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REMARKS ABOUT A TRANSITIVE VERSION OF PERFECTLY MEAGER SETS

Abstract

We show that if X has the property that every continuous image into Baire space is bounded and 2^ω is not a continuous image of X, then X is always of first category in some additive sense. This gives an answer to an oral question of L. Bukovský, whether every wQN set has the latter property.

1 Notation and Definitions

$\mathcal{MGR}(P)$ denotes the family of first category subsets of P. If $s \in 2^{<\omega}$ then $\mathcal{N}_s = \{t \in 2^\omega : t \supseteq s\}$ is a basic clopen set in 2^ω. Every clopen subset of 2^ω is a finite union of $\mathcal{N}_s$. We denote by $\Delta^0_1(2^\omega)$ the class of clopen subsets of 2^ω.

Definition 1 [NSW] *$X \subseteq 2^\omega$ is perfectly meager in the transitive sense iff for every perfect $P \subseteq 2^\omega$ one can find $\overline{F_n} = F_n \subseteq 2^\omega$ such that*

$$X \subseteq \bigcup_{n<\omega} F_n$$

and

$$\forall_{h\in 2^\omega} \left(\bigcup_{n<\omega} F_n\right) \cap (P+h) \in \mathcal{MGR}(P+h).$$

We use the abbreviation AFC' for this property.

Key Words: Hurewicz property, $S_1(\Gamma,\Gamma)$, AFC' set, wQN set, Σ set, $\mathcal{H}$ property
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Definition 2 **[BRR]** *$X \subseteq 2^\omega$ is wQN iff for every sequence*

$$f_n : X \to \mathbf{R}_+$$

of continuous functions converging to 0 we can find subsequence

$$f_{n_k} : X \to \mathbf{R}_+$$

such that one can find partition

$$\{X_n\}_{n<\omega}$$

of X such that for every $n < \omega$ $\{f_{n_k}\}_{k<\omega}$ converges uniformly to 0 on X_n.

Definition 3 **[BRR]** *$X \subseteq 2^\omega$ is Σ set iff for every sequence*

$$f_n : X \to \mathbf{R}_+$$

of continuous functions converging to zero one can find subsequence

$$f_{n_k} : X \to \mathbf{R}_+$$

such that

$$\forall_{x \in X} \sum_{k<\omega} f_{n_k}(x) < \infty.$$

We use also abbreviation (CB) to denote the property of X such that:

1. The image of the set X by every continuous function into ω^ω is bounded, and
2. The space 2^ω is not continuous image of X.

The property of X that every Borel image of X into ω^ω is bounded was considered in [BJ] and was there denoted by the property $\mathcal{H}$.

The property of X that every continuous image of X into ω^ω is bounded is known as a Hurewicz property.

2 Remarks

In [NSW] authors proved, answering a question in [S], that every algebraic sum of sets of strong measure zero and strong first category has the Marczewski property s_0. In fact, authors proved, that every algebraic sum of so called AFC' set and strong measure zero set has the property s_0. In [NSW] it is proved that every γ set has the property AFC'. It is obvious that every AFC' set is also perfectly meager. We know, that every wQN set is perfectly meager and every γ set is wQN. So it is natural question about relation between the class of wQN sets and AFC' sets.

3 Main Theorem

Lemma 1 *For every finite sequence of perfect subsets of 2^ω : $P_1, \ldots, P_k \subseteq 2^\omega$ there exists a partition of 2^ω into two disjoint clopen sets: U_0, U_1 such that*

$$\forall_{h\in 2^\omega}\forall_{1\le i\le k}\forall_{0\le j\le 1}(h+P_i)\cap U_j \neq \emptyset.$$

PROOF. Moving each one of the sets P_i $(1 \le i \le k)$ we can assume, that each of them contains a null sequence: $\underline{0} = (0, 0, \ldots)$.

Now there exists $n < \omega$ and $f_i \in P_i \setminus \{\underline{0}\}$ such, that $\{f_i|n\}_{i=1,\ldots,k}$ is a sequence of linear independent vectors over the field $Z_2 = \{0, 1\}$. (We treat 2^n as a linear space over the field Z_2). So we complete $\{f_i|n\}_{i=1,\ldots,k}$ to a base of the space 2^n with vectors $e_{k+1}, \ldots, e_n \in 2^n$.

Moreover, we put $e_i = f_i|n$ $(i = 1, \ldots, k)$. So $(e_1, \ldots, e_n)$ is a base of 2^n over Z_2.

Now we consider

$$V := \left\{ \sum_{i=1}^{n} \alpha_i e_i : \alpha_i \in Z_2 \wedge |\{i : \alpha_i = 1\}| \text{ is even} \right\}.$$

Obviously V is a linear subspace of 2^n over a field Z_2, moreover is has a codimension one. Also it is clear, that V does not contain $e_i = f_i|n$ $(1 \le i \le k)$.

Now we see, that for every $s \in 2^n$

$$s \in V \quad \text{iff} \quad s + f_i|n \notin V.$$

It is easy to see now, that if we define:

$$U_0 := \bigcup_{s\in V} \mathcal{N}_s \text{ and } U_1 := \bigcup_{s\in 2^n\setminus V} \mathcal{N}_s$$

then U_0 and U_1 will be disjoint clopen sets having the properties of Lemma 1. □

Lemma 2′ *Let P be a perfect set and let $\{B_i\}_{i<\omega}$ be an enumeration of the base of P with $B_0 = P$. There is a system $\{U_s : s \in 2^{<\omega}\}$ of clopen subsets of 2^ω, $U_\emptyset = 2^\omega$, $\{U_{s^\frown\langle 0\rangle}, U_{s^\frown\langle 0\rangle}\}$ is a partition of U_s such that*

(1) $\forall_{s\in 2^{<\omega}}\forall_{h\in 2^\omega}\forall_{j=0,1}\forall_{i\le|s|}(B_i + h)\cap U_s \neq \emptyset \rightarrow (B_i + h)\cap U_{s^\frown\langle j\rangle} \neq \emptyset.$

PROOF. By induction on length of $s \in 2^{<\omega}$ we define the sets U_s. We set $U_\emptyset = 2^\omega$ and assuming that U_s are constructed for all $s \in 2^k$ we find an integer

n_k such that for every $s \in 2^k$ there is a set $S_s \subseteq 2^{n_k}$ such that $U_s = \bigcup_{t\in S_s} \mathcal{N}_t$. For $s \in 2^k$ we set

$$\begin{aligned} T_k &= \{h \in 2^\omega : \forall_{n\geq n_k} h(n) = 0\}, \\ R_s &= \{(B_i + h) \cap U_s : (B_i + h) \cap U_s \neq \emptyset,\ i \leq k,\ h \in T_k\}. \end{aligned}$$

Let $\{U_0^s, U_1^s\}$ be a clopen partition of 2^ω with properties ensured by Lemma 1 for the finite system of perfect sets R_s and let us set

$$U_{s\frown\langle j\rangle} = U_s \cap U_j^s, \qquad j = 0, 1.$$

Now if $h \in 2^\omega$ is arbitrary, let $h_{(k)}, h^{(k)} \in 2^\omega$ be such that $h_{(k)}|n_k = h|n_k$, $h^{(k)}|[n_k, \infty) = h|[n_k, \infty)$, and $h_{(k)} + h^{(k)} = h$. Hence $h_{(k)} \in T_k$. If $(B_i + h) \cap U_s \neq \emptyset$, then also $(B_i + h_{(k)}) \cap U_s \neq \emptyset$, because $U_s + h^{(k)} = U_s$. Therefore

$$(B_i + h) \cap U_{s\frown\langle j\rangle} = [((B_i + h_{(k)}) \cap U_s) + h^{(k)}] \cap U_j^s \neq \emptyset$$

and condition (1) is fulfilled. □

Lemma 2′ has the following equivalent reformulation.

Corollary 2″ *For every perfect set $P \subseteq 2^\omega$ there is a continuous mapping $\Phi : 2^\omega \to 2^\omega$ such that for each $h \in 2^\omega$ the restriction $\Phi|(P + h)$ is an open mapping from $P + h$ onto 2^ω.*

PROOF. Let $\{U_s : s \in 2^{<\omega}\}$ be a system of clopen subsets of 2^ω as is stated in Lemma 2′. Let us define $\Phi : 2^\omega \to 2^\omega$ by $\Phi(x) = y$ iff $x \in \bigcap_{n<\omega} U_{y|n}$. Taking $i = 0$ in condition (1) we obtain

(2) $\forall_{h\in 2^\omega}\forall_{s\in 2^{<\omega}} (P + h) \cap U_s \neq \emptyset$

which easily implies that the mapping $\Phi|(P + h)$ is onto 2^ω. Similarly, condition (1) implies that

$$\Phi(B_i + h) = \bigcup\{\mathcal{N}_s : s \in 2^i,\ (B_i + h) \cap U_s \neq \emptyset\}. \qquad \square$$

Corollary 2‴ *If $X \subseteq 2^\omega$ and 2^ω is not a continuous image of X, then for every perfect set $P \subseteq 2^\omega$ there is a sequence $\{U_i\}_{i<\omega}$ of disjoint clopen subsets of 2^ω such that*

(1) $\forall_{h\in 2^\omega}\forall_{j<\omega} (P + h) \cap U_j \neq \emptyset$,

(2) $\forall_{h\in 2^\omega} (P + h) \setminus \bigcup_{j<\omega} U_j \in \mathcal{MGR}(P + h)$,

(3) $X \subseteq \bigcup_{i<\omega} U_i$.

PROOF. Let $\Phi : 2^\omega \to 2^\omega$ be a continuous mapping such that $\Phi|(P+h)$ is open and onto 2^ω for every $h \in 2^\omega$. Take any $y \in 2^\omega \setminus \Phi(X)$. In particular, $\Phi^{-1}(2^\omega \setminus \{y\}) \cap (P+h)$ is open dense in $P+h$. Let $S \subseteq \{s \in 2^{<\omega} : y \notin \mathcal{N}_s\}$ be a maximal antichain in $2^{<\omega}$ and let $\{U_i\}_{i<\omega}$ be an enumeration of the set $\{\Phi^{-1}(\mathcal{N}_s) : s \in S\}$. □

Theorem 1 *Let $X \subseteq 2^\omega$ be a set with the property (CB). Then X has the property AFC'.*

PROOF. Let $P \subseteq 2^\omega$ be a perfect set. Let

$$\{B_i\}_{i<\omega}$$

be a clopen base in P. For every $j < \omega$ we apply Corollary $2'''$ to the perfect set B_j and we obtain a sequence

$$\{U_i^{(j)}\}_{i<\omega}$$

of clopen subsets of 2^ω. Put

$$N = \bigcap_{j<\omega} \bigcup_{i<\omega} U_i^{(j)}.$$

We define now:

$$\Psi : N \to \omega^\omega$$

by the condition, that

$$\Psi(x) = f \iff x \in \bigcap_{j<\omega} U_{f(j)}^{(j)}$$

for $x \in N$ and $f \in \omega^\omega$.

Now $\Psi(X)$ is bounded in ω^ω, so there exists a sequence $f_n \in \omega^\omega$, $n < \omega$, such that the closed sets

$$F_n = \bigcap_{j<\omega} \bigcup_{i<f_n(j)} U_i^{(j)} \subseteq N$$

cover X.

Take any $h \in 2^\omega$, $n < \omega$ and assume that $F_n \cap (P+h)$ is not meager in $P+h$, i.e. there exists $j < \omega$ such that $B_j + h \subseteq F_n$. By condition (1) of Corollary $2'''$ we can now choose $i > f_n(j)$ so that $(B_j + h) \cap U_i^{(j)} \neq \emptyset$ and then $F_n \cap U_i^{(j)} \neq \emptyset$ which is a contradiction because F_n is disjoint with every set $U_i^{(j)}$ for $i \geq f_n(j)$.

This gives us that

$$\forall_{n<\omega} F_n \cap (P+h) \in \mathcal{MGR}(P+h)$$

holds true and so X has the property AFC'. □

4 Conclusions

Conclusion 1 *Every Σ set is an AFC' set.*

Proof. We know from [BRR], that every continuous image of an wQN set into ω^ω is bounded. Modifying the proof one can obtain, that continuous image of a Σ into ω^ω is also bounded. From [BRR] we know that continuous image of a Σ set is also a Σ set. One can show, that 2^ω is not a Σ set, so we obtain, that every Σ set has the property (CB). □

Conclusion 2 *Every wQN subset of 2^ω is an AFC' set.*

Proof. By [BRR] every wQN set is a Σ set. □

Conclusion 3 *Every $\mathcal{H}$ set is AFC'.*

Proof. Every $\mathcal{H}$ set is a wQN set. □

Conclusion 4

$$non(AFC') \geq \mathbf{b}.$$

Proof. Obvious, because every X with the cardinality less than $\mathbf{b}$ has the property (CB). □

Conclusion 5 *Every $S_1(\Gamma,\Gamma)$ set is AFC'. Recall that X is $S_1(\Gamma,\Gamma)$ iff for every sequence $\mathcal{U}_n$ of open γ covers we can find $V_n \in \mathcal{U}_n$ such that $\{V_n\}_{n<\omega}$ is also a γ cover. Under γ cover we mean every cover $\mathcal{U}$ of X such that $|\mathcal{U}| \geq \omega$ and*

$$\forall_{x\in X} |\{U \in \mathcal{U} : x \notin U\}| < \omega.$$

This notion was defined and considered in [JMSS].

Proof. In [JMSS] authors proved that every $S_1(\Gamma,\Gamma)$ set has the Hurewicz property and also that 2^ω has not the property $S_1(\Gamma,\Gamma)$ ([JMSS, Theorem 2]). Because every continuous image of set of the property $S_1(\Gamma,\Gamma)$ has also the property $S_1(\Gamma,\Gamma)$ ([JMSS, Theorem 3.1]), so we obtain the Conclusion 5. □

Conclusion 6 *There exists in ZFC an AFC' uncountable set.*

PROOF. In [JMSS, proof of Theorem 5.1] the existence of an uncountable set in $S_1(\Gamma, \Gamma)^*$ is proved and I. Recław [R] proved that every such set is a wQN set. □

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