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SOME PROPERTIES OF (Φ)-UNIFORMLY SYMMETRICALLY POROUS SETS

Abstract

We prove that each perfect linear set contains a perfect set which is (Φ)-uniformly symmetrically porous (Theorem 1). In the hyperspace of all nonempty compact sets in $[0, 1]$ (endowed with the Hausdorff distance), the (Φ)-uniformly symmetrically porous nonempty compact sets form a G_δ residual subspace (Theorem 2). We infer that the (Φ)-uniformly symmetrically porous perfect sets form a G_δ residual set in the space of all perfect sets in $[0, 1]$ (Theorem 3).

Each continuous increasing function Φ from $[0, 1]$ into $[0, 1]$ and fulfilling $\Phi(0) = 0$ is called a *porosity index*. The set of all porosity indices will be denoted by G. The notion of (Φ)-uniformly symmetric porosity, introduced in [H], constitutes (for the respective $\Phi \in G$) a sharper version of bilaterally strong porosity and strong symmetric porosity (cf. [H, Th. 3]). Our Theorems 1 and 2 describe the behaviour of (Φ)-uniformly symmetrically porous sets in connection with the families of perfect sets and compact sets. That generalizes results known earlier for other kinds of porosity. Theorems 1 and 2 are used in the proof of Theorem 3 which states that a typical perfect set in $[0, 1]$ is (Φ)-uniformly symmetrically porous.

From now on, fix an arbitrary $\Phi \in G$. Let $n \in \mathbb{N}$. Following [H], we denote by $R(\Phi, n)$ the set of all $E \subseteq \mathbb{R}$ for which there are numbers a_n, b_n such that for all $x \in E$ we have:
(i) $0 < a_n < b_n < 1/n$,
(ii) $\Phi(b_n - a_n) > a_n$,
(iii) $[x - b_n, x - a_n] \cap E = [x + a_n, x + b_n] \cap E = \emptyset$.

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Let $R(\Phi) = \bigcap_{n=1}^{\infty} R(\Phi, n)$. Each set $E \in R(\Phi)$ is called (Φ)-*uniformly symmetrically porous*.

The definitions of other kinds of porosity can be found in [Z]. For instance, if $E \subseteq \mathbb{R}$, $x \in \mathbb{R}$ and $r > 0$ are fixed, denote by $l(E, x, r)$ the length of the longest interval (a, b) such that $0 < a < b < r$ and

$$(x - b, x - a) \cap E = (x + a, x + b) \cap E = \emptyset\,.$$

Then E is called *strongly symmetrically porous* if

$$\limsup_{r \to 0+} l(E, x, r)/r = 1$$

for each $x \in E$. Observe that, if $\Phi(x) = x^2$, then, by (i), (ii), we get

$$a_n/(b_n - a_n) < b_n - a_n < b_n < 1/n\,.$$

Hence $\lim_{n\to\infty} a_n/(b_n - a_n) = 0$ or, equivalently $\lim_{n\to\infty}(b_n - a_n)/b_n = 1$. This, by (iii), means that (Φ)-uniform symmetric porosity implies strong symmetric porosity.

Theorem 1 *Every linear perfect set contains a perfect* (Φ)-*uniformly symmetrically porous set.*

Proof. Fix any two decreasing sequences $\{x_j\}_{j=1}^{\infty}$ and $\{z_j\}_{j=1}^{\infty}$ of real numbers tending to zero. As $\Phi \in G$, the sequence $\{\Phi(z_j)\}_{j=1}^{\infty}$ is decreasing and it tends to zero. Choose a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that $x_{n_j} < \Phi(z_j)$ for every j. Put $y_j = x_{n_j} + z_j$ for $j \in \mathbb{N}$. Next pick a subsequence $\{y_{m_j}\}_{j=1}^{\infty}$ such that $y_{m_j} < 1/j$ for every j. Define $a_j = x_{n_{m_j}}$ and $b_j = y_{m_j}$ for $j \in \mathbb{N}$. Then for each $j \in \mathbb{N}$ we have $0 < a_j < b_j < 1/j$ and $\Phi(b_j - a_j) > a_j$.

Now, let us turn to the main part of the proof. We may start with a perfect set $P \subseteq \mathbb{R}$ which is bounded. We will define by induction a family $\{P_s : s \in 2^{<\omega}\}$ of perfect subsets of P where $2^{<\omega}$ stands for the set of all finite sequences of zeros and ones. If $s \in 2^{<\omega}$, we will denote $\min P_s = \alpha_s$, $\max P_s = \beta_s$ and $h_n = \min\{\beta_s - \alpha_s : s \in 2^{<\omega},\ \mathrm{lh}\, s = n\}$ for $n \in \mathbb{N} \cup \{0\}$ where $\mathrm{lh}\, s$ is the length of s. Let $P_{\langle\,\rangle} = P$ and fix $n \geq 0$. Assume that perfect sets $P_s \subseteq P$ are defined for all $s \in 2^{<\omega}$ with $\mathrm{lh}\, s < n$. We shall define the sets $P_{s\frown 0}$ and $P_{s\frown 1}$ for $\mathrm{lh}\, s = n$ (where $s \frown i$ is the respective extension of the sequence s). Let k_n be the least positive integer satisfying $b_{k_n} + 2a_{k_n} < h_n$. Choose c_s (respectively, d_s) being a left-hand (respectively, right-hand) condensation point of $P_s \cap (\alpha_s, \alpha_s + a_{k_n})$ (respectively, $P_s \cap (\beta_s - a_{k_n}, \beta_s)$). Put

$P_{s\frown 0} = P_s \cap [\alpha_s, c_s]$ and $P_{s\frown 1} = P_s \cap [d_s, \beta_s]$ for all s with $\operatorname{lh} s = n$. Observe that for $t = s \frown i$ and $i \in \{0,1\}$, we have

$$\beta_t - \alpha_t < a_{k_n} \tag{1}$$

and

$$\alpha_{s\frown 1} - \beta_{s\frown 0} \geq \beta_s - \alpha_s - 2a_{k_n} \geq h_n - 2a_{k_n} > b_{k_n} + 2a_{k_n} - 2a_{k_n} = b_{k_n}. \tag{2}$$

Having all sets $P_s, s \in 2^{<\omega}$, we define

$$Q = \bigcap_{n=1}^{\infty} Q_n$$

where $Q_n = \bigcup\{P_s : s \in 2^{<\omega}, \operatorname{lh} s = n\}$. Then Q is a perfect subset of P. Now we will show that the sequences $\{a_{k_n}\}_{n=1}^{\infty}$ and $\{b_{k_n}\}_{n=1}^{\infty}$ witness that $Q \in R(\Phi)$. For each $n \in \mathbb{N}$, conditions (i) and (ii) are clear, by the choice of the numbers a_k and b_k. It suffices to prove (iii). Fix $n \in \mathbb{N}$ and $x \in Q$. Condition (iii) will be true, if

(iii') $[x - b_{k_n}, x - a_{k_n}] \cap Q_{n+1} = [x + a_{k_n}, x + b_{k_n}] \cap Q_{n+1} = \emptyset$.

Since $x \in Q$, therefore $x \in Q_{n+1}$ and thus $x \in P_t$ for some $t \in 2^{<\omega}, \operatorname{lh} t = n+1$. Now, (iii') can be derived from the following inequalities (compare (1) and (2)):

$$\begin{aligned} x - a_{k_n} &\leq \beta_t - a_{k_n} < \alpha_t, \\ x + a_{k_n} &\geq \alpha_t + a_{k_n} > \beta_t, \end{aligned}$$

(it means that the intervals $[x - b_{k_n}, x - a_{k_n}]$ and $[x + a_{k_n}, x + b_{k_n}]$ do not meet P_t),

$$\begin{aligned} (b_{k_n} - a_{k_n}) + \alpha_t - (x - a_{k_n}) &\leq b_{k_n} - a_{k_n} + a_{k_n} = b_{k_n}, \\ (b_{k_n} - a_{k_n}) + x + a_{k_n} - \beta_t &\leq b_{k_n} - a_{k_n} + a_{k_n} = b_{k_n}, \end{aligned}$$

(it means that the intervals $[x - b_{k_n}, x - a_{k_n}]$ and $[x + a_{k_n}, x + b_{k_n}]$ do not meet the set $P_s \subseteq Q_{n+1}$ closest to P_t). □

Note that the results analogous to Theorem 1 for strong symmetric porosity in $\mathbb{R}$ and for strong porosity in a Polish space were proved in [B, Th. 1.4, 1.6].

Let $\mathbb{K}$ denote the space of all nonempty compact sets in $[0,1]$ endowed with the Hausdorff metric ρ given by

$$\rho(H, F) = \max\big\{\sup_{x \in H} d(x, F), \sup_{y \in F} d(y, H)\big\}$$

where d is the usual metric on $[0,1]$. It is known that $\mathbb{K}$ forms a Polish space [K].

Theorem 2 *The* (Φ)*-uniformly symmetrically porous nonempty compact sets in* $[0,1]$ *form a* G_δ *dense subset of* $\mathbb{K}$*, and therefore it is a residual set in* $\mathbb{K}$*.*

Proof. Since the nonempty finite sets form a dense set (in $\mathbb{K}$) which is a subset of $R(\Phi)$, it suffices to show that $R(\Phi,n)\cap\mathbb{K}$ is open for each $n\in\mathbb{N}$. So, let $n\in\mathbb{N}$ and let $F\in R(\Phi,n)\cap\mathbb{K}$. Assume that numbers a_n and b_n fulfill (i),(ii) and (iii). Put

$$U=\bigcup_{x\in F}((x-b_n,x-a_n)\cup(x+a_n,x+b_n)).$$

Then U is open and bounded. We can express U as $\bigcup_j(c_j,d_j)$ where the intervals (c_j,d_j) are pairwise disjoint. Since $d_j-c_j\ge b_n-a_n$ for every j, therefore $U=\bigcup_{j=1}^p(c_j,d_j)$ for some $p\in\mathbb{N}$. Observe that $c_j\notin F$ and $d_j\notin F$ for $j=1,\dots,p$. For instance, we will show that $c_j\notin F$. If $c_j=x-b_n$ or $c_j=x+a_n$ for some $x\in F$, condition $c_j\notin F$ is clear by (iii). In the other case, there is a sequence $\{x_m\}_{m=1}^\infty\subseteq F$ such that

$$c_j=\lim_{m\to\infty}(x_m-b_n)\text{ or }c_j=\lim_{m\to\infty}(x_m+a_n).$$

By the compactness of F, choose a subsequence x_{i_m} tending to $x\in F$. Thus $c_j=x-b_n$ or $c_j=x+a_n$ which again by (iii), yields that $c_j\notin F$. Define

$$\varepsilon=(1/2)\min\{d(c_j,F),d(d_j,F):j=1,\dots,p\}.$$

Then the ball $B=\{H\in\mathbb{K}:\rho(H,F)<\varepsilon\}$ is contained in $R(\Phi,n)\cap\mathbb{K}$. Indeed, if $H\in B$ then $H\subseteq D$ where $D=\bigcup_{x\in F}(x-\varepsilon,x+\varepsilon)$. Consider any $y\in H$. Then $y\in(x-\varepsilon,x+\varepsilon)$ for some $x\in F$. Observe that

$$\begin{aligned}[y-b_n,y-a_n]\cup[y+a_n,y+b_n]&\subseteq[x-\varepsilon-b_n,x+\varepsilon-a_n]\cup[x-\varepsilon+a_n,x+\varepsilon+b_n]\\&\subseteq\bigcup_{j=1}^p[c_j-\varepsilon,d_j+\varepsilon].\end{aligned}$$

But $\bigcup_{j=1}^p[c_j-\varepsilon,d_j+\varepsilon]\cap D=\emptyset$, by the choice of ε. Since $H\subseteq D$, we have

$$[y-b_n,y-a_n]\cap H=[y+a_n,y+b_n]\cap H=\emptyset$$

which implies that $H\in R(\Phi,n)\cap\mathbb{K}$. □

Note that the result analogous to Theorem 2 for strongly porous sets was obtained in [L]. The version for strongly shell porous sets in a complete space X was shown in [V, Th. 2.1]. If $X=\mathbb{R}$, strong shell porosity coincides with strong symmetric porosity.

It is known that the family Perf of all perfect subsets of $[0,1]$ is a G_δ subset of $\mathbb{K}$ [K, §42, III, Th. 3]. Hence Perf is a Polish space, by the Alexandrov theorem [K, §33, VI]. Now we establish a result which follows from Theorems 1 and 2.

Theorem 3 *The perfect* (Φ)*-uniformly symmetrically porous sets in* $[0,1]$ *form a* G_δ *dense subset of* Perf*, and therefore it is a residual set in* Perf*.*

PROOF. Since $R(\Phi)\cap\mathbb{K}$ is a G_δ set in $\mathbb{K}$ (by Theorem 2), therefore $R(\Phi)\cap\text{Perf}$ is a G_δ set in Perf. It suffices to show that $R(\Phi)\cap\text{Perf}$ is dense in Perf. We will utilize the fact that the topology in $\mathbb{K}$ can equivalently be generated by the base consisting of sets of the form

$$U(J_0,J_1,\dots,J_m)=\{F\in\mathbb{K}:(F\subseteq J_0)\ \&\ (\forall j\in\{1,\dots,m\})(F\cap J_j\neq\emptyset)\}$$

where J_j, for $j=0,1,\dots,m$, are open in $[0,1]$ (see [K, §42, II]). So, we will find a set from $R(\Phi)\cap\text{Perf}$ in a nonempty set $U(J_0,J_1,\dots,J_m)$. We can assume that $J_1\cup\dots\cup J_m\subseteq J_0$ and that $J_1,\dots,J_m$ are pairwise disjoint (if the last condition is not fulfilled, we choose distinct points $x_j\in J_j$ for $j=1,\dots,m$ and pairwise disjoint intervals $J_j^\star\subseteq J_j$ with $x_j\in J_j^\star$). Now, for each $j\in\{1,\dots,m\}$, choose a perfect set $P_j\in R(\Phi),P_j\subseteq J_j^\star$ (we can use Theorem 1) and put $P=\bigcup_{j=1}^m P_j$. Then $P\in U(J_0,J_1,\dots,J_m)$. Additionally, $P\in R(\Phi)$ (here we use the pairwise disjointness of $J_1,\dots,J_m$) and obviously, $P\in\text{Perf}$. □

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