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ON THE STEINHAUS PROPERTY FOR INVARIANT MEASURES

Abstract

We consider some examples of invariant measures, defined on the real line, for which an analogue of the classical Steinhaus property does not hold.

Let $\mathbb{R}$ be the real line and let l be the standard Lebesgue measure on $\mathbb{R}$. It is well known that for every l–measurable subset X of $\mathbb{R}$ the equality

$$\lim_{h \to 0} l((X+h) \cap X) = l(X)$$

holds. (An analogous fact is also true for a Haar measure μ defined on an arbitrary locally compact topological group G.) From this fact it follows immediately that if $l(X) > 0$, then the difference set

$$X - X = \{x - y : x \in X,\ y \in X\}$$

contains a neighborhood of the point 0. (See, e.g., [1, Chapter 4], or [2, p. 198].) This property of an l–measurable set X with a strictly positive measure was first observed by Steinhaus and sometimes is called the Steinhaus property of X. The following result is an immediate consequence of the Steinhaus property. Let $\{X, Y\}$ be a partition of the real line into two Lebesgue measurable sets. Then at least one of the corresponding difference sets

$$X - X, \quad Y - Y$$

contains a neighbourhood of the point 0 and, therefore, has a nonempty interior.

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On the other hand, it can be proved (although, with the aid of the Axiom of Choice) that there exists a partition $\{A, B\}$ of the real line such that the difference sets $A - A$ and $B - B$ have empty interiors. For example, such a partition $\{A, B\}$ of $\mathbb{R}$ is constructed in [3]. Note also that an analogous partition $\{A, B\}$ of $\mathbb{R}$ is considered in detail in [4]; moreover, in [4] the sets A and B are Bernstein subsets of $\mathbb{R}$.

Since the Steinhaus property concerns measurable sets (in the sense of Lebesgue), it is reasonable to formulate the following question. Can the sets A and B of a partition $\{A, B\}$ be measurable with respect to some nontrivial σ–finite invariant measures defined on the real line? In this paper we shall show that the answer is positive. In particular, we shall establish that the sets A and B can be measurable with respect to a certain invariant extension of Lebesgue measure l (cf. [5] and [6]).

First of all we shall give a simple example of a partition $\{A, B\}$ of $\mathbb{R}$ having the property that $\mathrm{int}(A - A) = \emptyset$ and $\mathrm{int}(B - B) = \emptyset$.

Example. Let us consider the real line $\mathbb{R}$ as a vector space E over the field $\mathbb{Q}$ of all rational numbers. Take the one–element subset $\{1\}$ of the space E and extend this subset to a Hamel basis H of E. Denote by Γ the vector subspace of the space E generated by the set $H \setminus \{1\}$. In fact, Γ is a vector hyperplane in E. Obviously, we can represent $E = \mathbb{R}$ as the direct sum of $\mathbb{Q}$ and Γ, i.e. we have $E = \mathbb{Q} + \Gamma, \quad \mathbb{Q} \cap \Gamma = \{0\}$. Furthermore, it is clear that every rational number q can be uniquely represented in the form

$$q = n(q) + t(q),$$

where $n(q)$ is an integer and $0 \leq t(q) < 1$. Put

$$Q_1 = \text{the set of all } q \text{ for which } n(q) \text{ is an odd number;}$$
$$Q_2 = \text{the set of all } q \text{ for which } n(q) \text{ is a even number.}$$

Evidently, we get a partition $\{Q_1, Q_2\}$ of the set $\mathbb{Q}$. Define

$$A = Q_1 + \Gamma, \quad B = Q_2 + \Gamma.$$

Then $\{A, B\}$ is a partition of the real line and the difference sets $A - A$ and $B - B$ have empty interiors. Indeed, it is easy to check that

$$A - A = (Q_1 - Q_1) + \Gamma,$$

$$B - B = (Q_2 - Q_2) + \Gamma,$$

$$1 \notin Q_1 - Q_1, \quad 1 \notin Q_2 - Q_2.$$

Hence, we obtain

$$(A - A) \cap (1 + \Gamma) = \emptyset,$$

$$(B - B) \cap (1 + \Gamma) = \emptyset.$$

Taking into account the fact that Γ is an everywhere dense subgroup of the additive group of $\mathbb{R}$, we get that $1+\Gamma$ is an everywhere dense subset of $\mathbb{R}$ and, consequently, we have

$$\mathrm{int}(A - A) = \emptyset, \quad \mathrm{int}(B - B) = \emptyset.$$

Moreover, we can choose a Hamel basis H in such a way that the hyperplane Γ would be a Bernstein subset of the real line $\mathbb{R}$. In this case $1 + \Gamma$ is also a Bernstein subset of $\mathbb{R}$, and we see that A, B, $A - A$ and $B - B$ are Bernstein sets also.

Furthermore, it is not difficult to show that the hyperplane Γ is always an l–thick subset of $\mathbb{R}$, i.e. the inner Lebesgue measure of the set $\mathbb{R} \setminus \Gamma$ is equal to zero. Let us consider the family of all distinct translates of Γ. Obviously, this family is countable. Denote it by $\{\Gamma_n \ : \ n \in \omega\}$. It is clear that $\Gamma_n \cap \Gamma_m = \emptyset$ if $n \neq m$. Take the class S of all subsets Z of $\mathbb{R}$ which can be represented in the form

$$Z = \cup\{\Gamma_n \cap X_n : n \in \omega\},$$

where $\{X_n \ : \ n \in \omega\}$ is a countable family of l–measurable sets in $\mathbb{R}$. It is easy to check that S is a σ–algebra of sets and, moreover, S is invariant under the group of all translations of $\mathbb{R}$. Now, for each $Z \in S$, let us put

$$\nu(Z) = l(X_0) + l(X_1) + ... + l(X_n) + ... \, .$$

Since all the sets Γ_n $(n \in \omega)$ are l–thick and pairwise disjoint, the functional ν is well–defined on the σ–algebra S. Also it is not difficult to check that the following relations are fulfilled.

1) ν is a non-atomic σ–finite measure on $\mathbb{R}$.

2) ν is invariant under the group of all translations of $\mathbb{R}$ (moreover, ν is invariant under the group of all isometric transformations of $\mathbb{R}$).

3) $\mathrm{dom}(l) \subset \mathrm{dom}(\nu)$.

4) $\Gamma \in \mathrm{dom}(\nu)$, $\quad A \in \mathrm{dom}(\nu)$, $\quad B \in \mathrm{dom}(\nu)$.

5) $\nu(A) = \nu(B) = +\infty$.

Thus, we see that for two ν–measurable sets A and B the Steinhaus property does not hold, while $\{A, B\}$ is a partition of the real line.

We remark also that the ν–measurable set Γ is a Vitali subset of the real line, because for each $h \in \mathbb{R}$ we have $\mathrm{card}((\mathbb{Q} + h) \cap \Gamma) = 1$. This unusual property of the measure ν was mentioned in [5]. Of course, such a measure ν cannot be an invariant extension of Lebesgue measure l, since a certain Vitali set belongs to the domain of ν. Note that if X is an arbitrary Lebesgue measurable subset of the real line satisfying the inequality $l(X) > 0$, then $\nu(X) = +\infty$. From this fact it also follows that the measure ν cannot extend the Lebesgue measure l.

Now let us consider an arbitrary measure μ on $\mathbb{R}$ extending l and invariant under the group of all translations of $\mathbb{R}$. Take any partition $\{A, B\}$ of $\mathbb{R}$ such that $A \in \mathrm{dom}(\mu)$. (Hence, $B \in \mathrm{dom}(\mu)$ also.) The following question arises in a natural way. Is it true that at least one of the difference sets $A - A$ and $B - B$ has a nonempty interior? We shall show below that the answer to this question is negative. But first we consider a situation where we can answer the posed question in the affirmative.

It is easy to see that, for Lebesgue measure l, the Steinhaus property may be obtained as a direct consequence of the classical Lebesgue theorem on density points of l–measurable sets. More generally, we have the following

Proposition 1 *Let μ be a measure on $\mathbb{R}$ extending l and invariant under the group of all translations of $\mathbb{R}$. Let $\{A, B\}$ be a partition of $\mathbb{R}$ consisting of two μ–measurable sets. Suppose also that there exists a segment $I \subset \mathbb{R}$ such that $\mu(I \cap A) \neq l(I)/2$. Then at least one of the difference sets $A - A$ and $B - B$ contains a neighborhood of the point 0. (Hence, at least one of these difference sets has a nonempty interior.)*

Proof. Obviously, we can write $l(I) = \mu(I) = \mu(I \cap A) + \mu(I \cap B)$ and, consequently, $\mu(I \cap A) > l(I)/2$ or $\mu(I \cap B) > l(I)/2$. Without loss of generality we may assume that $\mu(I \cap A) > l(I)/2$. Let us show that in this case the difference set $A - A$ contains a neighbourhood of the point 0. Indeed, suppose that $A - A$ does not contain an open interval with the center 0. Then there exists a sequence $\{h_n : n \in \omega\}$ of elements of $\mathbb{R}$ satisfying the relations

$$\lim_{n \to \infty} h_n = 0,$$
$$(h_n + A) \cap A = \emptyset \quad (n \in \omega).$$

It is clear that, for some $\epsilon > 0$, we have $\mu(I \cap A) > l(I)/2 + \epsilon$. Let n be a natural number such that $\mu(I \cup (h_n + I)) < l(I) + \epsilon$. Then we get the inequalities

$$\begin{aligned} l(I) + \epsilon &> \mu(I \cup (h_n + I)) \\ &\geq \mu((I \cap A) \cup ((h_n + I) \cap (h_n + A))) \\ &= 2\mu(I \cap A) > l(I) + 2\epsilon, \end{aligned}$$

which give us a contradiction. □

Proposition 1 shows that if we want to construct a partition $\{A, B\}$ of $\mathbb{R}$ consisting of μ–measurable sets such that $\mathrm{int}(A - A) = \mathrm{int}(B - B) = \emptyset$,then we necessarily must have $\mu(I \cap A) = \mu(I \cap B) = l(I)/2$, for every segment I on the real line $\mathbb{R}$. From the last equalities it immediately follows also that

$$\mu(X \cap A) = \mu(X \cap B) = l(X)/2,$$

for every l–measurable subset X of $\mathbb{R}$ (note that several examples of invariant extensions of l satisfying the last relation are investigated in [6, p. 117]).

Now we shall construct a measure μ and a partition $\{A, B\}$ with the properties mentioned above.

Let $\mathbf{T}$ denote the one–dimensional torus, i.e. put

$$\mathbf{T} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

Obviously, $\mathbf{T}$ is a commutative divisible compact topological group with respect to the standard structures on $\mathbf{T}$. It is also clear that we may consider $\mathbf{T}$ as a closed interval $[0,\ 2\pi]$ in which the end–points 0 and 2π are identified. Equip $\mathbf{T}$ with a probability Lebesgue measure λ invariant under the group of all rotations of $\mathbf{T}$. Now we apply the method similar to the method of a well known paper [7], where a nonseparable invariant extension of Lebesgue measure is constructed. Let $\{h_n : n \in \omega\}$ be an arbitrary countable, everywhere dense subset of $\mathbb{R}$ linearly independent over the field $\mathbb{Q}$ of all rational numbers. Using the method of transfinite recursion we can define a homomorphism f from the abstract group $\mathbb{R}$ into the abstract group $\mathbf{T}$ such that

1) the graph of f, i.e. the set $\{(x, f(x)) : x \in \mathbb{R}\}$, is an $(l \times \lambda)$–thick subset of the product space $\mathbb{R} \times \mathbf{T}$. (In other words, the inner $(l \times \lambda)$–measure of the complement of this graph is equal to zero.);

2) $f(h_n) = \pi$ for each $n \in \omega$.

The construction of the homomorphism f with properties 1) and 2) is standard and does not present any difficulties. In fact, we use here a general theorem of the theory of commutative groups which states that any partial homomorphism from a commutative group G_1 into a divisible commutative group G_2 can be extended to a homomorphism from G_1 into G_2[1]. This theo-

[1](see, e.g., A. G. Kurosh, *The Theory of Groups*, "Nauka", Moscow, 1967, p. 551 - 552)

rem has many useful corollaries; for example,

a) any commutative group can be embedded in a divisible commutative group;

b) any infinite commutative group admits a nontrivial topologization;

c) any commutative group is algebraically isomorphic with an everywhere dense subgroup of a commutative compact topological group.

Now, starting with the homomorphism f mentioned above, we can define the required invariant extension μ of the measure l. Namely, for each $(l \times \lambda)$–measurable set Z, let us put

$$Z^* = \{x \in \mathbb{R} : (x, f(x)) \in Z\}.$$

Furthermore, put

$$S = \{Z^* : Z \in \mathrm{dom}(l \times \lambda)\},$$
$$\mu(Z^*) = (l \times \lambda)(Z) \quad (Z^* \in S).$$

It is not difficult to check that S is a σ–algebra of subsets of $\mathbb{R}$ and the functional μ is well–defined on S. It can be also checked that μ is a measure on this σ–algebra. Moreover, μ extends l and is invariant under the group of all isometric transformations of $\mathbb{R}$. As remarked in [8], the Steinhaus property does not hold for such a measure μ. More exactly, let us put

$$A = \{x \in \mathbb{R} : (x, f(x)) \in \mathbb{R} \times [0, \pi)\},$$
$$B = \{x \in \mathbb{R} : (x, f(x)) \in \mathbb{R} \times [\pi, 2\pi)\}.$$

From the definition of the sets A and B it immediately follows that

(1) $\{A, B\}$ is a partition of the real line $\mathbb{R}$;

(2) A and B are μ–measurable subsets of $\mathbb{R}$;

(3) $\mu(A) = \mu(B) = +\infty$;

(4) $h_n + A = B$ and $h_n + B = A$ for each $n \in \omega$.

In particular, we have

$$(h_n + A) \cap A = \emptyset, \quad (h_n + B) \cap B = \emptyset$$

for all $n \in \omega$. Hence, taking into account the fact that $\{h_n : n \in \omega\}$ is an everywhere dense subset of $\mathbb{R}$, we obtain that the difference sets $A - A$ and $B - B$ have empty interiors.

Slightly changing the above argument we can get a more general result. Namely, we have the following

Proposition 2 *There exists a measure μ on the real line $\mathbb{R}$, extending l and invariant under the group of all isometric transformations of $\mathbb{R}$, and a partition $\{A, B\}$ of $\mathbb{R}$ consisting of two μ–measurable sets such that all the sets A, B, $A-A$ and $B-B$ are totally imperfect subsets of $\mathbb{R}$, i.e. they are Bernstein subsets of $\mathbb{R}$.*

Finally, let us recall that the partition $\{A, B\}$ has also the following property. For every segment I on the real line $\mathbb{R}$, the equalities

$$\mu(I \cap A) = \mu(I \cap B) = l(I)/2$$

are fulfilled. Note in connection with this fact that a much stronger property of some subsets of the real line is discussed in detail in the paper [9]. For other properties of the measure μ constructed above, see [8].

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