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Michael Bensimhoun, Dov Gruner str. 211/9
Jerusalem, Israel. email: M.Bensimhoun@gmail.com

A COUNTEREXAMPLE FOR THE CHANGE OF VARIABLE FORMULA IN KH INTEGRALS

Abstract

It was shown in [1] and [4] that if $F(x)$ and $\Psi(x)$ are Riemann integrals of the form $\int_a^x f\mathrm{d}x$ and $\int_b^x \psi\mathrm{d}x$ resp., then $\psi \cdot f \circ \Psi$, if defined, is Riemann integrable. Furthermore, the change of variable formula applies, giving $\int_b^x \psi \cdot f \circ \Psi \,\mathrm{d}x = F(\psi(x)) - F(\psi(b))$. It is natural to try to generalize this theorem to the Kurzweil-Henstock integral (this question was also dealt with in a paper by the author [2]); in other words, assuming that F and Ψ are KH integrals of f and ψ resp., one would expect that $\psi \cdot f \circ \Psi$ be KH integrable. We show in this paper that this is false, and produce a counterexample based on the middle-third Cantor set and some rudiments of fractal geometry. In other words, by a well known theorem, we prove that the composition of two ACG functions needs not be ACG (in fact, we prove more generally that the composition of two absolutely continuous functions needs not be ACG). Of course, examples that show that the composition of two absolutely continuous functions needs not be absolutely continuous exist in the context of the Lebesgue integral, but since KH integrals need not be absolutely continuous, one cannot infer from these examples the validity of the above claim in the context of KH integration. On the other hand, the subtle method developed in this paper seems to be new, is entirely constructive, and we believe it could be applied to other interesting constructions.

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1 Hausdorff dimension - short review

Let S be a set contained in $\mathbb{R}^n$, and $d \in [0, +\infty]$. For every $r > 0$, define

$$H_r^d(S) = \inf\Big\{\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^d \colon \bigcup_{i=1}^{\infty} U_i \supset S,\, \operatorname{diam}(U_i) < r\Big\}$$

(the infimum is taken over all countable covers of S by open sets U_i satisfying $\operatorname{diam}(U_i) < r$). Notice that $H_r^d(S)$ is monotone decreasing in r, since the larger r is, the more collections of sets are permitted. Thus, the limit $\lim_{r\to 0} H_r^d(S)$ exists in $\mathbb{R}^+ \cup \{+\infty\}$. The *d-dimensional Hausdorff* measure of S is defined by

$$H^d(S) = \sup_{r>0} H_r^d(S) = \lim_{r\to 0} H_r^d(S).$$

For example, if $d = \ln 2/\ln 3$, it is not difficult to show that the d-dimensional Hausdorff measure of the middle-third Cantor set is 1.

The important point is that, if for some $0 < d < +\infty$, S has a strictly positive finite d-dimensional Hausdorff measure, then for every $d' < d$, the d'-dimensional Hausdorff measure of S is $+\infty$, and for every $d' > d$, it is equal to 0. Indeed, if $(U_i)_i$ is an open cover by sets satisfying $\operatorname{diam}(U_i) < r$ and if $d' > d$, then

$$\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^{d'} = \sum_{i=1}^{\infty} \operatorname{diam}(U_i)^{d'-d} \operatorname{diam}(U_i)^d \le r^{d'-d} \sum_{i=1}^{\infty} \operatorname{diam}(U_i)^d,$$

hence

$$\operatorname*{Inf}_{(U_i)_i} \sum_{i=1}^{\infty} \operatorname{diam}(U_i)^{d'} \le r^{d'-d} H_r^d(S) \xrightarrow{r\to 0} 0.$$

This shows the second assertion, and the first one follows immediately.

This unique finite positive number d that realizes the jump of dimensional measures, if exists, is called the *Hausdorff dimension* of S, and is a basic tool of fractal geometry.

2 Construction of the example

We use the formalism of differential elements and variational equivalence, set in [2]. In particular, an expression like $dh_1 \approx dh_2$ means that the differentials dh_1 is variationally equivalent to dh_2 (hence, either dh_1 and dh_2 are both KH integrable and their integrals coincide, or are they both non-integrable).

Theorem 2.1. *There exist absolutely integrable functions f and ψ, $[0,1] \to [0,1]$, such that, if $\Psi\colon [0,1] \to [0,1]$ is defined by $\Psi(x) = \int_0^x \psi(t)\,\mathrm{d}t$, then the differential element $f \circ \Psi\, d\Psi$ ($\approx f \circ \Psi\, \psi\, \mathrm{d}x$) is not KH-integrable in $[0,1]$. In addition, f is differentiable in $]0,1]$, Ψ is 1-Lipschitz, and ψ is bounded and differentiable a.e.*

Proof. Put $F(x) = x^d$, with $0 < d < \ln(2)/\ln(3)$, and

$$f(x) = \begin{cases} F'(x) = dx^{d-1}, & x \in]0,1], \\ 0, & x = 0. \end{cases}$$

It is clear that f is integrable, differentiable in $]0,1]$, with $F(x) = \int_0^x f(t)\mathrm{d}t$. In order to define the function ψ, we need the middle-third classical Cantor set. Let us define

$$I_1 =]1/3, 2/3[,\ I_{1,1} =]1/9, 2/9[,\ I_{1,2} =]7/9, 8/9[,$$

and in general, we define $I_{n_1,n_2,\dots,n_k}$ ($n_i \in \{1,2\}$, $n_1 = 1$) inductively: Having defined

$$I_{n_1,n_2,\dots,n_k} = \left] \frac{\alpha}{3^k}, \frac{\alpha+1}{3^k} \right[,$$

we then define

$$I_{n_1,n_2,\dots,n_k,1} = \left] \frac{3\alpha-2}{3^{k+1}}, \frac{3\alpha-1}{3^{k+1}} \right[\quad \text{and} \quad I_{n_1,n_2,\dots,n_k,2} = \left] \frac{3\alpha+4}{3^{k+1}}, \frac{3\alpha+5}{3^{k+1}} \right[.$$

Thus, the middle-third Cantor set is

$$\mathcal{C} = [0,1] \setminus \bigcup_{k\in\mathbb{N}} \bigcup I_{n_1,n_2,\dots,n_k}.$$

To simplify, we denote by Ω the set of generic dyadic indices of the form $n_1, n_2, \dots, n_k$ like above (that is, $n_i \in \{1,2\}$, $n_1 = 1$), by N an element of Ω, and denote $|N| = k$ ($|N|$ can be seen as the hierarchical level of the interval I_N). Furthermore, for every dyadic index N, we call a_N and b_N the bounds of I_N ($I_N =]a_N, b_N[$), and we let

$$m_N = \frac{a_N + b_N}{2}.$$

Notice that a_N and $b_N = a_N + 1/3^{|N|}$ belong to $\mathcal{C}$ for every N.

Define $\Psi\colon [0,1] \to [0,1]$ by

$$\Psi(x) = \begin{cases} 0, & x \in \mathcal{C}, \\ 1/3^{|N|}, & x = m_N\ (N \in \Omega), \\ \text{linear}, & x \in [a_N, m_N] \text{ and } x \in [m_N, b_N]\ (N \in \Omega). \end{cases}$$

Pictorially, inside each I_N, Ψ forms a peak at the middle of I_N, of height $1/3^{|N|}$. Observe that $|\Psi(y) - \Psi(x)| \leq |y - x|$ for every $x, y \in [0,1]$ (that is, Ψ is 1-Lipschitz). In particular, Ψ is continuous in $[0,1]$. Furthermore, it is clearly differentiable a.e. since the Cantor set $\mathcal{C}$ is negligible. Let ψ be the derivative of Ψ defined a.e., and extend it in any manner on the points where it is not defined. Since Ψ is Lipschitz, it is easy to see that Ψ is variationally equivalent to 0 on every negligible set of $[0,1]$, and hence on $\mathcal{C}$. It follows from [3], thm. 5.9, that Ψ is the integral of ψ in $[0,1]$, that is, $\Psi(x) = \int_0^x \psi(t)\,\mathrm{d}t$. Furthermore, it is clear that $\psi(x) = 1$ on the left half of each I_k, and $\psi(x) = -1$ on its right half, so ψ is bounded and differentiable a.e., and is absolutely integrable.

With these settings, we now show that $f \circ \Psi\, d\Psi$ is not integrable in $[0,1]$. Assume for a contradiction that $f \circ \Psi\, d\Psi$ is integrable in $[0,1]$. According to [5], it must hold that

$$\int_0^x f \circ \Psi \,\mathrm{d}\Psi = F \circ \Psi(x).$$

Hence, by [3], thm. 5.9 again, $d(F \circ \Psi) \approx 0$ on every negligible set, and in particular on $\mathcal{C}$; in other words, for every $\varepsilon > 0$, there exists $\delta\colon [0,1] \to \mathbb{R}_+$ such that, for every partial division D subordinated to δ and anchoring in $\mathcal{C}$,

$$\sum_{([u,v],\xi)\in D} |F(\Psi(v)) - F(\Psi(u))| = \sum_{([u,v],\xi)\in D} |\Psi(v)^d - \Psi(u)^d| < \varepsilon. \tag{1}$$

We prove that this is false. Fix $\varepsilon > 0$, and let δ_ε be as in (1).

For every dyadic index N and $k \in \mathbb{N}^*$, let us define

$$J_k = \Big\{ \Big[a_N - \frac{1}{3^k}\,,\, m_N\Big] \colon N \in \Omega \text{ and } |N| = k \Big\},$$
$$K_k = \Big\{ \Big[m_N\,,\, b_N + \frac{1}{3^k}\Big] \colon N \in \Omega \text{ and } |N| = k \Big\},$$

$$L_k = J_k \cup K_k, \quad \text{and} \quad L = \bigcup_{k\in\mathbb{N}} L_k.$$

Notice that for every $k \in \mathbb{N}^*$,

$$\mathcal{C} \subset \bigcup_{M\in L_k} M.$$

We also point out that for every $M \in L_k$,

$$\operatorname{diam}(M) = \frac{1}{3^k} + \frac{b_N - a_N}{2} = \frac{3}{2} \times \frac{1}{3^k}.$$

Furthermore, the numbers $a_N - 1/3^{|N|}$ and $b_N + 1/3^{|N|}$ belong to $\mathcal{C}$ for every index N. Hence, for every $M \in L_k$, $M = [u, v]$,

$$|\Psi(v)^d - \Psi(u)^d| = \Psi(m_N)^d = \left(\frac{1}{3^k}\right)^d = |\Psi(v) - \Psi(u)|^d$$

(since one of the two numbers $\Psi(u)$ or $\Psi(v)$ is 0). In conclusion,

$$|\Psi(v)^d - \Psi(u)^d| = \left(\frac{2}{3}\right)^d \operatorname{diam}(M)^d = \left(\frac{2}{3}\right)^d |v - u|^d. \tag{2}$$

Finally, given $M \in L_k$ and $M' \in L_{k'}$, with $k > k'$, either $M \subseteq M'$, or $M \cap M' = \emptyset$. (3)

Claim: *For every $\delta\colon [0,1] \to \mathbb{R}_+$, there exists a countable cover D of $\mathcal{C}$ by non-overlapping tagged intervals $\hat{I}$ of the form (M, ξ), where $M \in L$ and $\hat{I}$ is subordinated to δ and anchors in $\mathcal{C}$.*

Proof. Construct D inductively in the following way: put $D_0 = \emptyset$, and for all $k \geq 1$, let T_k be the set of elements $M \in L_k$ such that there exists $\xi_M \in M \cap \mathcal{C}$ that fulfills $\operatorname{diam}(M) \leq \delta(\xi)$. Put

$$D_k = D_{k-1} \cup \{(M, \xi_M)\colon M \in T_k \text{ and } M \cap J = \emptyset,\ \forall J \in D_{k-1}\}.$$

Define D by

$$D = \bigcup_{k \in \mathbb{N}} D_k.$$

The intervals of T_k are pairwise disjoint for every $k \in \mathbb{N}$, hence so are the elements of D, as is easily seen. Furthermore, every point c of $\mathcal{C}$ must ultimatively fall inside some tagged interval of D. Indeed given $c \in C$, choose the first k for which $(3/2)(1/3)^k \leq \delta(c)$, and choose $M \in L_k$ with $c \in M$. Then either $(M, c) \in D_k$, or $M \subseteq M'$ for some $(M', \xi) \in D_{k'}$ with $k' \leq k$ by (3). Hence, D fulfills the conditions of the claim. □

Now comes the fractal geometry argument: Choose some number $K > (\frac{3}{2})^d \varepsilon$. It is well known that the Hausdorff dimension of the middle-third Cantor set is $\ln 2/\ln 3$. Therefore, since $0 < d < \ln 2/\ln 3$ by hypothesis, there exists $r > 0$ such that

$$\operatorname*{Inf}_S \sum_{[u,v] \in S} |v - u|^d > K, \tag{4}$$

where the infimum is taken over all the countable covers S of $\mathcal{C}$ by intervals $[u, v]$ of length less than r. For all $x \in [0, 1]$, let $\delta(x) = \min(\delta_\varepsilon(x), r)$. By the claim above, there exists a countable cover D of $\mathcal{C}$ by non-overlapping tagged intervals of the form (M, ξ), where $M \in L$, $\xi \in \mathcal{C} \cap M$, and $\operatorname{diam}(M) < \delta(\xi)$. In particular, $\operatorname{diam}(M) < r$, therefore (4) implies

$$\sum_{([u,v],\xi)\in D} |v-u|^d > K.$$

But by (2),

$$|\Psi(v)^d - \Psi(u)^d| = \left(\frac{2}{3}\right)^d |v-u|^d,$$

for every $([u,v],\xi) \in D$. Hence

$$\sum_{([u,v],\xi)\in D} |\Psi(v)^d - \Psi(u)^d| > \left(\frac{2}{3}\right)^d K > \varepsilon. \tag{5}$$

Since the inequality is strict, we see that (5) holds in fact for some finite collection of elements of D subordinated to δ, and hence to δ_ε. Thus, we can form a partial division of $[0,1]$ anchoring in $\mathcal{C}$ and subordinated to δ_ε, such that (5) holds, in contradiction to (1). □

Corollary 2.2. *The composition of two absolutely continuous functions needs not be ACG (and* a fortiori *the composition of two ACG functions needs not be ACG).*

PROOF. This follows immediately from the theorem above, from the well known fact that a function is a KH-primitive if and only if it is ACG, and from [3], thm. 5.9 (since the function $F \circ \Psi$ differentiates to $\psi \cdot f \circ \Psi$ a.e.). □

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