the diagonal of I^2). So that one can interpret any pointerval of $C(I)$ as an ordinary point in $I^2_{y\geq x}$.

Suppose that $[a,b]_c \in C(I)$ is an arbitrary pointerval. The following one-to-one correspondence is what we are looking for: $[a,b]_c \leftrightarrow (a,b)$, where (a,b) is a point in $I^2_{y\geq x}$, whose first and second coordinates are a and b respectively. Automatically, when we talk about a map from the TTS into itself, we mean the correspondent induced map from $C(I) \to C(I)$.

Moreover, we would like to add that if $\dim C(I)$ denotes the topological dimension of $C(I)$, then obviously $\dim C(I) = 2$. In addition, we should specify that if $\mathcal{F} : C(I) \to C(I)$ is an arbitrary induced map and L denotes the diagonal of the space $I^2_{y\geq x}$, which we will call *TTS* (*triangular translated space*), then obviously L is an invariant subset of $I^2_{y\geq x}$ under the action of $\mathcal{F}$.

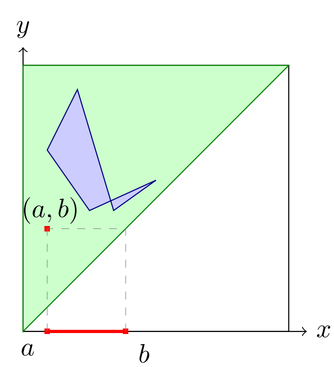

fig.1

Therefore, with the help of this translation, any set of pointervals in $C(I)$ can be uniquely considered as a set in the space $I^2_{y\geq x}$. If the translated set in $I^2_{y\geq x}$

is connected, the original set of pointervals in $C(I)$ is said to be connected. The word "connected" in the previous sentence can be replaced by "convex", "dense" and etc. Since we are paying special attention to the connected sets of pointervals in the last section, we prefer to call a connected set of pointervals by the shorter term *conterval*. Similarly as for pointervals, we will call a conterval consisting of one pointerval *degenerate*, while the adjective *nondegenerate* will be used otherwise.

Finally, we would like to specify that since any pointerval is, in fact, a single element, it does not make any sense to talk about the "intersection" or "disjointness" of two pointervals. Nevertheless, we sometimes do that. Hence, when we say, for e.g, two pointervals $J_c(1), J_c(2)$ are pairwise "disjoint", we mean that $J(1), J(2)$ as subsets of I are pairwise disjoint.

3 Map-connectedness

Let X be a compact topological space and $T : X \to X$ a continuous map. Also, let $Y \subseteq X$ be a nonempty strongly T-invariant (i.e. $T(Y) = Y$ closed subset of X). Then, Y is called *T-connected*, if it contains no proper closed subset which is mapped into its own interior; in other words, if U is any nonempty open (in Y) subset of Y, from $T(\overline{U}) \subseteq U$ implies that $U = Y$. Dowker and Friedlander in [5], and Sharkovsky in [16] [1] showed that any ω-limit set of the dynamical system (X, T) is T-connected. Moreover, they proved there that if a subset Y of the dynamical system (X, T) has the property of T-connectedness, then Y is the ω-limit set of some point of a dynamical system which "includes" $(Y, T|_Y)$.

It is also known that, if T is the identity map, then T-connectedness reduces to connectedness in the usual sense and in general, T-connectedness is to some extent analogous to connectedness; this is illustrated by Theorem III in [5], which extends Sierpinski's well-known decomposition theorem [14] from continua to T-connected sets.

Let us go back to Theorem (A), and once more have a quick look at it. Assume that $M \subset I$ is the set described there and $T : I \to I$ is any continuous map. According to the Dowker-Friedlander-Sharkovsky theorem, M is not an ω-limit set since it is not T-connected. Now, it is easy to see that an "analogous" set M_1 in the space $C(I)$ (i.e. M_1 is a union of finite number of nondegenerate contervals together with at most countable set of pointervals with positive distance with at least one of the above mentioned contervals) is never T-connected for any continuous map $T : C(I) \to C(I)$.

[1] Sharkovsky used the term "*weakly incompressibility*" instead of "T-connectedness".

Since, in the current paper, we consider induced maps $T_{ind} : C(I) \to C(I)$, it is necessary to show that there are some other forms of compact subsets of $C(I)$ which are not T_{ind}-connected, for instance a finite number of nondegenerate pointervals and at most countable set of degenerate pointervals. Apropos, we have the following property:

Lemma 1. *Let (X, T) and (Y, S) be dynamical systems, where T and S are continuous and T is surjective. Also, let (Y, S) be a factor of (X, T), i.e. there is a continuous surjection $\pi : X \to Y$ such that $\pi \circ T = S \circ \pi$. Then, T-connectedness of X implies S-connectedness of Y.*

Proof. Assume that Y is not S-connected. From here, there should exist a nonempty open subset B of Y with $S(\overline{B}) \subseteq B$ and $B \neq Y$. To finish the proof we only need to show that the following properties *(i)* $T\big(\overline{\pi^{-1}(B)}\big) \subseteq \pi^{-1}(B)$ and *(ii)* $\pi^{-1}(B) \neq X$ hold for the nonempty open set $\pi^{-1}(B)$. Firstly, since $\overline{\pi^{-1}(B)} \subseteq \pi^{-1}(\overline{B})$, we have $T\big(\overline{\pi^{-1}(B)}\big) \subseteq T\big(\pi^{-1}(\overline{B})\big) \subseteq \pi^{-1}\big(S(\overline{B})\big) \subseteq \pi^{-1}(B)$. Secondly, surjectivity of π together with $B \neq Y$ imply that $\pi^{-1}(B) \neq X$. □

In particular, it means that each dynamical extension of a T-connected set from I to $C(I)$ is T_{ind}-connected.

In the following sections, taking into account that the system $\big(I \times C(I), \tilde{F}\big)$ includes (I^2, F) and therefore is more complicated, we will search for other possible forms of sets $\big($other than the set described in Theorem (A)$\big)$ which cannot be realized as an ω-limit set.

4 Sets including a finite number of nondegenerate pointervals

Let $M \subset C(I)$ be a set containing a finite number (at least one) of nondegenerate pointervals. Also, assume that $a \in I$ is an arbitrary point. We are interested to determine what forms the set M can have, so that the set $\{a\} \times M$ could be realized as an ω-limit set for some nondegenerate pointerval in $I \times C(I)$. The present section is devoted to exploring this question.

As a matter of fact, there are only a few possible forms that the set M generally can have. For ease, we will consider these forms in the remainder of the section one by one and will discuss them in turn. Firstly, let us begin with the case where M in addition to a certain number of nondegenerate pointervals, also has some degenerate pointervals. Obviously, whatever form this subset of degenerate pointervals has, the set M does not satisfy the property of map-connectedness. The reason is that if M is map-connected, then, inasmuch as it contains a finite number of nondegenerate pointervals, there should be a degenerate pointerval whose image is a nondegenerate pointerval, which is obviously impossible. Since this is independent from the map considered on M, one can see that any attempt to realize the set $\{a\} \times M$ as an ω-limit set will end in failure. Therefore, M cannot include any degenerate pointerval. There remains two possibilities: either M consists of pairwise "disjoint" nondegenerate pointervals or some of these nondegenerate pointervals "intersect" some others. If the former is the case, it is easy to see that the set $\{a\} \times M$ can definitely be realized as an ω-limit set; but if the latter one is true, then $\{a\} \times M$ can be an ω-limit set for some nondegenerate pointerval in $I \times C(I)$ iff each pointerval of M "intersects" exactly one and only one other pointerval of M and additionally for any two pointervals of M, non of them is a subset of the other one. In fact, if M has any other form except from this one, one can verify that the property of map-connectedness does not hold and hence, the set $\{a\} \times M$ is not an ω-limit set. So, we can sum up this section by stating that, if a set $M \subset C(I)$ contains at least one and at most a finite number of nondegenerate pointervals, then $\{a\} \times M$, where a is an arbitrary single point of I, is an ω-limit set for some nondegenerate pointerval in $I \times C(I)$ iff M is either

1. a set of pairwise "disjoint" nondegenerate pointervals, or
2. a set of nondegenerate pointervals such that any pointerval has an "intersection" with exactly one and only one other pointerval of M and for any two pointervals $J_c(i), J_c(j) \in M$ we have $J_c(i) \backslash J_c(j) \neq \varnothing, J_c(j) \backslash J_c(i) \neq \varnothing$.

5 Sets with countably infinite number of pointervals

Suppose that $M \subset C(I)$ is a countable set which contains an infinite number of nondegenerate pointervals and $a \in I$ is a single point. The question is then whether there exists any nondegenerate pointerval $(x, J_c) \in I \times C(I)$ such that $\omega_{\tilde{F}}(x, J_c) = \{a\} \times M$, for some $\tilde{F} \in S_\Delta\big(I \times C(I)\big)$. In principle, the above described set M can have a vast variety of forms, nevertheless, in the present work, we do not mean to provide a complete characterization for all of these forms. We will just discuss some of the most important forms and will answer the previously mentioned question for them. Before answering the question, we will give the definition of a *homoclinic* set, which is essentially used in the remainder of the paper (see [4]).

Let X be a compact topological space and $\Gamma \subseteq X$ be a compact subset of it. Also, let the finite set $A = \{a_0, ..., a_{k-1}\} \neq \varnothing$ be a set of points of Γ and $f : \Gamma \to \Gamma$ be a continuous map such that A is a k-cycle of f with

$f(a_i) = a_{i-1}$ for $i > 0$ and $f(a_0) = a_{k-1}$. Assume that there is a system $\{\Gamma^i_n\}^\infty_{n=0}, i = 0, ..., k-1$, of nonempty pairwise disjoint compact subsets of Γ such that $\Gamma \setminus \cup_{i,n} \Gamma^i_n = A$ and $\lim_{n\to\infty} \Gamma^i_n = a_i, \forall i$. If $f(\Gamma^i_n) = \Gamma^{i-1}_n$ for $i > 0$ and any n, $f(\Gamma^0_n) = \Gamma^{k-1}_{n-1}$ for $n > 0$, and $f(\Gamma^0_0) = a_{k-1}$, then Γ is called a *homoclinic set (of order k) with respect to f.*

Remark. If Γ is homoclinic of order k with respect to f, then for each i, the set $\Gamma^i = \{a_i\} \cup^\infty_{n=0} \Gamma^i_n$ is homoclinic of order 1 with respect to $g = f^k$; more precisely, we have $g(\Gamma^i_n) = \Gamma^i_{n-1}$ for $n > 0$ and $g(\Gamma^i_0) = g(a_i) = a_i$ (see [4]).

Let $D \subset C(I)$ be the closure of a nonempty set consisting of a countably infinite number of nondegenerate pointervals. In addition, assume that D can be written as $D = \cup^\infty_{i=1} D_c(i)$, where $D_c(i)$s are pairwise "disjoint" finite sets of pointervals (pointervals of each portion $D_c(i)$ can "intersect" each other in whatever way). Obviously, the compact countable set D contains both degenerate and nondegenerate pointervals, since any sequence of portions of D converges to a degenerate pointerval. Furthermore, if A is any set of pointervals, say $A = \{J_c(1), J_c(2), ...\}$, then we write $\cup A = \cup^\infty_{i=1} J(i)$. In this section we want to show that D, mentioned above, is homoclinic with respect to some continuous map $G : D \to D$. One may ask why we assume disjointness for the subsets of pointervals? In other words, is disjointness a necessary condition for D in order to be a homoclinic set? Indeed the answer is positive, i.e. without the assumption of disjointness the set D can not be a homoclinic set. Nevertheless, this is not a necessary assumption for D in order to be map-connected. Later, in Examples 7 and 8 we will show this.

Before stating the following lemmas we need some notations. Let B be a countable compact set. Define a transfinite sequence $(B_\alpha)_{\alpha<\Omega}$ of subsets of B as follows: $B_0 = B$, $B_\gamma = \cap_{\alpha<\gamma} B_\alpha$ if γ is a limit ordinal, and B_γ is the derivative (the set of limit points) of $B_{\gamma-1}$ otherwise. Clearly, for any countable compact set such B there is an ordinal $\beta < \Omega$ such that B_β is nonempty and finite, and $B_{\beta+1} = \varnothing$. We denote such β by $\mathrm{Ker}(B)$.

Here we would like to mention that in the following, in Lemmas 2, 3 and Theorem 4, we use some ideas from [4].

Lemma 2. *Let $D \subset C(I)$ be the compact set described above and* $\mathrm{Ker}(D) = \eta$ *and $D_\eta = d_c$, where $d_c \in C(I)$ is a degenerate pointerval ($d \in I$). Then, there is a sequence $(A_n)^\infty_{n=1}$ of pairwise disjoint compact subsets of D such that $D \setminus \cup_n A_n = d_c$ and $\lim_{n\to\infty} A_n = d_c$ and moreover $\forall n : \eta >$* $\mathrm{Ker}(A_{n+1}) \geq \mathrm{Ker}(A_n)$.

Proof. We assume that $d \in (0, 1)$ and it is not an endpoint for any nondegenerate pointerval of D (the proof is similar for these cases). Let $\big(I(n)\big)^\infty_{n=1}$ be a strictly decreasing sequence of compact intervals of I with end points in $I\backslash \cup D$ and such that $\cap_{n=1}^{\infty} I(n) = \{d\}$. Take n_1 such that $C\big(I\backslash I(n_1)\big) \cap D \neq \varnothing$. Since $\lim_{n\to\infty} \mathrm{Ker}\left(C\big(I\backslash I(n)\big) \cap D\right)$ is η if η is a limit ordinal, and is $\eta - 1$ otherwise, there exists $n_2 > n_1$ such that $\eta > \mathrm{Ker}\left(C\big(I(n_1)\backslash I(n_2)\big) \cap D\right) \geq \mathrm{Ker}\left(C\big(I\backslash I(n_1)\big) \cap D\right)$. By induction we get an increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that for any $k > 0$, $\eta > \mathrm{Ker}\left(D \cap C\big(I(n_k)\backslash I(n_{k+1})\big)\right) \geq \mathrm{Ker}\left(D \cap C\big(I(n_{k-1})\backslash I(n_k)\big)\right)$, where $I(n_0) = I$. Then, put $A_k = \left(D \cap C\big(I(n_{k-1})\backslash I(n_k)\big)\right)$. If for some k, the set $(A_k)_{\mathrm{Ker}(A_k)}$ has more than one element, replace A_k in the sequence by a string $A_k^1, ..., A_k^{m(k)}$, where A_k^i are portions of A_k such that $\mathrm{Ker}(A_k^i) = \mathrm{Ker}(A_k)$ and $(A_k^i)_{\mathrm{Ker}(A_k)} = \{a_k^i\}$ for each i. $\square$

In the remainder, for any two subsets A and B, by $A \succ B$ we mean that there is a continuous map of A onto B. If A and B are subsets of $C(I)$, then $A \succ B$ means that there is a continuous induced map from A onto B.

Lemma 3. *Let $A, B \subset C(I)$ be nonempty compact sets consisting of a countable number of pairwise disjoint finite subsets of nondegenerate pointervals. Also, let $\alpha = \mathrm{Ker}(A) \geq \mathrm{Ker}(B) = \beta$ and $B_\beta = b_c$. Then, $A \succ B$.*

Proof. We use transfinite induction. It is easy to check that this is true for $\alpha = 0$. Now, assume $\mathrm{Ker}(A) = \alpha(0) > 0$ and the statement holds for any $\alpha < \alpha(0)$. We are tending to show that it is also true for $\alpha(0)$. First, consider the case $A_{\alpha(0)} = a_c$. Applying Lemma 2 to B we will get the sequence $(B_n)_{n=1}^{\infty}$ of corresponding compact subsets of B. For each n denote $\mathrm{Ker}(B_n) = \beta(n)$. Then, apply Lemma 2 to A and let $(D_n)_{n=1}^{\infty}$ be the corresponding sequence of compact subsets of A. For each k, $\lim_{n\to\infty} \mathrm{Ker}(D_n) \geq \beta(k)$, hence there is $n(k)$ such that $\mathrm{Ker}(D_{n(k)}) \geq \beta(k)$. Hence, for $k = 1$ there is $n(1)$ such that $\mathrm{Ker}(D_{n(1)}) \geq \beta(1)$. If B_1 has any degenerate pointerval, so definitely $D_1 \cup ... \cup D_{n(1)} \succ B_1$. But if B_1 does not have any degenerate pointerval, while $D_1 \cup ... \cup D_{n(1)}$ has at least one degenerate pointerval, then certainly $D_1 \cup ... \cup D_{n(1)} \succ B$. Similarly, we can find $n(k)$ for any k such that $\mathrm{Ker}(D_{n(k)}) \geq \beta(k)$ and $D_{n(k-1)+1} \cup ... \cup D_{n(k)} \succ B_k$ or $D_{n(k-1)+1} \cup ... \cup D_{n(k)} \succ B$. Clearly, the obtained sequence $\big(n(k)\big)_{k=1}^{\infty}$ is an increasing sequence. Now, take $A_k = D_{n(k-1)+1} \cup ... \cup D_{n(k)}$. Then, $\alpha(0) > \mathrm{Ker}(A_n) \geq \beta(n)$ for each n, and consequently by the hypothesis, $A_n \succ B_n$ $\big($note that by Lemma 2, for any n, the set $(B_n)_{\beta(n)}$ has no more than one element$\big)$ or otherwise $A_n \succ B$. Let φ_n be the corresponding map. Define φ by $\varphi(J_c) = \varphi_n(J_c)$ if $J_c \in A_n$, and $\varphi(a_c) = b_c$. Since $A = \cup_n A_n \cup \{a_c\}$ and $B = \cup_n B_n \cup \{b_c\}$, φ is a map from A onto B, and since $\lim_{n\to\infty} A_n = a_c$ and $\lim_{n\to\infty} B_n = b_c$, the map φ is continuous. Finally, if $A_{\alpha(0)} = \{a_c(0), ..., a_c(k-1)\}$ with $k > 1$, divide A into compact portions $A^0, ..., A^{k-1}$ such that $A^i_{\alpha(0)} = \{a_c(i)\}, \forall i$. Since $A \supset A^i \succ B, \forall i$, we have $A \succ B$. □

Theorem 4. *Let $M \subset C(I)$ be a nonempty compact set containing a countably infinite number of pairwise "disjoint" finite subsets of nondegenerate pointervals. Then, M is homoclinic with respect to an induced continuous map $\mathcal{F}$.*

Proof. Let $\mathrm{Ker}(M) = \alpha$, and $M_\alpha = \{a_c(0), ..., a_c(k-1)\}$. Also, let $I^0, ..., I^{k-1}$ be pairwise disjoint compact intervals in I "covering" M and such that I^i is a neighborhood (in I) of $a(i)$ for any i. Denote $M^i = \big(M \cap C(I^i)\big)$. Then, $\mathrm{Ker}(M^i) = \alpha$ and $M^i_\alpha = a_c(i)$. Apply Lemma 2 to every M^i and let $(D^i_n)_{n=1}^\infty$ be the corresponding sequence of compact subsets of $M^i, \forall i$. We may assume that $D^i_n \neq \varnothing$ for any i and n. To finish the proof it suffices to define sets M^i_n with properties needed for a homoclinic trajectory. First, put $M^0_0 = D^0_1$; then, clearly $M^0_0 \succ a_c(k-1)$. Now we must find the set M^1_0 such that $M^1_0 \succ M^0_0$. Since $T(M^i) = \alpha, \forall i$ and according to Lemma 2 there is $n(1) \geq 1$ such that $T(D^1_{n(1)}) \geq T(D^0_1)$. Put $M^1_0 = \cup_{n=1}^{n(1)} D^1_n$. Obviously $M^1_0 \succ M^0_0$. Similarly, we can find $n(2) \geq n(1)$ for which $M^2_0 = \cup_{n=1}^{n(2)} D^2_n \succ M^1_0$. In fact, by induction, for any set M^i_n, we are always able to find the set M^{i+1}_n such that $M^{i+1}_n \succ M^i_n$. Thus, one can find an induced map $\mathcal{F} : M \to M$ for which $\mathcal{F}\big(a_c(i)\big) = a_c(i-1)$, for $i \geq 1$ and $\mathcal{F}\big(a_c(0)\big) = a_c(k-1)$. Since $\lim_{n\to\infty} M^i_n = a_c(i)$, for any i, the map $\mathcal{F}$ is continuous. □

Let $M \subseteq C(I)$ and $\varepsilon > 0$. The finite set $\{x_c(1), ..., x_c(n)\} \subset M$ is called an ε-net for M provided that for any $x_c \in M$ there is $x_c(i), i = 1, ..., n$ with $\mathrm{dist}\big(x_c, x_c(i)\big) < \varepsilon$.

Now, if $F : M \to M$ is continuous and $\varepsilon > 0$, then the finite sequence $x_c(1), ..., x_c(n)$ of pointervals of M is called ε-recurrent chain, or shortly, ε-chain for F, if $\mathrm{dist}\Big(F\big(x_c(i)\big), x_c(i+1)\Big) < \varepsilon (\mathrm{mod}\ n)$, for any $i = 1, ..., n$.

It is needed to be mentioned that we are not going to prove the next lemma here, since the proof is extremely analogous to the one for the ordinary triangular map given in [12].

Lemma 5. (See Lemma 2 and its proof in [12]) *Let $M \subset C(I)$ be closed and $a \in I$ be a single point. Suppose that $F : M \to M$ is a continuous induced map such that for any $\varepsilon > 0$, there exists an ε-chain for F, which is an ε-net for M. Then, there exists $\tilde{F} \in S_\Delta\big(I \times C(I)\big)$ and a pointerval $(x, J_c) \in I \times C(I)$ such that $\omega_{\tilde{F}}(x, J_c) = \{a\} \times M$.*

It is obvious that if $M \subset C(I)$ is homoclinic with respect to some induced map F, then for any $\varepsilon > 0$ there is an ε-net for M, which is an ε-chain for F. Hence, from here, Theorem 4 and Lemma 5 we have

Corollary 6. *Let M be as in Theorem 4 and let a be a single point in I. Then, there is a nondegenerate pointerval $(x, J_c) \in I \times C(I)$ and there is an induced triangular map $\tilde{F} \in S_\Delta\big(I \times C(I)\big)$ for which $\omega_{\tilde{F}}(x, J_c) = \{a\} \times M$.*

As one can see, in most of the proofs of this section, we essentially used the fact that the considered set could be divided into an infinite countable number of disjoint subsets of nondegenerate pointervals. In actual fact, this assumption guaranteed the existence of an infinite number of, roughly speaking, "holes" arbitrarily close to the accumulation point, and this turns to be vitally necessary to construct the homoclinic trajectory. Example 7 illustrates this necessity.

Example 7. Assume that $M_1 = \{J_c(1), J_c(2), ...\} \subset C(I)$ is a set of nondegenerate pointervals such that $J(n) \cap J(n+1) \neq \varnothing$, $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} J_c(n) = p_c$, where p_c is a degenerate pointerval in $C(I)$. We will show that $M := \overline{M_1}$ is not a map-connected set under the action of any continuous induced map from M into itself. By contradiction let us assume that $\mathcal{F} : M \to M$ is a continuous induced map such that M is $\mathcal{F}$-connected. So, there should certainly exists n for which $\mathcal{F}\big(J_c(n)\big) = p_c$. But this implies that $\mathcal{F}\big(J_c(n)\big) = p_c$, for all $n \in \mathbb{N}$. Hence the preimage of any nondegenerate pointerval of M does not belong to M and this leads to contradiction.

The next example shows that while the absence of the holes results in the absence of the homoclinic trajectory, however it is still possible for such a set to be an ω-limit set for some pointerval.

Example 8. Let $D = \cup_{m=1}^{\infty} \cup_{k=0}^{3^{m-1}-1} [\frac{3k+1}{3^m}, \frac{3k+2}{3^m}] \subset I$ $\big($pay attention that for the ternary Cantor set C_3 created from the segment $[0,1]$, we have that $C_3 = [0,1] \backslash \cup_{m=1}^{\infty} \cup_{k=0}^{3^{m-1}-1} (\frac{3k+1}{3^m}, \frac{3k+2}{3^m})\big)$. It is obvious that D could be written as a countable union of an infinite number of pairwise disjoint nondegenerate closed intervals, say $[a_i, b_i] \subset I$; i.e. $D = \cup_{i=1}^{\infty} [a_i, b_i]$ such that $[a_k, b_k] \cap [a_l, a_l] = \varnothing, k \neq l$. Suppose that $M = \overline{\{[a_i, b_i]_c\}_{i=1}^{\infty}}$ (the set M' is the Cantor ternary set). We show that there exists an induced continuous surjection from M onto itself, with respect to which M is map-connected. Let $f : I \to I$ be the following surjection: $f(x) = 3\min\{x, 1-x\}$ for any $x \in [0, \frac{1}{3}) \cup (\frac{2}{3}, 1]$ and $f(x) = 1$ for any $x \in [\frac{1}{3}, \frac{2}{3}]$. The induced continuous map $\mathcal{F} : M \to M$ of f is the desired surjection. It is not difficult to see that for any nondegenerate pointerval, say $J_c(i) \in M$, and for any $\varepsilon > 0$ there exists $J_c(j) \in M$ such that

$\operatorname{dist}\big(J_c(j), 1_c\big) < \varepsilon$ and $J_c(j) \in \cup_{n=0}^{\infty} \mathcal{F}^{-n}\big(J_c(i)\big)$. In addition, there certainly exists an $n \in \mathbb{N}$ with $f^n\big(J_c(i)\big) = [\frac{1}{3}, \frac{2}{3}]_c$. Assume that on each $\{x\} \times M$, where $x \in I$, acts the same map as $\mathcal{F}$. Now, if a is an arbitrary point in I, it is easy to find a nondegenerate pointerval in $I \times C(I)$, for which $\{a\} \times M$ is an ω-limit set.

6 Connected sets of pointervals

Until now we have limited ourselves to investigate, at first, finite and later countable sets of pointervals. But, clearly these are not the only possible sets of pointervals one can imagine. It seems that it is necessary to cover a wider range of sets in our study in order to obtain a thorougher description of ω-limit sets of triangular induced systems. To this end, we will survey some sets including contervals (connected set of pointervals) and discuss when they can form an ω-limit set for a pointerval in $\big(I \times C(I)\big)$.

First, let us give the definition of an ε-route between two points, where ε is a positive real number. Let X be a compact metric space and a and b two points in it. The finite ordered sequence $x_1, x_2, ..., x_n$ is called an ε-route from a to b provided that $x_1 = a,\ x_n = b$ and $d(x_i, x_{i+1}) < \varepsilon,\ \forall i = 1, 2, ..., n-1$.

Next, we will show that if A is a closed connected subset of the compact metric space X, then for any two arbitrary points a and b in A and any $\varepsilon > 0$, there exists an ε-route in A, which connects a, b. It is clear that the set A is a compact subset of X. Hence, for any $\varepsilon > 0$, there is a finite ε-cover for A, which implies that there necessarily exists an ε-route inside A connecting the points a and b. In the following theorem we will see that this fact guarantees that once we have a closed connected set on a fibre $\{a\} \times C(I)$, where a is a single point in I, we are always able to realize this set as an ω-limit set without approaching any point outside the desired closed connected set.

Theorem 9. *Let C be a closed conterval in $C(I)$ and let $a \in I$ be a single point. Then, there exists a map $\tilde{F} \in S_{\Delta}\big(I \times C(I)\big)$ and a pointerval $(x, J_c) \in I \times C(I)$ such that $\omega_{\tilde{F}}(x, J_c) = \{a\} \times C$.*

Proof. It is well known that any compact metric space has a countable dense subset. So, Let $D = \big(J_c(i)\big)_{i=0}^{\infty}$ be a dense sequence in C. For any $\varepsilon_i > 0, i \geq 1$ there exists a circular ε_i-route $\mathfrak{R}_i = \big(E_c^i(l)\big)_{l=0}^{n_i}$, $n_i \in \mathbb{N}$ in C_1 such that $\mathfrak{R}_i \supseteq \{J_c(0), ..., J_c(i)\}$ and $E_c^i(0) = E_c^i(n_i) = J_c(0)$. Take the sequence $\mathcal{S} = \left(\big(E_c^i(l)\big)_{l=0}^{n_i}\right)_{i=1}^{\infty}$. For ease, we rename the elements of this sequence as follows: $\mathcal{S} = \big(E_c(k)\big)_{k=0}^{\infty}$. Now, assume that $(a_k)_{k=0}^{\infty}$ is a sequence in I such that $a_k \to a,\ k \to \infty$. Let $\mathcal{K} = \{(a_k, E_c(k))\}_{k=0}^{\infty} \cup (\{a\} \times C)$ and $\tilde{G} : \mathcal{K} \to \mathcal{K}$ be the following map: $\tilde{G}|_{\{a\}\times C} = id$, and $\tilde{G}(a_k, E_c(k)) = (a_{k+1}, E_c(k+1))$, for any$k = 0, 1, \dots$. The set $\mathcal{K}$ is obviously compact and $\tilde{G}$ is continuous. Therefore, using the Extension Lemma, one can continuously extend $\tilde{G}$ to a map $\tilde{F} \in S_\Delta(I \times C(I))$, for which obviously the equality $\omega_{\tilde{F}}(a_0, E_c(0)) = \{a\} \times C$ holds. $\square$

In Theorem 9, the set$\{a\} \times C$ was realized as an ω-limit set with the help of the identity map. In spite of the fact that it does not seem very obvious, but surprisingly, one can show that in some particular cases the set $\{a\} \times C(I)$ can be realized as an ω-limit set for a pointerval in $I \times C(I)$ only if the map defined on $\{a\} \times C$ or its second iterate is the identity map on $\{a\} \times C$.

In order to find necessary and sufficient conditions under which a set consisting of a finite number of closed nondegenerate contervals could be realized as an ω-limit set, it is easier to begin with the case where the considered set, say M, consists of only two closed nondegenerate contervals C_1 and C_2. For ease, we assume that $(\cup C_1) \cap (\cup C_2) = \varnothing$. The first more or less obvious thing we show here (via the following examples) is that not any two closed nondegenerate pointervals can form an ω-limit set. In fact, several factors together play a significant role to make such a set able to be an ω-limit set. Roughly speaking, there must be a kind of "similarity" between the ways the pointervals of each conterval C_1 and C_2 are arranged together. This loosely expressed statement could be better understood through the next simple examples.

We remind that $M = C_1 \cup C_2$, where C_1 and C_2 are closed contervals in $C(I)$. Assume that C_1 has a degenerate pointerval in it (i.e. it intersects the diagonal L in the TTS), whereas C_2 does not. Clearly, M is not map-connected since the diagonal is invariant in $I^2_{y\geq x}$. Therefore, if M is map-connected and one of the contervals intersects the diagonal, then the second one has to intersect it as well; however, obviously, this is not a sufficient condition.

As the second example, let both C_1 and C_2 have no intersection with L. Again, this is not sufficient for M to be realized as an ω-limit set. For instance, suppose that C_1 and C_2 are as in fig.2. Taking any two distinct pointervals in C_1, one of them necessarily includes the other one. On the other hand, for any two arbitrary pointervals in C_2, say $J_c(1), J_c(2)$, we have $J_c(1) \nsubseteq J(2)$ and $J_c(1) \nsupseteq J(2)$, which means that there does not exist any continuous induced map from C_1 onto C_2. So, the set M obviously cannot be a map-connected set.

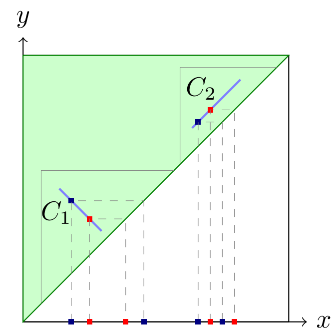

fig.2

In fact, the set M mentioned in the second example cannot be realized as an ω-limit set since C_1 and C_2 are, roughly speaking, of different structures. That is, the pointervals located inside each of C_1 and C_2 have essentially "dissimilar positions" with respect to each other. But the question is then what do we mean exactly by "dissimilar positions"? To clarify this, we have to dig up the terminology in a more mathematical language.

Let $A_i, i = 1, ..., n$ be some sets in $C(I)$ such that $(\cup A_{i_1}) \cap (\cup A_{i_2}) = \varnothing,\ \forall i_1 \neq i_2$. We say $A_1, A_2, ...$and A_n are of the *same arrangement* provided that there exist n continuous induced surjections $\Theta_i : A_i \to A_{i+1} (\text{mod } n)$ generated by continuous surjections $\theta_i : \cup A_i \to \cup A_{i+1} (\text{mod } n)$ such that $\Theta_n ... \Theta_1 = id$ or $(\Theta_n ... \Theta_1)^2 = id$. Obviously, if $E^i_c(1), E^i_c(2) \in A_i$ with $E^i(1) \cap E^i(2) \neq \varnothing$, then $\theta_i\big(E^i(1)\big) \cap \theta_i\big(E^i(2)\big) \neq \varnothing$.

Clearly, if any set $M \in C(I)$, consisting of a finite number of closed non-degenerate contervals $A_i, i = 1, ..., n$ such that $(\cup A_{i_1}) \cap (\cup A_{i_2}) = \varnothing,\ \forall i_1 \neq i_2$, is map-connected with respect to some continuous induced map, then all the contervals of M have the same arrangement. Moreover, the next theorem, which is a corollary of 9, shows that having the same arrangement is sufficient for the contervals of the set M to be realized as an ω-limit set for some pointerval in $I \times C(I)$.

Theorem 10. *Let $M \subseteq C(I)$ be a set consisting of closed contervals $C_1, ..., C_n,\ \ n \in \mathbb{N}$ such that $(\cup C_i) \cap (\cup C_j) = \varnothing, \forall i, j = 1, ..., n$ and $i \neq j$. Also, let $C_i, i = 1, ..., n$ be of the same arrangement. If $a \in I$ is an arbitrary point, then, there exist $\tilde{F} \in S_\Delta\big(I \times C(I)\big)$ and $(x, J_c) \in I \times C(I)$ such that $\omega_{\tilde{F}}(x, J_c) = \{a\} \times M$.*

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