RESEARCH

Real Analysis Exchange
Vol. 36(2), 2010/2011, pp. 307–324

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MAXIMAL CLASSES FOR THE FAMILY OF $[\lambda, \varrho]$-CONTINUOUS FUNCTIONS

Abstract

In this paper we give the definition of $[\lambda, \varrho]$-continuity of real-valued functions defined on an open interval, which is an example of path continuity. We give some properties of $[\lambda, \varrho]$-continuous functions. The aim of the paper is to find the maximal additive class and the maximal multiplicative class for the family of $[\lambda, \varrho]$-continuous functions.

1 Preliminaries

First, we shall collect some of the notions and definitions which appear frequently in the sequel. We apply standard symbols and notations. By $\mathbb{R}$ we denote the set of real numbers, by $\mathbb{N}$ we denote the set of positive integers. The symbol $|\cdot|$ stands for the Lebesgue measure on $\mathbb{R}$. Let f be a real-valued function defined on a open interval $I = (a, b)$. We will denote by $D_{ap}(f)$, $\big(D^+_{ap}(f),\ D^-_{ap}(f)\big)$ the set of all point at which function f is not approximately continuous (at which f is not approximately continuous from the right or the left, respectively).

Let E be a measurable subset of $\mathbb{R}$ and let $x \in \mathbb{R}$. The numbers

$$\underline{d}^+(E, x) = \liminf_{t\to 0^+} \frac{|E \cap [x, x+t]|}{t} \quad \text{and} \quad \overline{d}^+(E, x) = \limsup_{t\to 0^+} \frac{|E \cap [x, x+t]|}{t}$$

Mathematical Reviews subject classification: Primary: 26A15 ; Secondary: 54C30
Key words: density of a set at a point, continuous functions, approximately continuous functions, path continuity, maximal additive class, maximal multiplicative class
Received by the editors April 20, 2010
Communicated by: Brian S. Thomson

are called the right lower density of E at x and right upper density of E at x, respectively. The left lower and upper densities of E at x are defined analogously. If

$$\underline{d}^{+}(E,x)=\overline{d}^{+}(E,x)\qquad\big(\underline{d}^{-}(E,x)=\overline{d}^{-}(E,x)\big),$$

then we call this number the right density (left density) of E at x and denote it by $d^{+}(E,x)$ ($d^{-}(E,x)$). The numbers

$$\overline{d}(E,x)=\limsup_{\substack{t\to0^{+}\\k\to0^{+}\\t+k\neq0}}\frac{|E\cap[x-t,x+k]|}{k+t}\quad\text{and}\quad\underline{d}(E,x)=\liminf_{\substack{t\to0^{+}\\k\to0^{+}\\t+k\neq0}}\frac{|E\cap[x-t,x+k]|}{k+t}$$

are called the upper and lower density of E at x, respectively. If $\overline{d}(E,x)=\underline{d}(E,x)$, we call this number the density of E at x and denote it by $d(E,x)$.

Let us observe that

$$\overline{d}(E,x)=\max\{\overline{d}^{+}(E,x),\overline{d}^{-}(E,x)\}\quad\text{and}\quad\underline{d}(E,x)=\min\{\underline{d}^{+}(E,x),\underline{d}^{-}(E,x)\}.$$

Moreover, it is clear that

$$\overline{d}^{+}(E,x)=1-\underline{d}^{+}(\mathbb{R}\setminus E,x)\quad\text{and}\quad\underline{d}^{+}(E,x)=1-\overline{d}^{+}(\mathbb{R}\setminus E,x).$$

Similarly,

$$\overline{d}^{-}(E,x)=1-\underline{d}^{-}(\mathbb{R}\setminus E,x)\quad\text{and}\quad\underline{d}^{-}(E,x)=1-\overline{d}^{-}(\mathbb{R}\setminus E,x).$$

A.M.Bruckner, R.J. O'Malley and B.S.Thomson in [1] investigated the notion of path system and developed a framework within which a number of generalized derivatives can be expressed. We use this idea for studying some notion of generalized continuity.

Definition 1.1. *[3] Let E be a measurable subset of $\mathbb{R}$ and $0<\lambda\le\varrho<1$. We say that a point $x\in\mathbb{R}$ is a point of $[\lambda,\varrho]$-density of E if $\underline{d}(E,x)>\lambda$ and $\overline{d}(E,x)>\varrho$.*

Definition 1.2. *[3] Let $0<\lambda\le\varrho<1$. A real-valued function f defined on an open interval I is called $[\lambda,\varrho]$-continuous at $x\in I$, provided that there is a measurable set $E\subset I$ such that x is a point of $[\lambda,\varrho]$-density of E, $x\in E$ and $f_{|E}$ is continuous at x. If f is $[\lambda,\varrho]$-continuous at each point of I, we say that f is $[\lambda,\varrho]$-continuous.*

We will denote the class of all $[\lambda,\varrho]$-continuous functions by $\mathcal{C}_{[\lambda,\varrho]}$.

Definition 1.3. *[1] A real-valued function f defined on an open interval I is called approximately continuous at $x \in I$ provided that there is a measurable set $E \subset I$ such that $\underline{d}(E, x) = 1$, $x \in E$ and $f_{|E}$ is continuous at x. If f is approximately continuous at each point of I we say that f is approximately continuous.*

By $\mathcal{A}$ we denote the class of all real-valued approximately continuous functions defined on an open interval I.

Corollary 1.1. $\mathcal{A} \subset \mathcal{C}_{[\lambda,\varrho]}$ *for each* $0 < \lambda \leq \varrho < 1$.

2 Auxiliary lemmas

First we recall some standard properties of the density of a set at a point.

Lemma 2.1. *Let E and F be any measurable subsets of $\mathbb{R}$ and $x \in \mathbb{R}$. Then*

1. $\underline{d}^+(E, x) + \underline{d}^+(F, x) \leq \underline{d}^+(E \cap F, x) + 1$.
2. $\underline{d}^+(E, x) + \overline{d}^+(F, x) \leq \overline{d}^+(E \cap F, x) + 1$.
3. $\underline{d}^+(E \cup F, x) \leq \underline{d}^+(E, x) + \overline{d}^+(F, x)$.
4. *If* $F \subset E$ *and* $\underline{d}^+(E, x) = \overline{d}^+(E, x)$, *then*
$$\underline{d}^+(E \backslash F, x) = d^+(E, x) - \overline{d}^+(F, x) \quad \textit{and} \quad \overline{d}^+(E \backslash F, x) = d^+(E, x) - \underline{d}^+(F, x).$$
5. *If* $\overline{d}^+(E, x) = 0$, *then* $\overline{d}^+(E \cup F, x) = \overline{d}^+(F, x) = \overline{d}^+(F \setminus E, x)$ *and* $\underline{d}^+(E \cup F, x) = \underline{d}^+(F, x) = \underline{d}^+(F \setminus E, x)$.
6. *If* $\overline{d}^+(E \setminus F, x) = \overline{d}^+(F \setminus E, x) = 0$, *then* $\underline{d}^+(E \cap F, x) = \underline{d}^+(E, x) = \underline{d}^+(F, x)$ *and* $\overline{d}^+(E \cap F, x) = \overline{d}^+(E, x) = \overline{d}^+(F, x)$.

Proof. We prove only the first inequality. The rest of the proofs are similar.

Given measurable sets $A, B \subset \mathbb{R}$ the equality $|A \cup B| = |A| + |B| - |A \cap B|$ is true. Therefore
$$\big|[x, x+t]\big| \geq \big|(E \cup F) \cap [x, x+t]\big| = \big|E \cap [x, x+t]\big| + \big|F \cap [x, x+t]\big| - \big|E \cap F \cap [x, x+t]\big|.$$
Hence
$$1 \geq \frac{\big|(E \cup F) \cap [x, x+t]\big|}{t} = \frac{\big|E \cap [x, x+t]\big|}{t} + \frac{\big|F \cap [x, x+t]\big|}{t} - \frac{\big|E \cap F \cap [x, x+t]\big|}{t}$$

for each $t > 0$. It implies that

$$\underline{d}^+(E \cap F, x) + 1 = \liminf_{t\to 0^+}\left(1 + \frac{\left|E \cap F \cap [x, x+t]\right|}{t}\right) \geq$$
$$\geq \liminf_{t\to 0^+}\left(\frac{\left|E \cap [x, x+t]\right|}{t} + \frac{\left|F \cap [x, x+t]\right|}{t}\right) \geq \liminf_{t\to 0^+}\frac{\left|E \cap [x, x+t]\right|}{t} +$$
$$+ \liminf_{t\to 0^+}\frac{\left|F \cap [x, x+t]\right|}{t} = \underline{d}^+(E, x) + \underline{d}^+(F, x).$$

□

Certainly, similar lemma holds for the left densities.
Afterwards, we will need same auxiliary lemmas.

Lemma 2.2. *Let $x \in \mathbb{R}, 0 < a < 1$ and let E be a measurable set. For each $k \in \mathbb{N}$ such that $\frac{1}{k} < a$ there is a sequence of intervals $\{I_n = [a_n, b_n] \colon n \geq 1\}$ such that $x < \ldots < b_{n+1} < a_n < \ldots$, $d^+\left(\bigcup_{n=1}^{\infty} I_n, x\right) = a$ and $\overline{d}^+\left(E \cap \bigcup_{n=1}^{\infty} I_n, x\right) \geq \frac{1}{k}\overline{d}^+(E, x)$.*

PROOF. Observe, that if

$$\overline{d}^+\left(E \cap \bigcup_{n=1}^{\infty} I_n, x\right) \geq \frac{1}{k}\overline{d}^+(E, x)$$

for some k, then for every $k_1 \geq k$ we get $\overline{d}^+\left(E \cap \bigcup_{n=1}^{\infty} I_n, x\right) \geq \frac{1}{k_1}\overline{d}^+(E, x)$, too. Therefore we may assume that k is the smallest natural number for which $\frac{1}{k} < a$. Then $a < \frac{2}{k}$.

Let $c_n = x + \frac{1}{n}$ for $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} \frac{\left|[c_{n+1}, c_n]\right|}{\left|[x, c_{n+1}]\right|} = \lim_{n\to\infty} \frac{\frac{1}{n(n+1)}}{\frac{1}{n+1}} = 0$. Let

$$U_n^i = \left[c_{n+1} + \tfrac{i-1}{k}(c_n - c_{n+1}), c_{n+1} + (a + \tfrac{i-1}{k})(c_n - c_{n+1})\right]$$

for $i = 1, \ldots, k-1$ and $U_n^k = \left[c_n - a(c_n - c_{n+1}), c_n\right]$.

It is obvious that

$$|U_n^1| = |U_n^2| = \ldots = |U_n^k| = a \cdot |[c_{n+1}, c_n]|$$

and

$$[c_{n+1}, c_n] = \bigcup_{i=1}^{k} U_n^i.$$

Hence

$$|E\cap U_n^1|+|E\cap U_n^2|+\ldots+|E\cap U_n^k|\geq |E\cap[c_{n+1},c_n]|.$$

Therefore for each $n\geq 1$ there exists a closed interval $J_n\subset[c_{n+1},c_n]$ such that

$$|J_n|=a\cdot|[c_{n+1},c_n]| \text{ and } |J_n\cap E|\geq\frac{1}{k}|E\cap[c_{n+1},c_n]|.$$

First, we shall show that $d^+\left(\bigcup\limits_{n=1}^{\infty}J_n,x\right)=a$.

Let $z\in(x,c_1)$. There is $n\geq 1$ such that $z\in[c_{n+1},c_n]$. Then

$$\Big|\bigcup_{i=1}^{\infty}J_i\cap[x,z]\Big|=\Big|\bigcup_{i=1}^{\infty}J_i\cap[x,c_{n+1}]\Big|+\Big|\bigcup_{i=1}^{\infty}J_i\cap[c_{n+1},z]\Big|=$$
$$=\Big|\bigcup_{i=n+1}^{\infty}J_i\Big|+\big|J_n\cap[c_{n+1},z]\big|\leq a\cdot|[x,c_{n+1}]|+|[c_{n+1},c_n]|$$

and

$$\frac{\big|\bigcup\limits_{i=1}^{\infty}J_i\cap[x,z]\big|}{z-x}\leq\frac{\big|\bigcup\limits_{i=1}^{\infty}J_i\cap[x,z]\big|}{c_{n+1}-x}\leq a+\frac{|[c_{n+1},c_n]|}{|[x,c_{n+1}]|}.$$

On the other hand,

$$\Big|\bigcup_{i=1}^{\infty}J_i\cap[x,z]\Big|\geq\Big|\bigcup_{i=1}^{\infty}J_i\cap[x,c_{n+1}]\Big|\geq a\cdot|[x,c_{n+1}]|=a|[x,z]|-|[c_{n+1},c_n]|$$

and

$$\frac{\big|\bigcup\limits_{i=1}^{\infty}J_i\cap[x,z]\big|}{z-x}\geq\frac{a\cdot|[x,z]|-|[c_{n+1},c_n]|}{z-x}\geq a-\frac{|[c_{n+1},c_n]|}{|[x,c_{n+1}]|}.$$

Suppose that $\lim\limits_{m\to\infty}z_m=x$ and $z_m\in[c_{n_m+1},c_{n_m}]$ for $m\geq 1$. Then $\lim\limits_{m\to\infty}n_m=\infty$. Since $\lim\limits_{m\to\infty}\frac{|[c_{n_m+1},c_{n_m}]|}{|[x,c_{n_m+1}]|}=0$, we obtain that $\lim\limits_{m\to\infty}\frac{\left|\bigcup\limits_{n=1}^{\infty}J_n\cap[x,z_m]\right|}{z-x}=a$, and it follows that $d^+\left(\bigcup\limits_{n=1}^{\infty}J_n,x\right)=a$.

At the end, we will prove that $\overline{d}^+\left(E\cap\bigcup\limits_{n=1}^{\infty}J_n,x\right)\geq\frac{1}{k}\overline{d}^+(E,x)$. Again, let

$z \in (x, c_1)$ and $z \in [c_{n+1}, c_n]$. Then

$$\frac{\left|\bigcup_{i=1}^{\infty} J_i \cap E \cap [x,z]\right|}{z-x} \geq \frac{1}{k} \cdot \frac{\sum_{i=n+1}^{\infty}\left|[c_{i+1}, c_i] \cap E\right|}{z-x} = \frac{1}{k} \cdot \frac{\left|[x, c_{n+1}] \cap E\right|}{z-x} \geq$$
$$\geq \frac{1}{k}\cdot\frac{\left|[x,z]\cap E\right|}{z-x} - \frac{1}{k}\cdot\frac{c_n - c_{n+1}}{z-x} \geq \frac{1}{k}\cdot\frac{\left|[x,z]\cap E\right|}{z-x} - \frac{\frac{1}{n}-\frac{1}{n+1}}{k\cdot\frac{1}{n}} = \frac{1}{k}\cdot\frac{\left|[x,z]\cap E\right|}{z-x} - \frac{1}{k(n+1)}.$$

There is a sequence $(y_m)_{m=1}^{\infty}$ converging to x from right such that $\lim_{m\to\infty} \frac{\left|E\cap[x,y_m]\right|}{y_m - x} = \overline{d}^+(E,x)$. For each m there is n_k such that $y_m \in [c_{n_m+1}, c_{n_m}]$. Certainly, $\lim_{m\to\infty} n_m = \infty$. Hence

$$\lim_{m\to\infty} \frac{\left|\bigcup_{n=1}^{\infty} J_n \cap E \cap [x, y_m]\right|}{y_m - x} \geq \lim_{m\to\infty}\left(\frac{1}{k}\cdot\frac{\left|[x,y_m]\cap E\right|}{y_m - x} - \frac{1}{k(n_m+1)}\right) = \frac{1}{k}\overline{d}^+(E,x).$$

Therefore $\overline{d}^+\left(E \cap \bigcup_{n=1}^{\infty} J_n, x\right) \geq \frac{1}{k}\overline{d}^+(E,x)$.

We have proved that $d^+\left(\bigcup_{n=1}^{\infty} J_n, x\right) = a$ and $\overline{d}^+\left(E\cap\bigcup_{n=1}^{\infty} J_n, x\right) \geq \frac{1}{k}\overline{d}^+(E,x)$, but the elements of the sequence do not have to be disjoint.

Let $\{I_n \colon n \geq 1\}$ be a sequence of closed disjoint intervals such that $I_n \subset \operatorname{int} J_n$ for all $n \in \mathbb{N}$ and $\overline{d}^+\left(\bigcup_{n=1}^{\infty}(J_n \setminus I_n), x\right) = 0$. By Lemma 2.1, property 5, it is immediate that

$$\underline{d}^+\left(\bigcup_{n=1}^{\infty} I_n, x\right) = \underline{d}^+\left(\bigcup_{n=1}^{\infty} J_n \setminus \left(\bigcup_{n=1}^{\infty}(J_n\setminus I_n)\right), x\right) = \underline{d}^+\left(\bigcup_{n=1}^{\infty} J_n, x\right) = a$$

and

$$\overline{d}^+\left(\bigcup_{n=1}^{\infty} I_n, x\right) = \overline{d}^+\left(\bigcup_{n=1}^{\infty} J_n \setminus \left(\bigcup_{n=1}^{\infty}(J_n\setminus I_n)\right), x\right) = \overline{d}^+\left(\bigcup_{n=1}^{\infty} J_n, x\right) = a.$$

Hence, $d^+\left(\bigcup_{n=1}^{\infty} I_n, x\right) = a$.

Furthermore, $\overline{d}^+\left(E\cap\bigcup_{n=1}^{\infty} I_n, x\right) = \overline{d}^+\left(E\cap\bigcup_{n=1}^{\infty} J_n, x\right) \geq \frac{1}{k}\overline{d}^+(E,x)$. We thus get a required sequence of closed disjoint intervals $\{I_n \colon n \geq 1\}$ which completes the proof of the lemma. □

Lemma 2.3. *Let F be a measurable set and let $x \in \mathbb{R}$. There is a sequences of intervals $\{I_n = [a_n, b_n] \colon x < \ldots < b_{n+1} < a_n < \ldots, n \geq 1\}$ such that*

$$\overline{d}^+\Big(F \setminus \bigcup_{n=1}^{\infty} I_n, x\Big) = \overline{d}^+\Big(\bigcup_{n=1}^{\infty} I_n \setminus F, x\Big) = 0.$$

Proof. Let $x_m = x + \frac{1}{2^m}$ and $F_m = F \cap (x_{m+1}, x_m)$. For each $m \in \mathbb{N}$ there exists a closed set $\widetilde{F}_m$ such that $\widetilde{F}_m \subset F_m$ and $|F_m \setminus \widetilde{F}_m| < \frac{1}{4^m}$. Let $\{U_m^i\}_{i=1}^{\infty}$ be the set of all connected components of the set $(x_{m+1}, x_m) \setminus \widetilde{F}_m$. For every m there exists i_m such that $\left|\bigcup_{i=i_m+1}^{\infty} U_m^i\right| \leq \frac{1}{4^m}$. Therefore, the set $[x_{m+1}, x_m] \setminus \bigcup_{i=1}^{i_m-1} U_m^i$ is a union of a finite number of closed intervals $F_m^1, F_m^2, \ldots, F_m^{i_m}$ such that $\widetilde{F}_m \subset \bigcup_{i=1}^{i_m} F_m^i$ and $\left|\bigcup_{i=1}^{i_m} F_m^i \setminus \widetilde{F}_m\right| \leq \frac{1}{4^m}$. As required sequence $\{I_n \colon n \geq 1\}$ we take the family of all intervals $\{F_m^i \colon 1 \leq i \leq i_m,\ m \geq 1\}$ enumerated according to their natural order in $\mathbb{R}$ from the right to the left. We have

$$\left|\bigcup_{i=1}^{i_m} F_m^i \setminus F_m\right| \leq \left|\bigcup_{i=1}^{i_m} F_m^i \setminus \widetilde{F}_m\right| < \frac{1}{4^m}.$$

On the other hand,

$$\left|F_m \setminus \bigcup_{i=1}^{i_m} F_m^i\right| \leq |F_m \setminus \widetilde{F}_m| + \left|\widetilde{F}_m \setminus \bigcup_{i=1}^{i_m} F_m^i\right| = |F_m \setminus \widetilde{F}_m| < \frac{1}{4^m}.$$

Fix any $y \in [x, x_1]$. There is $m_0 \in \mathbb{N}$ such that $y \in \big[x_{m_0+1}, x_{m_0}\big]$. Then

$$\frac{\big|(F \setminus \bigcup_{n=1}^{\infty} I_n) \cap [x, y]\big|}{y - x} \leq \frac{\big|\bigcup_{m=m_0}^{\infty} (F \setminus \bigcup_{n=1}^{n_m} F_m^i) \cap [x_{m+1}, x_m]\big|}{y - x} \leq \frac{\sum\limits_{m=m_0}^{\infty} \frac{1}{4^m}}{x_{m_0+1} - x} =$$
$$= \frac{\frac{1}{4^{m_0}}}{\frac{1}{2^{m_0+1}}(1 - \frac{1}{4})} = \frac{2^{m_0+1}}{3 \cdot 4^{m_0-1}}.$$

Hence $\overline{d}\big(F \setminus \bigcup\limits_{n=1}^{\infty} I_n, x\big) = 0.$

Besides,

$$\frac{\left|\left(\bigcup_{n=1}^{\infty} I_n \setminus F\right) \cap [x,y]\right|}{y-x} \le \frac{\left|\bigcup_{m=1}^{\infty}\left(\bigcup_{i=1}^{i_m} F_m^i \setminus \widetilde{F}_m\right) \cap [x_{m+1}, x_m]\right|}{y-x} \le \frac{\sum\limits_{m=m_0}^{\infty} \frac{1}{4^m}}{x_{m_0+1}-x} =$$
$$= \frac{\frac{1}{4^{m_0}}}{\frac{1}{2^{m_0+1}}(1-\frac{1}{4})} = \frac{2^{m_0+1}}{3 \cdot 4^{m_0-1}}.$$

Hence $\overline{d}\left(\bigcup\limits_{n=1}^{\infty} I_n \setminus F, x\right) = 0$ and the proof is completed. □

At the end, we present the equivalent condition for a function to belong to $\mathcal{C}_{[\lambda,\varrho]}$.

Theorem 2.1. *[3, Theorem 2.1] Let $0 < \lambda \le \varrho < 1$, and let $f\colon I \to \mathbb{R}$ be a measurable function. Then f is $[\lambda,\varrho]$-continuous at x if and only if*

$$\lim_{\varepsilon\to 0^+} \underline{d}\left(\{y \in I \colon |f(x)-f(y)| < \varepsilon\}, x\right) > \lambda$$

and

$$\lim_{\varepsilon\to 0^+} \overline{d}\left(\{y \in I \colon |f(x)-f(y)| < \varepsilon\}, x\right) > \varrho.$$

Corollary 2.1. $\bigcap\limits_{0<\lambda\le\varrho<1} \mathcal{C}_{[\lambda,\varrho]} = \mathcal{A}$.

3 The maximal additive class

Definition 3.1. *Let $\mathcal{F}$ be a family of real functions defined on an open interval I. A set $\mathcal{M}_a(\mathcal{F}) = \{g\colon I \to \mathbb{R}\colon \forall_{f\in\mathcal{F}}\ f+g \in \mathcal{F}\}$ is called the maximal additive class for $\mathcal{F}$.*

Remark 3.1. *Let $f\colon I \to \mathbb{R}$, $f(x) = 0$ for $x \in I$ be a constant function. Clearly, if $f \in \mathcal{F}$ then $\mathcal{M}_a(\mathcal{F}) \subset \mathcal{F}$.*

In [1] maximal additive classes and maximal multiplicative classes for Darboux functions and for Darboux Baire 1 functions are described.
In this section we characterize the maximal additive class for $\mathcal{C}_{[\lambda,\varrho]}$.

Theorem 3.1. *Let $0 < \lambda \le \varrho < 1$ and $I = (a,b)$. If $g\colon I \to \mathbb{R}$, $g \in \mathcal{C}_{[\lambda,\varrho]} \setminus \mathcal{A}$ then there exists a function $f \in \mathcal{C}_{[\lambda,\varrho]}$ such that $f + g \notin \mathcal{C}_{[\lambda,\varrho]}$.*

Proof. Let $g \in \mathcal{C}_{[\lambda,\varrho]} \setminus \mathcal{A}$ and $x \in D_{ap}(f)$. Without loss of generality we may assume that g is not approximately continuous at right at x. Then $\overline{d}^+(\{y \in I\colon |g(x)-g(y)| \geq \varepsilon\}, x) = c > 0$ for some $\varepsilon > 0$. There is a positive integer k such that $\lambda + \frac{c}{2k} < 1$ and $\frac{2-c}{2k} < \lambda$. Then $\frac{1}{k} < \lambda + \frac{c}{2k}$. Applying Lemma 2.2 to $\{y\colon |g(y)-g(x)| \geq \varepsilon\}$ and $a = \lambda + \frac{c}{2k}$, we can find a sequence of intervals $\{I_n = [a_n, b_n]\colon i \geq 1\}$ such that $x < \ldots < b_{n+1} < a_n < \ldots < b$, $d^+\big(\bigcup_{n=1}^{\infty} I_n, x\big) = \lambda + \frac{c}{2k}$ and $\overline{d}^+\Big(\{y\colon |g(y)-g(x)| \geq \varepsilon\} \cap \bigcup_{n=1}^{\infty} I_n, x\Big) \geq \frac{c}{k}$. Let $\{K_n = [c_n, d_n]\colon n \geq 1\}$ be a sequence of intervals such that $I_n \subset \operatorname{int} K_n$ for all $n \in \mathbb{N}$ and $\overline{d}^+\left(\bigcup_{n=1}^{\infty}(K_n \setminus I_n), x\right) = 0$. Let a function $f\colon I \to \mathbb{R}$ be defined by

$$f(y) = \begin{cases} 0 & \text{if } y \in (a,x] \cup [d_1, b) \cup \bigcup_{n=1}^{\infty} I_n, \\ -g(y)+g(x)+\varepsilon & \text{if } y \in \bigcup_{n=1}^{\infty} [d_{n+1}, c_n], \\ \text{linear in each connected component of } \bigcup_{n=1}^{\infty} K_n \setminus \bigcup_{n=1}^{\infty} \operatorname{int} I_n. \end{cases}$$

Since $g \in \mathcal{C}_{[\lambda,\varrho]}$, it is obvious that f is $[\lambda,\varrho]$-continuous at every point except at x. From inequalities

$$\underline{d}\big(\{y \in I\colon f(y) = f(x) = 0\}, x\big) \geq \underline{d}\left((a,x] \cup \bigcup_{n=1}^{\infty} I_n, x\right) = \underline{d}^+\left(\bigcup_{n=1}^{\infty} I_n, x\right) \geq \lambda + \tfrac{c}{2k} > \lambda$$

and

$$\overline{d}\big(\{y \in I\colon f(y) = f(x) = 0\}, x\big) \geq \overline{d}\left((a,x] \cup \bigcup_{n=1}^{\infty} K_n, x\right) = \overline{d}^-((a,x], x) = 1 > \varrho,$$

we deduce that f is $[\lambda,\varrho]$-continuous at x. Hence $f \in \mathcal{C}_{[\lambda,\varrho]}$.

On the other hand, we have $(f+g)(x) = g(x)$ and

$$\big\{y \in I\colon |(f+g)(y) - g(x)| < \varepsilon\big\} \cap \left([x,b) \setminus \bigcup_{n=1}^{\infty} K_n\right) = \emptyset.$$

We will show that $f+g$ is not $[\lambda,\varrho]$-continuous at x. Set $E = \{y\colon |(f+$

$g)(y) - g(x)| < \varepsilon\}$. Then we obtain

$$\underline{d}^+(E,x) \le \underline{d}^+\Big(E \cap \bigcup_{n=1}^{\infty} I_n, x\Big) + \overline{d}^+\Big(E \cap \bigcup_{n=1}^{\infty} (K_n \setminus I_n), x\Big) +$$

$$+\overline{d}^+\Big(E \cap \bigcup_{n=1}^{\infty} [d_{n+1}, c_n], x\Big) = \underline{d}^+\Big(\{y \in I \colon |g(y) - g(x)| < \varepsilon\} \cap \bigcup_{n=1}^{\infty} I_n, x\Big) + 0 + 0 =$$

$$= d^+\Big(\bigcup_{n=1}^{\infty} I_n, x\Big) - \overline{d}^+\Big(\{y \in I \colon |g(y) - g(x)| \ge \varepsilon\} \cap \bigcup_{n=1}^{\infty} I_n, x\Big) \le \lambda + \frac{c}{2k} - \frac{c}{k} < \lambda.$$

Therefore $f+g$ is not $[\lambda, \varrho]$-continuous at x. Hence $f+g \notin \mathcal{C}_{[\lambda,\rho]}$ and the proof is completed.

□

Lemma 3.1. *Let $f, g \colon I \to \mathbb{R}$ and $x \in I$. If both functions, f and g, are $[\lambda, \varrho]$-continuous at x and at least one of them is approximately continuous at x then $f+g$, fg, $\min\{f,g\}$ and $\max\{f,g\}$ are $[\lambda, \varrho]$-continuous at x.*

PROOF. Without loss of generality we may assume that f is approximately continuous at x. Therefore there exists a measurable set E such that $x \in E$, $\underline{d}(E,x) = 1$ and $f_{|E}$ is continuous at x. Since g is $[\lambda, \varrho]$-continuous at x, there is a measurable set F such that $x \in F$, x is a point of $[\lambda, \varrho]$-density of F and $g_{|F}$ is continuous at x. Therefore functions $f+g$, fg, $\min\{f,g\}$ and $\max\{f,g\}$ restricted to $E \cap F$ are continuous at x, $E \cap F$ is a measurable set,

$$\underline{d}(E \cap F, x) \ge \underline{d}(E,x) + \underline{d}(F,x) - 1 > 1 + \lambda - 1 = \lambda$$

and

$$\overline{d}(E \cap F, x) \ge \underline{d}(E,x) + \overline{d}(F,x) - 1 > 1 + \varrho - 1 = \varrho.$$

It follows that $f+g$, fg, $\min\{f,g\}$ and $\max\{f,g\}$ are $[\lambda, \varrho]$-continuous at x. □

Corollary 3.1. *If $f, g \colon I \to \mathbb{R}$, $f, g \in \mathcal{C}_{[\lambda,\rho]}$ and $D_{ap}(f) \cap D_{ap}(g) = \emptyset$, then $f+g$, fg, $\min\{f,g\}$ and $\max\{f,g\}$ belong to $\mathcal{C}_{[\lambda,\rho]}$.*

Corollary 3.2. *If $f, g \colon I \to \mathbb{R}$, $f \in \mathcal{C}_{[\lambda,\rho]}$ and $g \in \mathcal{A}$, then $f+g, fg, \min\{f,g\}, \max\{f,g\} \in \mathcal{C}_{[\lambda,\varrho]}$.*

Theorem 3.2. $\mathcal{M}_a(\mathcal{C}_{[\lambda,\varrho]}) = \mathcal{A}$.

PROOF. By Theorem 3.1, we get $\mathcal{C}_{[\lambda,\varrho]} \cap \mathcal{M}_a(\mathcal{C}_{[\lambda,\varrho]}) \subset \mathcal{A}$. By Corollary 3.2, we conclude that $\mathcal{A} \subset \mathcal{M}_a(\mathcal{C}_{[\lambda,\varrho]})$. The last needed inclusion, $\mathcal{M}_a(\mathcal{C}_{[\lambda,\varrho]}) \subset \mathcal{C}_{[\lambda,\varrho]}$, follows from Remark 3.1. □

4 The maximal multiplicative class

Definition 4.1. *Let $\mathcal{F}$ be a family of real functions defined on an open interval I. A set $\mathcal{M}_m(\mathcal{F}) = \{g\colon \forall_{f\in\mathcal{F}}\ fg \in \mathcal{F}\}$ is called the maximal multiplicative class for $\mathcal{F}$.*

In this section we characterize the maximal multiplicative class for $\mathcal{C}_{[\lambda,\varrho]}$.

Lemma 4.1. *Let $g \in \mathcal{C}_{[\lambda,\varrho]} \setminus \mathcal{A}$ and $x \in D_{ap}(g)$. If $g(x) \neq 0$ then there exists $f \in \mathcal{C}_{[\lambda,\varrho]}$ such that $fg \notin \mathcal{C}_{[\lambda,\varrho]}$.*

Proof. Without loss of generality we may assume that g is not approximately continuous from the right at x. Let $g(x) = t \neq 0$. Choose $0 < \varepsilon < |t|$ such that $\overline{d}^+\big(\{y\colon |g(y)-t| \geq \varepsilon\}, x\big) = c > 0$. There exists a positive integer k such that $\lambda + \frac{c}{2k} < 1$ and $\frac{2-c}{2k} < \lambda$. Then $\frac{1}{k} < \lambda + \frac{c}{2k}$. Applying Lemma 2.2, we can find a sequence $\{I_n = [a_n, b_n]\colon x < \ldots < b_{n+1} < a_n < \ldots < b, n \geq 1\}$ such that $d^+\Big(\bigcup_{n=1}^{\infty} I_n, x\Big) = \lambda + \frac{c}{2k}$ and $\overline{d}^+\big(\{y\colon |g(y)-t| \geq \varepsilon\} \cap \bigcup_{n=1}^{\infty} I_n, x\big) \geq \frac{c}{k}$.

Let $\{K_n = [c_n, d_n]\colon n \geq 1\}$ be a sequence of pairwise disjoint intervals satisfying conditions $I_n \subset \operatorname{int} K_n$ for $n \in \mathbb{N}$ and $\overline{d}^+\Big(\bigcup_{n=1}^{\infty}(K_n \setminus I_n), x\Big) = 0$. A function $f\colon I \to \mathbb{R}$ is defined in the following way

$$f(y) = \begin{cases} 1 & \text{if } y \in (a,x] \cup [d_1, b) \cup \bigcup_{n=1}^{\infty} I_n, \\ 0 & \text{if } y \in \bigcup_{n=1}^{\infty} [d_{n+1}, c_n], \\ \multicolumn{2}{l}{\text{linear in each connected component of } \bigcup_{n=1}^{\infty} K_n \setminus \bigcup_{n=1}^{\infty} \operatorname{int} I_n.} \end{cases}$$

Certainly, f is continuous at each point except x. Since

$$\underline{d}(\{y\colon f(y) = f(x) = 1\}, x) \geq \underline{d}\Big((-\infty, x] \cup \bigcup_{n=1}^{\infty} K_n, x\Big) = \underline{d}^+\Big(\bigcup_{n=1}^{\infty} I_n, x\Big) = \lambda + \frac{c}{2k}$$

and

$$\overline{d}(\{y\colon f(y) = f(x) = 1\}, x) \geq \overline{d}\Big((a,x] \cup \bigcup_{n=1}^{\infty} K_n, x\Big) = \overline{d}\big((a,x], x\big) = 1 > \rho,$$

we obtain that $f \in \mathcal{C}_{[\lambda,\varrho]}$.

On the other hand, we have $(fg)(x) = g(x)$ and

$$\big\{y \in I\colon |(fg)(y) - g(x)| < \varepsilon\big\} \cap \bigcup_{n=1}^{\infty} [d_{n+1}, c_n] = \emptyset.$$

We will show that fg is not $[\lambda,\varrho]$-continuous at x. Set $E=\{y\in I\colon |(fg)(y)-g(x)|<\varepsilon\}$. Then we obtain

$$\underline{d}^+(E,x)\le \underline{d}^+\Big(E\cap\bigcup_{n=1}^{\infty} I_n,x\Big)+\overline{d}^+\Big(E\cap\bigcup_{n=1}^{\infty}(K_n\setminus I_n),x\Big)+$$
$$+\overline{d}^+\Big(E\cap\bigcup_{n=1}^{\infty}[d_{n+1},c_n],x\Big)=\underline{d}^+\Big(\{y\in I\colon |g(y)-g(x)|<\varepsilon\}\cap\bigcup_{n=1}^{\infty} I_n,x\Big)+0+0=$$
$$=d^+\Big(\bigcup_{n=1}^{\infty} I_n,x\Big)-\overline{d}^+\Big(\{y\in I\colon |g(y)-g(x)|>\varepsilon\}\cap\bigcup_{n=1}^{\infty} I_n,x\Big)\le\lambda+\frac{c}{2k}-\frac{c}{k}<\lambda.$$

Therefore fg is not $[\lambda,\varrho]$-continuous at x. Thus $fg\notin\mathcal{C}_{[\lambda,\rho]}$, and the proof is completed. □

Definition 4.2. *Let $0<\lambda\le\varrho<1$. Let $\mathbf{P}(\lambda,\varrho)$ be a set of all functions $f\colon I\to\mathbb{R}$ satisfying the following conditions*

(P1) $D_{ap}(f)\subset N_f$, *where* $N_f=\{x\in I\colon f(x)=0\}$,

(P2) *for each $x\in D_{ap}(f)$ and for each measurable set E such that $E\supset N_f$ and $\underline{d}(E,x)>\lambda$,*
$\overline{d}(E,x)>\varrho$ we have

$$\lim_{\varepsilon\to 0^+}\underline{d}(E\cap\{y\colon |f(y)-f(x)|<\varepsilon\},x)>\lambda$$

and

$$\lim_{\varepsilon\to 0^+}\overline{d}(E\cap\{y\colon |f(y)-f(x)|<\varepsilon\},x)>\varrho.$$

Corollary 4.1. *Let $0<\lambda\le\varrho<1$. Then $\mathcal{A}\subset\mathbf{P}(\lambda,\varrho)$.*

Theorem 4.1. *$\mathcal{M}_m(\mathcal{C}_{[\lambda,\varrho]})=\mathbf{P}(\lambda,\varrho)$ for each $0<\lambda\le\varrho<1$.*

Proof. Let $g\in\mathbf{P}(\lambda,\varrho)$ and $f\in\mathcal{C}_{[\lambda,\varrho]}$. Fix any $x\in I$. There exists a measurable set E such that $x\in E$, $\underline{d}(E,x)>\lambda$, $\overline{d}(E,x)>\varrho$ and $f_{|E}$ is continuous at x. First, assume that g is approximately continuous at x. Then, by Lemma 3.1, fg is $[\lambda,\varrho]$-continuous at x.

Now, consider the second case, $x\in D_{ap}(g)$. Applying $(P1)$, we obtain $g(x)=0$. Since $f_{|E}$ is continuous at x, we conclude that there exist real numbers r, M such that $|f(y)|<M$ for $y\in E\cap[x-r,x+r]$. It follows, in view of $(P2)$, that

$$\lim_{\varepsilon\to 0^+}\underline{d}(\{y\colon |(fg)(y)|<\varepsilon\},x)\ge\lim_{\varepsilon\to 0^+}\underline{d}(\{y\colon |g(y)|<\tfrac{\varepsilon}{M}\}\cap E,x)=$$
$$=\lim_{\varepsilon\to 0^+}\underline{d}(\{y\colon |g(y)|<\varepsilon\}\cap E,x)>\lambda$$

and

$$\lim_{\varepsilon\to 0^+} \overline{d}(\{y\colon |(fg)(y)| < \varepsilon\}, x) \geq \lim_{\varepsilon\to 0^+} \overline{d}(\{y\colon |g(y)| < \tfrac{\varepsilon}{M}\} \cap E, x) = \\ = \lim_{\varepsilon\to 0^+} \overline{d}(\{y\colon |g(y)| < \varepsilon\} \cap E, x) > \varrho.$$

By Theorem 2.1, fg is $[\lambda, \varrho]$-continuous at x. Hence $fg \in \mathcal{C}_{[\lambda,\varrho]}$. Thus we have proven that $\mathbf{P}(\lambda, \varrho) \subset \mathcal{M}_m(\mathcal{C}_{[\lambda,\varrho]})$.

Now, let us assume that $g \in \mathcal{M}_m(\mathcal{C}_{[\lambda,\varrho]})$. If $x \in D_{ap}(g)$ then, by Lemma 4.1, we get $g(x) = 0$. Therefore g fulfils condition $(P1)$. Take any measurable set E such that $\underline{d}(E, x) > \lambda$ and $\overline{d}(E, x) > \varrho$. By Lemma 2.3 (and corresponding lemma for left-sided density) we can find two sequences of intervals $\{I_n = [a_n, b_n]\colon \ldots < b_n < a_{n+1} < \ldots < \ldots x, n \geq 1\}$ and $\{J_k = [c_k, d_k]\colon x < \ldots < d_{k+1} < c_k < \ldots, n \geq 1\}$ such that

$$\overline{d}\Big(E \setminus \Big(\bigcup_{n=1}^{\infty} I_n \cup \bigcup_{k=1}^{\infty} J_k\Big), x\Big) = \overline{d}\Big(\Big(\bigcup_{n=1}^{\infty} I_n \cup \bigcup_{k=1}^{\infty} J_k\Big) \setminus E, x\Big) = 0.$$

Let $\bar{I}_n = [\bar{a}_n, \bar{b}_n]$ and $\bar{J}_k = [\bar{c}_k, \bar{d}_k]$ be pairwise disjoint closed intervals such that $I_n \subset \operatorname{int} \bar{I}_n$, $J_k \subset \operatorname{int} \bar{J}_k$ for all $n, k \in \mathbb{N}$ and $\overline{d}\Big(\bigcup\limits_{n=1}^{\infty} (\bar{I}_n \setminus I_n) \cup \bigcup\limits_{k=1}^{\infty} (\bar{J}_k \setminus J_k), x\Big) = 0$. By Lemma 2.1, we have $\underline{d}\Big(\bigcup\limits_{n=1}^{\infty} I_n \cup \bigcup\limits_{k=1}^{\infty} J_k, x\Big) = \underline{d}(E, x) > \lambda$ and $\overline{d}\Big(\bigcup\limits_{n=1}^{\infty} I_n \cup \bigcup\limits_{k=1}^{\infty} J_k, x\Big) = \overline{d}(E, x) > \varrho$. Since for each $k \in \mathbb{N}$

$$\lim_{\alpha\to\infty} \big|([\bar{d}_{k+1}, \bar{c}_k] \cap \{y\colon |g(y) \cdot \alpha| < 1\}) \setminus N_g\big| = 0,$$

we get that for each $k \in \mathbb{N}$ there exists a number α_k, such that

$$\big|([\bar{d}_{k+1}, \bar{c}_k] \cap \{y\colon |g(y) \cdot \alpha_k| < 1\}) \setminus N_g\big| < \frac{\bar{d}_{k+1} - x}{2^k}. \tag{1}$$

Moreover,

$$N_g \cap \bigcup_{k=1}^{\infty} [\bar{d}_{k+1}, \bar{c}_k] \subset E \setminus \bigcup_{k=1}^{\infty} J_k. \tag{2}$$

From (1) and (2), it is easy to verify that

$$\overline{d}^+\Big(\bigcup_{k=1}^{\infty} ([\bar{d}_{k+1}, \bar{c}_k] \cap \{y\colon |g(y) \cdot \alpha_k| < 1\}) \setminus N_g, x\Big) = 0.$$

Similarly, we can find a sequence $\{\beta_n\colon n \geq 1\}$ such that

$$\overline{d}^{-}\Big(\bigcup_{k=1}^{\infty}([\bar{b}_n, \bar{a}_{n+1}] \cap \{y\colon |g(y)\cdot\beta_n| < 1\}) \setminus N_g, x\Big) = 0.$$

Let a function $f\colon I \to \mathbb{R}$ be defined by

$$f(y) = \begin{cases} 1 & \text{if } y \in \bigcup_{n=1}^{\infty} I_n \cup \bigcup_{k=1}^{\infty} J_k \cup (a, \bar{a}_1] \cup [\bar{d}_1, b) \cup \{x\}, \\ \alpha_k & \text{if } y \in [\bar{d}_{k+1}, \bar{c}_k],\ k = 1, 2, \ldots, \\ \beta_n & \text{if } y \in [\bar{b}_n, \bar{a}_{n+1}],\ n = 1, 2, \ldots, \\ \text{linear in } [\bar{a}_n, a_n],\ [b_n, \bar{b}_n],\ [\bar{c}_k, c_k] \text{ and } [d_k, \bar{d}_k], k = 1, 2, \ldots,\ n = 1, 2, \ldots. \end{cases}$$

Directly from the definition of f, it follows that it is continuous at each point except x. If $E_1 = \bigcup_{n=1}^{\infty} I_n \cup \bigcup_{k=1}^{\infty} J_k \cup (-\infty, \bar{a}_1] \cup [\bar{d}_1, \infty) \cup \{x\}$ then f restricted to E_1 is constant, so in particular, it is continuous at x. Moreover,

$$\underline{d}(E_1, x) \geq \underline{d}\Big(\bigcup_{n=1}^{\infty} I_n \cup \bigcup_{k=1}^{\infty} J_k, x\Big) = \underline{d}(E, x) > \lambda$$

and

$$\overline{d}(E_1, x) \geq \overline{d}\Big(\bigcup_{n=1}^{\infty} I_n \cup \bigcup_{k=1}^{\infty} J_k, x\Big) = \overline{d}(E, x) > \varrho.$$

Therefore f is $[\lambda, \varrho]$-continuous at x. Hence $f \in \mathcal{C}_{[\lambda,\varrho]}$. Moreover, $fg(x) = 0$.

Put $E_\varepsilon = \{y \in I\colon |(fg)(y) - (fg)(x)| < \varepsilon\} = \{y \in I\colon |(fg)(y)| < \varepsilon\}$ for $0 < \varepsilon < 1$. Since $g \in \mathcal{M}_m(\mathcal{C}_{[\lambda,\varrho]})$, we get $\lim_{\varepsilon\to 0^+} \underline{d}(E_\varepsilon, x) > \lambda$ and $\lim_{\varepsilon\to 0^+} \overline{d}(E_\varepsilon, x) > \varrho$. On the other hand,

$$\underline{d}(E_\varepsilon, x) \leq \underline{d}\Big(E_\varepsilon \cap \Big(\bigcup_{n=1}^{\infty} I_n \cup \bigcup_{k=1}^{\infty} J_k\Big), x\Big) + \overline{d}\Big(E_\varepsilon \cap \Big(\bigcup_{n=1}^{\infty} [\bar{b}_n, \bar{a}_{n+1}] \cup \bigcup_{k=1}^{\infty} [\bar{d}_{k+1}, \bar{c}_k]\Big), x\Big) +$$

$$+ \overline{d}\Big(E_\varepsilon \cap \Big(\bigcup_{n=1}^{\infty} (\bar{I}_n \setminus I_n) \cup \bigcup_{k=1}^{\infty} (\bar{J}_k \setminus J_k)\Big), x\Big) = \underline{d}\Big(E_\varepsilon \cap \Big(\bigcup_{n=1}^{\infty} I_n \cup \bigcup_{k=1}^{\infty} J_k\Big), x\Big) =$$

$$= \underline{d}\Big(\{y \in I\colon |g(y)| < \varepsilon\} \cap \Big(\bigcup_{n=1}^{\infty} I_n \cup \bigcup_{k=1}^{\infty} J_k\Big), x\Big) = \underline{d}\big(\{y \in I\colon |g(y)| < \varepsilon\} \cap F, x\big)$$

and

$$\overline{d}(E_\varepsilon, x) \leq \overline{d}\Big(E_\varepsilon \cap \Big(\bigcup_{n=1}^{\infty} I_n \cup \bigcup_{k=1}^{\infty} J_k\Big), x\Big) + \overline{d}\Big(E_\varepsilon \cap \Big(\bigcup_{n=1}^{\infty} [\bar{b}_n, \bar{a}_{n+1}] \cup \bigcup_{k=1}^{\infty} [\bar{d}_{k+1}, \bar{c}_k]\Big), x\Big) +$$

$$+ \overline{d}\Big(E_\varepsilon \cap \Big(\bigcup_{n=1}^{\infty} (\bar{I}_n \setminus I_n) \cup \bigcup_{k=1}^{\infty} (\bar{J}_k \setminus J_k)\Big), x\Big) = \overline{d}\Big(E_\varepsilon \cap \Big(\bigcup_{n=1}^{\infty} I_n \cup \bigcup_{k=1}^{\infty} J_k\Big), x\Big) =$$

$$= \overline{d}\Big(\{y \in I : |g(y)| < \varepsilon\} \cap \Big(\bigcup_{n=1}^{\infty} I_n \cup \bigcup_{k=1}^{\infty} J_k\Big), x\Big) = \overline{d}\big(\{y \in I : |g(y)| < \varepsilon\} \cap F, x\big)$$

for each $0 < \varepsilon < 1$. Hence $\lim_{\varepsilon \to 0^+} \underline{d}\big(\{y \in I : |g(y)| < \varepsilon\} \cap F, x\big) \geq \lim_{\varepsilon \to 0^+} \underline{d}(E_\varepsilon, x) > \lambda$ and $\lim_{\varepsilon \to 0^+} \overline{d}\big(\{y \in I : |g(y)| < \varepsilon\} \cap F, x\big) \geq \lim_{\varepsilon \to 0^+} \overline{d}(E_\varepsilon, x) > \varrho$. It follows that condition $(P2)$ is fulfilled. $\square$

Corollary 4.2. *If a function g satisfies condition $(P1)$ and for each $x \in D_{ap}(g)$ we have $\underline{d}(N_g, x) > \lambda$ and $\overline{d}(N_g, x) > \varrho$ then $g \in \mathcal{M}_m(\mathcal{C}_{[\lambda,\varrho]})$.*

Corollary 4.3. $\mathcal{A} \subsetneq \mathcal{M}_m(\mathcal{C}_{[\lambda,\varrho]})$.

Example 4.1. Fix any $\lambda \in (0, 1)$. We will show that the sharp inequality $\underline{d}(N_g, x) > \lambda$ in Corollary 4.2 is essential. We will construct a function $g : \mathbb{R} \to \mathbb{R}$ such that g is discontinuous only at $x = 0$ belongs to $C_{[\lambda,\varrho]}$ and does not belong to $\mathcal{M}_m(\mathcal{C}_{[\lambda,\varrho]})$. Let $\{I_n = [a_n, b_n] : 0 < \ldots < b_{n+1} < a_n < \ldots, n \geq 1\}$ be a sequence of intervals such that $d^+\Big(\bigcup_{n=1}^{\infty} I_{3n}, 0\Big) = \lambda$ and

$$d^+\Big(\bigcup_{n=1}^{\infty} I_{3n-1}, 0\Big) = d^+\Big(\bigcup_{n=1}^{\infty} I_{3n-2}, 0\Big) = \frac{1-\lambda}{2}.$$

Then

$$\underline{d}^+\Big(\bigcup_{n=1}^{\infty} I_n, 0\Big) \geq \underline{d}^+\Big(\bigcup_{n=1}^{\infty} I_{3n}, 0\Big) + \underline{d}^+\Big(\bigcup_{n=1}^{\infty} I_{3n-1}, 0\Big) + \underline{d}^+\Big(\bigcup_{n=1}^{\infty} I_{3n-2}, 0\Big) = 1.$$

Thus $\underline{d}^+\big(\bigcup_{n=1}^{\infty} I_n, 0\big) = 1$. Define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0] \cup [b_1, \infty) \cup \bigcup_{n=1}^{\infty} I_{3n}, \\ 1 & \text{if } x \in \bigcup_{n=1}^{\infty} I_{3n-1}, \\ \frac{1}{n} & \text{if } x \in \bigcup_{n=1}^{\infty} I_{3n-2}, \\ \text{linear on the intervals } [b_{n+1}, a_n], \ n = 1, 2, \ldots. \end{cases}$$

It is clear that g is continuous at each point except 0 and $N_g = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} I_{3n}$. Hence $\underline{d}(N_g, 0) = \lambda$ and $\overline{d}(N_g, 0) = 1$. Let $E = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} (I_{3n} \cup I_{3n-2})$. Then $g_{|E}$ is continuous at 0, $\overline{d}(E, 0) = \overline{d}^{-}((-\infty, 0], 0) = 1$ and

$$\underline{d}(E,0) = \underline{d}^{+}\Big(\bigcup_{n=1}^{\infty}(I_{3n} \cup I_{3n-2}), 0\Big) \geq \underline{d}^{+}\Big(\bigcup_{n=1}^{\infty} I_{3n}, 0\Big) + \underline{d}^{+}\Big(\bigcup_{n=1}^{\infty} I_{3n-2}, 0\Big) = \frac{1+\lambda}{2} > \lambda.$$

Hence g is $[\lambda, \varrho]$-continuous at 0 and $g \in \mathcal{C}_{[\lambda,\varrho]}$. Besides, $D_{ap}(g) \subset N_g$. On the other hand, let $F = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} (I_{3n} \cup I_{3n-1})$. Then $N_g \subset F$, $\overline{d}(F, 0) = \overline{d}^{-}((-\infty, 0], 0) = 1$ and

$$\underline{d}(F,0) = \underline{d}^{+}\Big(\bigcup_{n=1}^{\infty}(I_{3n} \cup I_{3n-1}), 0\Big) \geq \underline{d}^{+}\Big(\bigcup_{n=1}^{\infty} I_{3n}, 0\Big) + \underline{d}^{+}\Big(\bigcup_{n=1}^{\infty} I_{3n-1}, 0\Big) = \frac{1+\lambda}{2} > \lambda.$$

But

$$\underline{d}(F \cap \{x \in \mathbb{R} \colon |g(x)| < \varepsilon\}, 0) = \underline{d}^{+}\Big(\bigcup_{n=1}^{\infty} I_{3n}, 0\Big) = \lambda$$

for each $0 < \varepsilon < 1$. It follows that condition $(P2)$ is not fulfilled. Hence $g \notin \mathcal{M}_m(\mathcal{C}_{[\lambda,\varrho]})$.

5 $Min_{\mathcal{F}}$ and $Max_{\mathcal{F}}$

Definition 5.1. *Let $\mathcal{F}$ be a family of real functions defined on an open interval I. Then we define* $\mathbf{Min}_{\mathcal{F}} = \{g \colon I \to \mathbb{R} \colon \forall_{f \in \mathcal{F}} \min\{f, g\} \in \mathcal{F}\}$ *and* $\mathbf{Max}_{\mathcal{F}} = \{g \colon \forall_{f \in \mathcal{F}} \max\{f, g\} \in \mathcal{F}\}$.

Lemma 5.1.

1. $\mathbf{Min}_{\mathcal{C}_{[\lambda,\varrho]}} = \{-f \colon f \in \mathbf{Max}_{\mathcal{C}_{[\lambda,\varrho]}}\}$.
2. $\mathbf{Min}_{\mathcal{C}_{[\lambda,\varrho]}} \subset \mathcal{C}_{[\lambda,\varrho]}$ *and* $\mathbf{Max}_{\mathcal{C}_{[\lambda,\varrho]}} \subset \mathcal{C}_{[\lambda,\varrho]}$.

Proof. 1. It follows immediately from equality $\max\{f, g\} = -\min\{-f, -g\}$ and property $f \in \mathcal{C}_{[\lambda,\varrho]} \Rightarrow -f \in \mathcal{C}_{[\lambda,\varrho]}$.

2. Let $f \in \mathbf{Min}_{\mathcal{C}_{[\lambda,\varrho]}}$ and fix $x \in I$. Take the constant functions $g(y) = f(x)+1$ for $y \in I$. Then $g \in \mathcal{C}_{[\lambda,\varrho]}$, $\min\{f, g\} \in \mathcal{C}_{[\lambda,\varrho]}$ and $\min\{f(x), g(x)\} = f(x)$. Moreover,

$$\{y \in I \colon |\min\{f(y), g(y)\} - f(x)| < \varepsilon\} = \{y \in I \colon |f(y) - f(x)| < \varepsilon\}$$

for all $0<\varepsilon<1$. Hence f is $[\lambda,\varrho]$-continuous at x which gives an inclusion $\mathbf{Min}_{\mathcal{C}_{[\lambda,\varrho]}} \subset \mathcal{C}_{[\lambda,\varrho]}$. Moreover, $\mathbf{Max}_{\mathcal{C}_{[\lambda,\varrho]}} = -\mathbf{Min}_{\mathcal{C}_{[\lambda,\varrho]}} \subset -\mathcal{C}_{[\lambda,\varrho]} = \mathcal{C}_{[\lambda,\varrho]}$.

□

Theorem 5.1. $\mathbf{Max}_{\mathcal{C}_{[\lambda,\varrho]}} = \mathcal{A}$.

Proof. By Corollary 3.2, we get $\mathcal{A} \subset \mathbf{Max}_{\mathcal{C}_{[\lambda,\varrho]}}$.

Let $g \notin \mathcal{A}$ and g is not approximately continuous at $x \in I$. Without loss of generality we may assume that g is not approximately continuous at right at x. Therefore $\overline{d}^{+}(\{y \in I\colon |g(y)-f(x)| \geq \varepsilon\}, x) = c > 0$ for some $0<\varepsilon<1$. As earlier, we choose a positive integer k such that $\lambda + \frac{c}{2k} < 1$, $\frac{2-c}{2k} < \lambda$ and $\frac{1}{k} < \lambda + \frac{c}{2k}$. Applying Lemma 2.2 to $\{y\colon |g(y)-g(x)| \geq \varepsilon\}$ and $a = \lambda + \frac{c}{2k}$, we can find a sequence of intervals $\{I_n = [a_n,b_n]\colon i \geq 1\}$ such that $x < \ldots < b_{n+1} < a_n < \ldots$, $d^{+}\big(\bigcup_{n=1}^{\infty} I_n, x\big) = \lambda + \frac{c}{2k}$ and $\overline{d}^{+}\Big(\{y\colon |g(y)-g(x)| \geq \varepsilon\} \cap \bigcup_{n=1}^{\infty} I_n, x\Big) \geq \frac{c}{k}$. Let $\{K_n = [c_n,d_n]\colon n \geq 1\}$ be a sequence of pairwise disjoint intervals such that $I_n \subset \mathrm{int} K_n$ for all $n \in \mathbb{N}$ and $\overline{d}^{+}\Big(\bigcup_{n=1}^{\infty}(K_n \setminus I_n), x\Big) = 0$. Let a function $f\colon I \to \mathbb{R}$ be defined in the following way

$$f(y) = \begin{cases} g(x)-1 & \text{if } y \in (a,x] \cup [d_1,b) \cup \bigcup_{n=1}^{\infty} I_n, \\ g(x)+1 & \text{if } y \in \bigcup_{n=1}^{\infty} [d_{n+1}, c_n], \\ \text{linear in every connected component of } & \bigcup_{n=1}^{\infty} K_n \setminus \bigcup_{n=1}^{\infty} \mathrm{int} I_n. \end{cases}$$

It is obvious that f is $[\lambda,\varrho]$-continuous at each point except x. Inequalities

$$\underline{d}\big(\{y \in I\colon f(y)=f(x)=0\}, x\big) \geq \underline{d}\Big((a,x] \cup \bigcup_{n=1}^{\infty} I_n, x\Big) = \underline{d}^{+}\Big(\bigcup_{n=1}^{\infty} I_n, x\Big) \geq \lambda + \tfrac{c}{2k} > \lambda$$

and

$$\overline{d}\big(\{y \in I\colon f(y)=f(x)=0\}, x\big) \geq \overline{d}\Big((a,x] \cup \bigcup_{n=1}^{\infty} I_n, x\Big) = \overline{d}^{-}\Big((a,x], x\Big) = 1 > \varrho,$$

imply that f is $[\lambda,\varrho]$-continuous at x. Hence $f \in \mathcal{C}_{[\lambda,\varrho]}$.

We will show that $\max\{f,g\}$ is not $[\lambda,\varrho]$-continuous at x. Certainly, $\max\{f(x),g(x)\} = g(x)$. Set $E = \{y \in I\colon |\max\{f(y),g(y)\} - g(x)| < \varepsilon\}$.

Then $E \cap \bigcup_{n=1}^{\infty} [b_{n+1}, c_n] = \emptyset$. Moreover,

$$\underline{d}^+(E,x) \le \underline{d}^+\Big(E \cap \bigcup_{n=1}^{\infty} I_n, x\Big) + \overline{d}^+\Big(E \cap \bigcup_{n=1}^{\infty} (K_n \setminus I_n), x\Big) +$$
$$+ \overline{d}^+\Big(E \cap \bigcup_{n=1}^{\infty} [d_{n+1}, c_n], x\Big) = \underline{d}^+\Big(\{y \in I\colon |g(y) - g(x)| < \varepsilon\} \cap \bigcup_{n=1}^{\infty} I_n, x\Big) =$$
$$= d^+\Big(\bigcup_{n=1}^{\infty} I_n, x\Big) - \overline{d}^+\Big(\{y \in I\colon |g(y)-g(x)| \ge \varepsilon\} \cap \bigcup_{n=1}^{\infty} I_n, x\Big) \le \lambda + \frac{c}{2k} - \frac{c}{k} < \lambda.$$

Therefore $\max\{f,g\}$ is not $[\lambda,\varrho]$-continuous at x. Hence $\max\{f,g\} \notin \mathcal{C}_{[\lambda,\rho]}$ which completes the proof. □

Corollary 5.1. $\mathbf{Min}_{\mathcal{C}_{[\lambda,\varrho]}} = -\mathbf{Max}_{\mathcal{C}_{[\lambda,\varrho]}} = \mathcal{A}$.

References

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