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A NOTE ON THE DISTRIBUTION OF DIGITS IN TRIADIC EXPANSIONS

Abstract

We estimate the Hausdorff dimension of some Borel sets determined by the digits in triadic expansions.

1 Introduction

Let $x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{3^n}$, where $\varepsilon_n(x) \in \{0, 1, 2\}$, be the 3-adic expansion of $x \in [0, 1]$. Our purpose is to estimate the Hausdorff dimension of the set

$$M_{k,m}(q) = \left\{ x : \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \varepsilon_n^k(x) \varepsilon_{n+1}^m(x) = q \right\}, \tag{1}$$

where $k, m \in \{0, 1, 2\},\ k + m \geq 1$ and $q \in [0, 2^{k+m}]$. The proof is based on the construction of a suitable measure. It would be desirable to see the analogous problem for x expressed as a decimal in the scale $r = 4, 5, \ldots$ but we have not been able to do this; for $r = 2$ see [3], extended in [4]. Similar results have been obtained in [1], [2], [5], [7], [11], [13]. In section 3 we give a multifractal analysis of some measures related to this work.

Let $P = (p_{ij}),\ i, j = 0, 1, 2$, be a stochastic irreducible matrix, $P^{(0)} = (\pi_0, \pi_1, \pi_2)$ be a probability vector such that $P^{(0)}P = P^{(0)}$ and $E_N(x)$ be the interval of the form $[\frac{\kappa}{3^N}, \frac{\kappa+1}{3^N})$ containing $x,\ \kappa = 0, 1, \ldots, 3^N - 1$. We define the measure μ by its values on $E_N(x)$;

$$\mu(E_N(x)) = \pi_{\varepsilon_1(x)} \prod_{n=1}^{N-1} p_{\varepsilon_n(x)\varepsilon_{n+1}(x)}. \tag{2}$$

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It is well known [2] that if

$$M = \left\{ x : \lim_{N\to\infty} \frac{\log \mu(E_N(x))}{-N \log 3} = \delta_P \right\}, \tag{3}$$

where

$$\delta_P = \frac{-1}{\log 3} \sum_{i,j=0}^{2} \pi_i p_{ij} \log p_{ij},$$

then $\mu(M) = 1$ and the Hausdorff dimension of M, $\dim M$, is δ_P.

2 The Hausdorff Dimension of $M_{k,m}(q)$

In this section it is show that we can choose P in such a way that $M = M_{k,m}(q)$. From (2) we obtain

$$\log \mu(E_N(x)) = \log \pi_{\varepsilon_1(x)} + \sum_{n=1}^{N-1} \sum_{i,j=0}^{2} \delta_{\varepsilon_n(x),i}\, \delta_{\varepsilon_{n+1}(x),j} \log p_{ij},$$

where $\delta_{\cdot,\cdot}$ is the usual Kronecker symbol. We observe that

$$\delta_{\varepsilon_n(x),i} = (-1)^i \frac{\prod\limits_{\substack{j=0\\ j\neq i}}^{2} (j - \varepsilon_n(x))}{i!(2-i)!} = \sum_{k=0}^{2} c_{ki} \varepsilon_n^k(x), \quad c_{ki} \in \mathbb{R}. \tag{4}$$

Let $\log A_{km} = \sum\limits_{i,j=0}^{2} c_{ki} c_{mj} \log p_{ij}$. Then

$$\log \mu(E_N(x)) = \log \pi_{\varepsilon_1(x)} + \sum_{n=1}^{N-1} \sum_{k,m=0}^{2} \varepsilon_n^k(x)\, \varepsilon_{n+1}^m(x)\ \log A_{km}$$

and

$$\lim_{N\to\infty} \frac{1}{N} \log \mu(E_N(x)) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Bigg[\log A_{00} + \varepsilon_n(x) \log(A_{10}A_{01}) \tag{5}$$
$$+ \varepsilon_n^2(x) \log(A_{20}A_{02}) + \sum_{k,m=1}^{2} \varepsilon_n^k(x)\, \varepsilon_{n+1}^m(x)\ \log A_{km} \Bigg].$$

Set $E_{10} = E_{01} = A_{10}A_{01}$, $E_{20} = E_{02} = A_{20}A_{02}$ and $E_{km} = A_{km}$ for $km \neq 0$. We will make the following assumptions: if for some E_{ij} the product ij is 0 then $j = 0$ and if for some $M_{k,m}$ the product km is 0 then $m = 0$.

Theorem. *Theorem Let* $M_{k,m}(q)$ *be as in* (1). *Then* $\dim M_{k,m}(q) = \delta_P$, *where* $P = (p_{ij}),\ i,j = 0,1,2,$ *is a solution of the system*

$$E_{ij} = 1, \quad (i,j) \neq (k,m), \quad q = t_{km} = \sum_{i,j=0}^{2} i^k j^m \pi_i p_{ij}.$$

Note. We adopt the convention that $0^0 = 1$.

PROOF. Suppose the stochastic irreducible matrix $P = (p_{ij})$ is a solution of the above system. Then from (3) and (5) it follows that

$$M = \left\{ x : \lim_{N\to\infty} \frac{-1}{N\log 3} \sum_{n=1}^{N} \left[\log A_{00} + \varepsilon_n^k(x)\varepsilon_{n+1}^m(x)\log E_{km}\right] = \delta_P \right\}.$$

Since $\int \varepsilon_1^k(x)\varepsilon_2^m(x)d\mu(x) = t_{km}$, the ergodic theorem [2] shows that

$$M = M_{k,m}(t_{km}) = M_{k,m}(q),$$

which is the desired conclusion. We need only show the existence of a solution to the system. For simplicity of notation we write $x_i = p_{i0},\ y_i = p_{i1}$, $i = 0,1,2$. By (4) it is obvious that

$$\begin{aligned} &c_{00} = 1,\ c_{10} = -\frac{3}{2},\ c_{20} = \frac{1}{2},\ c_{01} = 0,\ c_{11} = 2,\ c_{21} = -1,\\ &c_{02} = 0,\ c_{12} = -\frac{1}{2},\ c_{22} = \frac{1}{2}. \end{aligned}$$

Hence in any case (with respect to k,m) we have *five of the following six relations, (a)–(f), and the relation (g)*:

(a) $$E_{10} = x_0^{-3}y_0^2(1-x_0-y_0)^{-\frac{1}{2}}x_1^2x_2^{-\frac{1}{2}} = 1$$

(b) $$E_{20} = x_0y_0^{-1}(1-x_0-y_0)^{\frac{1}{2}}x_1^{-1}x_2^{\frac{1}{2}} = 1$$

(c) $$\begin{aligned} E_{11} = {} & x_0^{\frac{9}{4}}y_0^{-3}(1-x_0-y_0)^{\frac{3}{4}}x_1^{-3}y_1^4(1-x_1-y_1)^{-1}\\ & \times x_2^{\frac{3}{4}}y_2^{-1}(1-x_2-y_2)^{\frac{1}{4}} = 1 \end{aligned}$$

(d) $$\begin{aligned} E_{21} = {} & x_0^{-\frac{3}{4}}y_0(1-x_0-y_0)^{-\frac{1}{4}}x_1^{\frac{3}{2}}y_1^{-2}(1-x_1-y_1)^{\frac{1}{2}}\\ & \times x_2^{-\frac{3}{4}}y_2(1-x_2-y_2)^{-\frac{1}{4}} = 1 \end{aligned}$$

(e) $$E_{12} = x_0^{-\frac{3}{4}}y_0^{\frac{3}{2}}(1-x_0-y_0)^{-\frac{3}{4}}x_1y_1^{-2}(1-x_1-y_1)$$

$$\times x_2^{-\frac{1}{4}} y_2^{\frac{1}{2}} (1-x_2-y_2)^{-\frac{1}{4}} = 1$$

$$(f) \qquad E_{22} = x_0^{\frac{1}{4}} y_0^{-\frac{1}{2}} (1-x_0-y_0)^{\frac{1}{4}} x_1^{-\frac{1}{2}} y_1 (1-x_1-y_1)^{-\frac{1}{2}}$$

$$\times x_2^{\frac{1}{4}} y_2^{-\frac{1}{2}} (1-x_2-y_2)^{\frac{1}{4}} = 1$$

$$(g) \qquad q = t_{km}$$

We give the proof only for the case $k = m = 1$; the other cases may be proved in much the same way. For the convenience of the reader we write the equations which we have in any case.

(i) $\mathbf{k = m = 1}$. We have the equations (a), (b), (d), (e), (f) and (g). An easy computation shows that we have (g) and

$$\frac{x_0(1-x_0-y_0)}{y_0^2} = \frac{x_1(1-x_1-y_1)}{y_1^2} = \frac{x_2(1-x_2-y_2)}{y_2^2}$$
$$= \frac{x_1^2}{x_0 x_2} = \frac{y_1^2}{y_0 y_2}, \quad x_0^2 = y_0 x_1.$$

Combining these we obtain

$$x_1 = \frac{x_0^2}{y_0},\ x_2 = \frac{x_0^2}{1-x_0-y_0}\,, \tag{6}$$

$$F_1(x_0,y_0,y_1) = y_1^2 + y_1\frac{x_0y_0}{1-x_0-y_0} - \frac{x_0y_0}{1-x_0-y_0}(1-\frac{x_0^2}{y_0}) = 0, \tag{7}$$

$$F_2(x_0,y_0,y_1) = y_1^4 + y_1^2\frac{x_0^2y_0}{1-x_0-y_0} - x_0^3(1-\frac{x_0^2}{1-x_0-y_0}) = 0, \tag{8}$$

$$y_2 = \frac{y_1^2 y_0}{x_0(1-x_0-y_0)}\,. \tag{9}$$

From (6) we see that x_0, y_0 must be such that $x_0^2 < y_0 < 1 - x_0 - x_0^2$ (and so $x_0 \in (0, \frac{1}{2})$). Let $\mathbf{x_0} \in (0, \frac{1}{2})$ and $y_0 \in (\mathbf{x_0^2}, 1 - \mathbf{x_0} - \mathbf{x_0^2})$. The equation (7) has a unique positive solution $y_1' = h'(\mathbf{x_0}, y_0)$. The same holds for (8) with $y_1'' = h''(\mathbf{x_0}, y_0)$. Then for $y_1' = y_1''$ we must have

$$H(\mathbf{x_0}, y_0) = {h'}^2(\mathbf{x_0}, y_0) - h''(\mathbf{x_0}, y_0) = 0.$$

Since $H(\mathbf{x_0}, \mathbf{x_0^2}) < 0$ and $H(\mathbf{x_0}, 1-\mathbf{x_0}-\mathbf{x_0^2}) > 0$, there exists $\mathbf{y_0} \in (\mathbf{x_0^2}, 1-\mathbf{x_0}-\mathbf{x_0^2})$, such that $H(\mathbf{x_0}, \mathbf{y_0}) = 0$. Hence for this $(\mathbf{x_0}, \mathbf{y_0})$ we have $y_1' = y_1'' = \mathbf{y_1}$. It is easy to check that $\mathbf{y_1} \in (0, 1 - \frac{\mathbf{x_0^2}}{\mathbf{y_0}})$ and $y_2 \in (0, 1 - \frac{\mathbf{x_0^2}}{1-\mathbf{x_0}-\mathbf{y_0}})$. The Jacobian $\frac{\partial(F_1,F_2)}{\partial(y_0,y_1)}(\mathbf{x_0}, \mathbf{y_0}, \mathbf{y_1})$ is positive $(\partial F_1/\partial y_0(\mathbf{x_0}, \mathbf{y_0}, \mathbf{y_1}) < 0)$ and so the implicit function theorem [10, p. 168] gives that there is a neighborhood B of $\mathbf{x_0}$ and uniquely determined continuous functions g_0, g_1, defined on B such that $y_0 = g_0(x_0)$, $y_1 = g_1(x_0)$ and

$$F_1(x_0, g_0(x_0), g_1(x_0)) = F_2(x_0, g_0(x_0), g_1(x_0)) = 0. \tag{10}$$

Since the Jacobian is positive everywhere in our domain, we can have the functions g_0, g_1 defined on $(0, \frac{1}{2})$ and satisfy (10). Thus by (6) and (9) we get continuous functions f_i, g_2 such that $x_i = f_i(x_0)$, $y_2 = g_2(x_0)$, $x_0 \in (0, \frac{1}{2})$. From this we conclude that t_{11} is also a continuous function of x_0. It is obvious that

$$y_0 = g_0(x_0) \to \frac{1}{4},\ x_i = f_i(x_0) \to 1,\ y_i = g_i(x_0) \to 0,\ i = 1, 2 \text{ as } x_0 \to \frac{1}{2},$$

which gives $t_{11} \to 0$, as $x_0 \to \frac{1}{2}$. If x_0 is near to 0, then $H(x_0, x_0^{\frac{1}{4}}) > 0$, $H(x_0, x_0) < 0$, and $F_2(x_0, y_0, \frac{\sqrt{x_0^{\frac{3}{2}}}}{y_0}) > 0$. Therefore,

$$g_i(x_0) \to 0,\ f_i(x_0) \to 0,\ i = 0, 1, 2, \text{ as } x_0 \to 0$$

and finally that $t_{11} \to 4$ as $x_0 \to 0$. Let $q \in (0, 4)$. By the above there is $x_0 \in (0, \frac{1}{2})$ and so a stochastic irreducible matrix P, such that $M = M_{1,1}(q)$ which is our assertion.

If $q = 4$, then by [7] we have $\dim M_{1,1}(4) = 0$. We can extend the proof to assume P such that $x_i = 0$, $y_i = 0$ and $\delta_P = 0$ (P is not irreducible).

If $q = 0$, then for P such that $x_0 = \frac{1}{2}$, $y_0 = \frac{1}{4}$, $x_i = 1$, $y_i = 0$, $i = 1, 2$ we take that M is a proper subset of $M_{1,1}(0)$ and so $\dim M_{1,1}(0) \geq \delta_P$. We apply another version of (3) (see [2, p. 144]) and use the results for $q \in (0, 4)$ to obtain $\dim M_{1,1}(0) = \delta_P$.

(**ii**) $\mathbf{k = 2}$, $\mathbf{m = 1}$. We have the equations (a), (b), (c), (e), (f) and (g) or equivalently (g) and

$$\frac{x_0(1 - x_0 - y_0)}{y_0^2} = \frac{x_1(1 - x_1 - y_1)}{y_1^2} = \frac{x_2(1 - x_2 - y_2)}{y_2^2}$$
$$= \frac{x_1^2}{x_0 x_2},\ \frac{x_1^4}{x_0^3 x_2} = \frac{y_1^4}{y_0^3 y_2}, \quad x_0^2 = y_0 x_1.$$

As in case (i), we obtain (6), (7) and

$$F_3(x_0, y_0, y_1) = y_1^8 + y_1^4 \frac{x_0^4 y_0}{1 - x_0 - y_0} - x_0^7 (1 - \frac{x_0^2}{1 - x_0 - y_0}) = 0,$$
$$y_2 = \frac{y_1^4 y_0}{x_0^3 (1 - x_0 - y_0)}. \tag{11}$$

We can now proceed analogously to the proof of case (i).

(iii) $\mathbf{k}=\mathbf{1}$, $\mathbf{m}=\mathbf{2}$. We have the equations (a), (b), (c), (d), (f) and (g) or equivalently (g) and

$$\frac{x_0^3(1-x_0-y_0)}{y_0^4}=\frac{x_1^3(1-x_1-y_1)}{y_1^4}=\frac{x_2^3(1-x_2-y_2)}{y_2^4}$$
$$=\frac{x_1^4}{x_0^3x_2},\ \frac{x_1^2}{x_0x_2}=\frac{y_1^2}{y_0y_2},\quad x_0^2=y_0x_1.$$

A simple computation gives (g), (6), (9) and

$$F_4(x_0,y_0,y_1)=y_1^4+y_1\frac{x_0^3y_0}{1-x_0-y_0}-\frac{x_0^3y_0}{1-x_0-y_0}\left(1-\frac{x_0^2}{y_0}\right)=0, \tag{12}$$

$$F_5(x_0,y_0,y_1)=y_1^8+y_1^2\frac{x_0^6y_0}{1-x_0-y_0}-x_0^7\left(1-\frac{x_0^2}{1-x_0-y_0}\right)=0. \tag{13}$$

The equation (12) has a unique positive solution $y_1'=h'(x_0,y_0)$. The same holds for (13) with $y_1''=h''(x_0,y_0)$. We must have $H(x_0,y_0)=h'^2(x_0,y_0)-h''(x_0,y_0)=0$. The rest of the proof runs as in case (i).

(iv) $\mathbf{k}=\mathbf{2}$, $\mathbf{m}=\mathbf{2}$. We have the equations (a), (b), (c), (d), (e) and (g) or equivalent (g) and

$$\frac{x_0^3(1-x_0-y_0)}{y_0^4}=\frac{x_1^3(1-x_1-y_1)}{y_1^4}=\frac{x_2^3(1-x_2-y_2)}{y_2^4}$$
$$=\frac{x_1^4}{x_0^3x_2}=\frac{y_1^4}{y_0^3y_2},\ x_0^2=y_0x_1.$$

As in previous cases we obtain (6), (11), (12) and

$$F_6(x_0,y_0,y_1)=y_1^{16}+y_1^4\frac{x_0^{12}y_0}{1-x_0-y_0}-x_0^{15}\left(1-\frac{x_0^2}{1-x_0-y_0}\right)=0.$$

The rest of the proof is similar to that in case (iii).

(v) $\mathbf{k}=\mathbf{2}$, $\mathbf{m}=\mathbf{0}$. We have the equations (a), (c), (d), (e), (f) and (g) or equivalent (g) and

$$\frac{x_0(1-x_0-y_0)}{y_0^2}=\frac{x_1(1-x_1-y_1)}{y_1^2}=\frac{x_2(1-x_2-y_2)}{y_2^2}=\frac{x_1^4y_0^2}{x_2x_0^5},$$
$$\frac{x_1^2}{x_0x_2}=\frac{y_1^2}{y_0y_2},\quad \frac{x_1^4}{x_0^3x_2}=\frac{y_1^4}{y_0^3y_2}.$$

We see at once that

$$x_0 = x_1 = x_2,\ y_0 = y_1 = y_2,$$
$$y_0^4 - x_0^3(1 - x_0 - y_0) = 0.$$

The proof is immediate.

(**vi**) $\mathbf{k = 1}$, $\mathbf{m = 0}$. We have the equations $(b) - (f)$ and (g). The result is well known, see [7, pp. 77].

Corollary. *Under the hypotheses of Theorem we have*

$$\dim M_{k,m}(q) = \frac{-1}{\log 3}\Big[\log x_0 + q \log \frac{g_1(x_0)}{x_0}\Big].$$

PROOF. By (a)–(f) we get $\log E_{km} = \log \frac{y_1}{x_0} = \log \frac{g_1(x_0)}{x_0}$. Since $\log A_{00} = \log x_0$ we obtain $\delta_P = \frac{-1}{\log 3}\Big[\log x_0 + q \log \frac{g_1(x_0)}{x_0}\Big]$, which completes the proof. $\square$

3 A Multifractal Analysis

The multifractal analysis of a Borel probability measure ν on $[0, 1]$, [6], [8], [9], [12], is the study of the Hausdorff dimension of the sets

$$E_c = \left\{x : \lim_{N\to\infty} \frac{\log \nu(E_N(x))}{-N \log 3} = c\right\}, \quad c \in \mathbb{R}.$$

Proposition. *Let* μ *be a measure as in* (2)*, where* $P = (p_{ij})$ *is such that* $E_{ij} = 1$*, for* $(i, j) \neq (k, m)$*,* $k, m \in \{0, 1, 2\}$*,* $k + m \geq 1$*. Then*

$$\dim E_c = \dim M_{k,m}(q),$$

where $q = \frac{c \log 3 + \log p_{00}}{\log(p_{00}p_{11}^{-1})}$.

PROOF. By assumption and (5) it follows that

$$E_c = \left\{x : \lim_{N\to\infty} \frac{-1}{N \log 3}\sum_{n=1}^{N}[A_{00} + \varepsilon_n^k(x)\varepsilon_{n+1}^m(x) \log E_{km}] = c\right\},$$

$\log E_{km} = \log(p_{00}^{-1}p_{11})$ and $A_{00} = \log p_{00}$. The proof is straightforward. It is clear that c must be such that $0 \leq \frac{c \log 3 + \log p_{00}}{\log(p_{00}p_{11}^{-1})} \leq 2^{k+m}$, otherwise the set E_c is empty. If $\log(p_{00}p_{11}^{-1}) = 0$, then our measure is that of Lebesque as is easy to check.

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